

No unbounded arbitrage, weak no market arbitrage and no arbitrage price system conditions; Equivalent conditions*

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Abstract

Page and Wooders (1996) prove that the no unbounded arbitrage (NUBA), a special case of a condition in Page (1987), is equivalent to the existence of a no arbitrage price system (NAPS) when no agent has non-null useless vectors. Allouch, Le Van and Page (2002) extend the NAPS introduced by Werner (1987) and show that this condition is equivalent to the weak no market arbitrage (WNMA) of Hart (1974). They mention that this result implies the one given by Page and Wooders (1996). In this note, we show that all these conditions are equivalent.

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1 Introduction

Allouch, Le Van and Page (2002) consider the problem of existence of competitive equilibrium in an unbounded exchange economy. They extend the definition of no arbitrage price system (NAPS) introduced by Werner (1987) to the case where some agent in the economy has only useless vectors. They show that an extension of NAPS condition of Werner (1987) is actually equivalent to the weak no market arbitrage (WNMA) condition introduced by Hart (1974). They mention that this result implies one given by Page and Wooders (1996) who prove that no unbounded arbitrage (NUBA) condition, a special case of Page (1987), is equivalent to NAPS when no agent has non-null useless vectors. The proof of the claims consist of two parts. One is very easy (NAPS implies WNMA or NAPS implies NUBA). The converse part is more difficult.

The purpose of this note is to show that when the statement NUBA implies NAPS (Page and Wooders, 1996) is true then we have WNMA implies NAPS (Allouch, Le Van and Page, 2002). But it is obvious that if the second statement holds then the first one also holds. The novelty of the result of this note is that the results are self-contained. While Allouch, Le Van and Page (2002) prove WNMA implies NAPS by using a difficult Lemma in Rockafellar (1970) and then deduce that NUBA implies NAPS when no agent has non-null useless vectors, we show that these conditions are actually circular. In some mathematical senses, these conditions let us to think of the circular tours of Brouwer and Kakutani fixed-point theorems (Zeidler. E, 1992). Moreover, proofs are simple and elementary.

We consider an unbounded exchange economy \mathcal{E} with m agents indexed by $i = 1, \dots, m$. For each agent there is an endowment $e^i \in \mathbb{R}^l$, a closed convex non-empty consumption set $X_i \subset \mathbb{R}^l$ and a upper semi-continuous, quasi-concave utility function u^i from X_i to \mathbb{R} .

For a subset $X \subset \mathbb{R}^l$, let denote $\text{int}X$ the interior of X , X^0 is the polar of X where $X^0 = \{p \in \mathbb{R}^l \mid p \cdot x \leq 0, \forall x \in X\}$ and $X^{00} = (X^0)^0$. If X is closed, convex and contains the origin then $X^{00} = X$.

For $x \in X_i$, agent i 's weak preferred set at x is

$$\widehat{P}^i(x) = \{y \in X_i \mid u^i(y) \geq u^i(x)\}.$$

Let $R_i(x)$ be recession cone of $\widehat{P}^i(x)$ (see Rockafellar, 1970). The set $R_i(x)$ is called the set of *useful vectors* for u^i is given as

$$R_i(x) = \{w \in \mathbb{R}^l \mid u^i(x + \lambda w) \geq u^i(x), \text{ for all } \lambda \geq 0\}.$$

It is easy to check that $R_i(x)$ is a closed convex cone.

The lineality space of i is defined by

$$L_i(x) = \{w \in \mathbb{R}^l \mid u^i(x + \lambda w) = u^i(x), \text{ for all } \lambda \in \mathbb{R}\} = R_i(x) \cap -R_i(x)$$

Elements in $L_i(x)$ will be called *useless vectors* at x . Note that $R_i(x)$ and $L_i(x)$ do not depend on x , let us set $R_i = R_i(e^i)$, $L_i = L_i(e^i)$. Denote L_i^\perp is the orthogonal space of L_i .

Let us first recall the *no-unbounded-arbitrage* condition denoted now on by NUBA introduced by Page (1987) and Page-Wooders (1996) which requires non-existence of an unbounded set of mutually compatible net trades that are utility non decreasing.

Definition 1 *The economy satisfies the NUBA condition if $\sum_{i=1}^m w^i = 0$ and $w^i \in R_i$ for all i implies $w^i = 0$ for all i .*

There exists a weaker condition, called the *weak-no-market-arbitrage* condition (WNMA), introduced by Hart (1974). This condition requires that all mutually compatible net trades which are utility non-decreasing be useless.

Definition 2 *The economy satisfies the WNMA condition if $\sum_{i=1}^m w^i = 0$ and $w^i \in R_i$ for all i implies $w^i \in L_i$ for all i .*

If $L_i = \{0\}$, $\forall i$, then WNMA is equivalent to NUBA.

We shall use the concepts of *no-arbitrage-price system* condition (NAPS) of Allouch, Le Van, Page (2002). Define the notion of *no-arbitrage price*:

$$\mathbf{Definition 3} \quad S_i = \left\{ \begin{array}{l} \{p \in L_i^\perp \mid p \cdot w > 0, \forall w \in (R_i \cap L_i^\perp) \setminus \{0\} \text{ if } R_i \setminus L_i \neq \emptyset\} \\ L_i^\perp \text{ if } R_i = L_i \end{array} \right\}$$

Observe that, when $L_i = \{0\}$, then we can write

$$S_i = \{p \in \mathbb{R}^l \mid p \cdot w > 0, \forall w \in R_i \setminus \{0\}\}.$$

Definition 4 *The economy \mathcal{E} satisfies the NAPS condition if $\cap_i S_i \neq \emptyset$.*

2 The equivalent conditions

As we mentioned above, the proofs of the implications NAPS \implies NUBA and NAPS \implies WNMA are easy. We now give elementary proofs for NUBA \implies NAPS and WNMA \implies NAPS.

The following lemma is useful in our proof:

Lemma 1 $WNMA \implies \sum_i (R_i \cap L_i^\perp)$ is closed.

In particular, if $L_i = \{0\}$ for all i , then $NUBA \implies \sum_i R_i$ is closed.

Proof: Assume that there exists a sequence $\sum_i w_n^i \rightarrow w$, with $w_n^i \in R_i \cap L_i^\perp$ for all i and n . We shall prove that the sum $\sum_i \|w_n^i\|$ is bounded, and then the vector w is in $\sum_i (R_i \cap L_i^\perp)$. Suppose that

$$\lim_{n \rightarrow \infty} \sum_i \|w_n^i\| = +\infty$$

Then we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sum_{i=1}^m \frac{w_n^i}{\sum_i \|w_n^i\|} &= 0, \\ \lim_{n \rightarrow +\infty} \sum_{i=1}^m \frac{\|w_n^i\|}{\sum_i \|w_n^i\|} &= 1. \end{aligned}$$

Therefore we can suppose that $\frac{w_n^i}{\sum_i \|w_n^i\|} \rightarrow w^i$ when $n \rightarrow +\infty$. Note that since R_i is a closed convex cone, we have $w^i \in R_i$ and $\sum_i w^i = 0$, $\sum_i \|w^i\| = 1$. But $WNMA$ implies that when $w^i \in L_i$, we also have $w^i \in L_i^\perp$. Hence, for all i , $w^i = 0$ that leads to a contradiction. ■

The following result has been proven by Page and Wooders (1996) where they used Dubovitskii-Milyutin (1965) Theorem. We give here an elementary proof to make the note self-contained.

Proposition 1 Assume $L_i = \{0\}$, $\forall i$. Then $NUBA \implies NAPS$.

Proof: Since $L_i = \{0\}$, it holds that $S_i \neq \emptyset \forall i$. Assume now that $\cap_i S_i = \emptyset$. Then $\cap_i \bar{S}^i$ is contained in a linear subspace $H \subset \mathbb{R}^l$ since $\text{int} \cap_i S_i = \text{int} \cap_i \bar{S}^i = \emptyset$.

It follows from $\bar{S}^i = -(R_i)^0$ that $\cap_i \bar{S}^i = -(\sum_i R_i)^0 \subset H$.

This implies

$$H^\perp \subset \left(\sum_i R_i \right)^{00}.$$

The sum $\sum_i R_i$ is closed by Lemma 1, hence $\sum_i R_i = \left(\sum_i R_i \right)^{00}$ since it is closed convex set and contains the origin. Hence, $H^\perp \subset \sum_i R_i$ and $\sum_i R_i$ contains a line.

Thus there exist $r \in H^\perp$, $r \neq 0$, $-r \in H^\perp$ and $(r^1, \dots, r^m) \neq 0$, $r^i \in R_i$ such that

$$r = \sum_{i=1}^m r^i$$

Since $-r \in \sum_i R_i$, there exist $(r'^1, \dots, r'^m) \neq 0$, $r'^i \in R_i$ such that

$$\sum_i r'^i = -r.$$

Therefore $\sum_i (r^i + r'^i) = 0$ and $r^i + r'^i \in R_i$ since R_i is the convex cone. By the NUBA condition, we have $r^i = -r'^i$. This means that, for some i , R_i contains a line and $S_i = \emptyset$: a contradiction. ■

Allouch, Le Van and Page (2002) prove the equivalence of NAPS and WNMA by using a lemma which is based on the concept of a support function (Corollary 16.2.2 in Rockafellar, 1970). From Proposition 1, we get the following proposition, the proof of which is elementary.

Proposition 2 $WNMA \implies \cap_i S_i \neq \emptyset$

Proof: Consider a new economy $\tilde{\mathcal{E}} = (\tilde{X}_i, \tilde{u}^i, \tilde{e}^i)$ defined by

$$\begin{aligned}\tilde{X}_i &= X_i \cap L_i^\perp, \quad \tilde{u}^i = u^i |_{\tilde{X}_i}, \quad \tilde{e}^i = (e^i)^\perp \\ \tilde{R}_i &= R_i \cap L_i^\perp\end{aligned}$$

We have $\tilde{L}_i = (R_i \cap L_i^\perp) \cap -(R_i \cap L_i^\perp) = \{0\}$. Hence, in the economy $\tilde{\mathcal{E}}$, WNMA is NUBA. Proposition 1 implies that $\cap_i \tilde{S}_i \neq \emptyset$ where

$$\tilde{S}_i = \{p \in \mathbb{R}^l \mid p \cdot w > 0, \forall w \in (R_i \cap L_i^\perp) \setminus \{0\}\}.$$

It is easy to see that $\tilde{S}_i = S_i + L_i$. Thus, if $(\cap_i \tilde{S}_i) \cap (\cap_i L_i^\perp) \neq \emptyset$, then $\cap_i S_i \neq \emptyset$.

We will show that $(\cap_i \tilde{S}_i) \cap (\cap_i L_i^\perp) \neq \emptyset$. On the contrary, suppose that $(\cap_i \tilde{S}_i) \cap (\cap_i L_i^\perp) = \emptyset$. By using a separation theorem (see, e.g., Theorem 11.3, Rockafellar 1970), note that $\cap_i \tilde{S}_i$ is open and $\cap_i L_i^\perp$ is a subspace, there exists a vector $w \neq 0$ such that:

$$w \cdot p > 0 = w \cdot l, \quad \forall p \in \cap_i \tilde{S}_i, \quad \forall l \in \cap_i L_i^\perp.$$

Therefore, we get

$$w \in \sum_{i=1}^m L_i.$$

Moreover, we have

$$w \cdot p \geq 0 \quad \forall p \in \overline{\cap_i \tilde{S}_i}.$$

Since for every i , \tilde{S}_i is open, and $\cap_i \tilde{S}_i \neq \emptyset$ we have $\overline{\cap_i \tilde{S}_i} = \cap_i \overline{\tilde{S}_i}$. From the Lemma 1, $\sum_i \tilde{R}_i$ is closed. We then have:

$$w \in -(\cap_i \tilde{S}_i)^0 = \left(\sum_i \tilde{R}_i\right)^{00} = \sum_i \tilde{R}_i$$

Therefore, for each i , there exists $l_i \in L_i$, $\tilde{w}^i \in \tilde{R}_i$ such that $w = \sum_i l_i = \sum_i \tilde{w}^i$, in other words, $\sum_i (l_i - \tilde{w}^i) = 0$. The WNMA implies that $l_i - \tilde{w}^i \in L_i$. Since $\tilde{R}_i = R_i \cap L_i^\perp$, it implies that $\tilde{w}^i \in L_i$ and $\tilde{w}^i \in L_i^\perp$. Thus $\tilde{w}^i = 0$ for all i and $w = 0$: we obtain a contradiction. The proof is complete. ■

The following result is trivial:

Proposition 3 *If Proposition 2 holds then Proposition 1 holds.*

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