Efficient and Optimal Capital Accumulation and Non Renewable Resource Depletion: The Hartwick Rule in a Two Sector Model

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April 2008
(Draft of July, 16th, 2008)

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Abstract

Usual resource models with capital accumulation focus upon simple one to one process transforming output either into some consumption good or into some capital good. We consider a bisectoral model where output from the consumption good production sector may be either consumed or used in producing some capital good through an irreversible capital accumulation process. The natural resource and the capital good are also inputs in both production sectors. In this framework we reconsider the usual results of the efficient and optimal growth theory under an exhaustible resource constraint. We show that the usual efficiency condition relates to the investment good production function and not to the consumption good production function as in the canonical model of Dasgupta and Heal. We give an economic interpretation of the efficiency conditions in our bisectoral setting. We show then that the standard Hotelling rule relating the growth rate of the consumption good to the growth rate of the marginal productivity of the resource remains valid independently of the multisectoral specification of the model. Last we explore different forms of the Hartwick rule in the context of efficient paths and optimal paths.

JEL classification : O30, O41, Q01, Q32

Keywords : Efficiency, Optimal Growth, exhaustible resource, Hartwick rule

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1 INTRODUCTION

The basic framework of numerous aggregate models with a man made capital and a non renewable resource has been laid down by Dasgupta and Heal (DH in the sequel) in their well known seminal paper (Dasgupta and Heal, 1974). In their setting there exists one and only one man made good which can be either consumed or used to increase the capital stock. Capital accumulation is a reversible process: It is possible to transform back the capital stock into consumption good at no cost. Furthermore, there is no physical depreciation of capital. We try here to build the minimally differentiated production model permitting to disentangle basic relationships which are blurred in a single production sector model. Thus we assume that there exist two production sectors. The consumption good production sector uses labor, capital and some non renewable resource while the capital good production sector uses the same inputs and possibly the consumption good. Furthermore, capital is a specific good which cannot be consumed and is depreciating over time.

In this framework we reconsider the usual results of the efficient and optimal growth theory derived from the canonical DH model. We assume that the exhaustible resource is an essential input in both production sectors. Without technical progress, it is well known that the feasibility of a sustained consumption path in the long run is a crucial problem with *essential* exhaustible resources. We put a particular emphasis upon physical efficiency issues. Efficiency is a fundamental problem, rooted in the issue of minimizing the use of the resource to sustain as long as possible a strictly positive consumption rate. This problem has received most attention in the sustainability literature. The possibility of a sustained utility level, or of a sustained constant consumption rate, has strong connections with different forms of Hartwick's rule[1].

To study the efficiency problem we resort to a standard two stages procedure: First solve the static efficiency problem having to be solved at each point of time before attacking the pure dynamical problem, the solution of

[1] See Dixit, Hammond and Hoel, (1980), Dasgupta and Mitra, (1983), Cass and Mitra, (1991), Mitra, (2002), Asheim, Buchholz and Withagen, (2003), Cairns and Long, (2006), for important contributions to this issue.

which is linking through time the optimized values of the sequence of static problems. The static problem may be given different formulations. Here we maximize the capital good production given an aggregate resource use, a given available capital stock and a given consumption rate having to be achieved. This is one possible way to describe the production frontier at time t, a frontier which is assumed to exist in most disaggregated models, like in the Dixit *et al.* (1980) paper. Next we solve the truly dynamic efficiency problem. The main result is that dynamic efficiency implies that the growth rate of the marginal productivity of the resource in the capital good production sector must be equal to the net marginal productivity of capital in the capital good production sector. This is a result which cannot be isolated in any model in which the consumption good and the capital good production sectors are merged together. This is also the kind of result which is difficult to isolate in disaggregated models like the Dixit *et al.* (1980) model in which sectors do not exist.

To select amongst the set of efficient trajectories, we resort to the standard criterion of maximisation of the sum of discounted utilities. We show that the Hotelling rule takes the following form. The growth rate of the discounted marginal utility of consumption has to be equalized to the growth rate of the marginal productivity of the resource in the sole consumption good production sector. Again this is a result which cannot be isolated in a one man made production good economy.

Turning to Hartwick's rule, we first consider forms of the rule for efficient paths, stating that the value, in terms of the natural resource, of the instantaneous change in asset endowments must be nil at each point of time. Adapting the proof strategy initiated by Michel (1982), we first show that the converse of this form of the rule should hold along any efficient path having to sustain a constant consumption level. Considering the rule itself, we show that an efficient path satisfying at each time this form of the rule can only sustain consumption paths which are step functions. Thus it appears that as a prescriptive rule, the rule strongly constrains the kind of consumption trajectories that may be efficiently achieved but does not impose that the consumption level should be constant through time. This stands in accordance with Asheim *et al.* (2003) analysis which exhibits possible discontinuities in the consumption path to narrow the prescriptive scope of the rule. What we prove is that the type of step functions they use are the only functions which are compatible with the rule.

We next consider the generalized Hartwick rule as defined in the Dixit *et al.* (1980) paper. We show first that in our model the converse of the rule should hold, that is along any optimal constant utility path, the instantaneous change in asset endowments, valued in discounted marginal utility terms, should be nil at each time. As for the rule itself, we depart from the smoothness assumptions of Dixit *et al.* and rely instead over our previous efficiency results. Since an optimal path must be efficient, following the rule forever implies that the corresponding optimal consumption path should be a step function. But assuming a strictly concave utility function, the optimal consumption path should be continuous, implying that the consumption level, hence the utility level, should be constant over time. Thus the crux of the problem lies within the efficiency constraints.

The paper is organized as follows. Section 2 describes the bisectoral model. Efficiency is studied in Section 3 while optimality is considered in Section 4. Section 5 concludes.

2 THE MODEL

We consider an economy in which the labor supply is inelastic and constant through time. Let l be the amount of labor available at each point of time.

The economy is producing two goods, *"gelly"* and *"capital"*. Gelly is the usual polymorphic good of most macroeconomic models and can be either consumed or used as an input in the capital good production sector. Capital is the other produced good which is required to produce both gelly and capital itself, but cannot be consumed.

Let g be the gelly production function and, by slight abuse of notation, the gelly production level. Producing gelly requires capital, labor and non renewable resource, denoted respectively by K^g, l^g and s^g:

$$g \quad = \quad g(K^g, l^g, s^g) \quad .$$

Assumptions $G.1$ and $G.2$ are both standard assumptions but are distinguished for analytical reasons:

Assumption G.1: $g : \mathcal{R}_+^3 \rightarrow \mathcal{R}_+$ is a function of class $\mathcal{C}^2$ strictly increasing, concave, satisfying the Inada condition, that is:

$$\lim_{K^g \downarrow 0} g_K = \lim_{l^g \downarrow 0} g_l = \lim_{s^g \downarrow 0} g_s = +\infty \quad ,$$

where g_K, g_l and g_s are the partial derivatives of g with respect to K^g, l^g and s^g respectively, and for each limit the two other factors are held constant and strictly positive. Furthermore each input is assumed to be essential: $g(K^g, l^g, s^g) = 0$ if any one input is equal to 0.

A more stringent condition is that g is homogeneous:

Assumption G.2: g satisfies $G.1$ and:

$$g(K^g, l^g, s^g) = g_K K^g + g_l l^g + g_s s^g \ , \ \ \forall (K^g, l^g, s^g) \in \mathcal{R}_{++}^3 \quad .$$

Let k be the output level of the capital good production sector and the production function of this sector. Capital production requires capital, labor, gelly and resource, denoted respectively by K^k, l^k, g^k and s^k, so that:

$$k = k(K^k, l^k, g^k, s^k) \quad .$$

The following assumptions $K.1$ and $K.2$ parallel the assumptions $G.1$ and $G.2$ for the capital good production sector.

Assumption K.1: $k : \mathcal{R}_+^4 \rightarrow \mathcal{R}_+$ is a function of class $\mathcal{C}^2$ strictly increasing, strictly quasi-concave, satisfying the Inada condition, that is:

$$\lim_{K^k \downarrow 0} k_K = \lim_{l^k \downarrow 0} k_l = \lim_{g^k \downarrow 0} k_g = \lim_{s^k \downarrow 0} k_s = +\infty \quad .$$

where k_K, k_l, k_g and k_s are the partial derivatives of k with respect to K^k, l^k, g^k and s^k respectively, and for each limit the three other factors are constant and positive. Furthermore each input is essential.

Assumption K.2: k satisfies $K.1$ and:

$$k(K^k, l^k, g^k, s^k) = k_K K^k + k_l l^k + k_g g^k + k_s s^k \ , \ \ \forall (K^k, l^k, g^k, s^k) \in \mathcal{R}_{++}^4 .$$

The concavity of function g and strict quasi-concavity of function k are both required to obtain a unique solution of the below static efficiency program (cf. subsection 3.1 and Appendix A.1).

We assume that capital is freely and instantaneously transferable from any production sector to the other one and that its attrition law is the standard radioactive decay law, the same in both sectors. Let δ be the proportional rate of capital wear and tear. Denoting by $K(t)$ the amount of capital available in the economy at time t, we have: $K(t) \equiv K^g(t) + K^k(t)$, and under the equal proportional wear and tear assumption:

$$\dot{K}(t) \quad = \quad k(K^k(t), l^k(t), g^k(t), s^k(t)) - \delta K(t) \quad .$$

K^0 is the initial capital stock: $K(0) \equiv K^0$. We assume that $K^0 > 0$. If not, under the above essentiality assumption, the only feasible consumption path would be the zero consumption path forever.

The labor can be costlessly and instantaneously reallocated from any production sector to the other one, so that the only constraints to be satisfied are:

$$l - l^g(t) - l^k(t) \geq 0 \quad , \quad l^g(t) \geq 0 \text{ and } l^k(t) \geq 0 \ , \ t \geq 0. \tag{2.1}$$

Let $S(t)$ be the stock of the non renewable resource available at time t and S^0 be the initial endowment, $S(0) = S^0$, then:

$$\dot{S}(t) \quad = \quad -s(t) \quad ,$$

where: $s(t) \equiv s^g(t) + s^k(t)$. Extraction costs are neglected.

We denote by $u(c)$ the instantaneous utility generated by the instantaneous consumption rate c.

Assumption U : $u : \mathcal{R}_{++} \rightarrow \mathcal{R}$ is a function of class $\mathcal{C}^2$ strictly increasing, strictly concave, satisfying the Inada condition: $\lim_{c \downarrow 0} u'(c) = +\infty$.

The welfare W is the sum of the instantaneous utilities discounted at some positive constant social rate ρ:

$$W \quad = \quad \int_0^{\infty} u(c(t)) e^{-\rho t} dt \quad .$$

A policy $\mathcal{P}$ is a path $\{(K^i(t), l^i(t), s^i(t), i = g, k\ ;\ g^k(t), c(t)), t \geq 0\}$. It is feasible starting from K^0 and S^0 iff $\forall t \geq 0$:

$$K^0 + \int_0^t \left[k(K^k(\tau), l^k(\tau), g^k(\tau), s^k(\tau)) - \delta K(\tau)\right] d\tau - K^g(t) - K^k(t) \geq 0\ ,$$

$$K^g(t) \geq 0 \ \text{ and } \ K^k(t) \geq 0\ ,$$

$$g(K^g(t), l^g(t), s^g(t)) - g^k(t) - c(t) \geq 0\ ,$$

$$c(t) \geq 0 \ \text{ and } \ g^k(t) \geq 0\ ,$$

$$S^0 - \int_0^t \left[s^g(\tau) + s^k(\tau)\right] d\tau \geq 0\ ,$$

$$s^g(t) \geq 0 \ \text{ and } \ s^k(t) \geq 0 \ \ ,$$

together with (2.1).

We denote by $\mathcal{P}(K^0, S^0)$ a policy which is feasible starting from (K^0, S^0).

The problem of the social planner is to choose a welfare maximizing policy. Since optimal policies must be efficient policies we characterize first such policies.

3 EFFICIENCY

Let us consider some feasible policy $\mathcal{P}^*(K^0, S^0)$. According to the usual definition of efficiency this policy is efficient if it does not exists any alternative feasible policy $\mathcal{P}'(K^0, S^0)$ such that $c'(t) \geq c^*(t)$, $t \geq 0$, with the strict inequality over some non degenerate time interval. Under $G.1$ and $K.1$, this definition is equivalent to the following one. $\mathcal{P}^*$ is efficient if, first for any time interval $[t_1, t_2]$, $0 \leq t_1 < t_2$, over which $c^*(t) > 0$, the restriction of the policy to the interval is minimizing the cumulated use of the resource amongst the set of subpolicies which are securing a consumption rate $c'(t) \geq c^*(t)$ over the whole interval, when starting from $K(t_1) = K^*(t_1)$ and ending at $K(t_2) = K^*(t_2)$. Clearly this is just a local necessary condition. For a global condition, we should add that, when considering any infinite duration time interval, $[t_1, \infty)$, $t_1 \geq 0$, the resource stock is exhausted.

The problem of minimizing the cumulated extraction is best understood when conceived as a two stages optimization problem. The first stage is a standard static optimization problem which has to be solved at each point of time. At any date, given the available capital and given that the available labor has to be wholly used since it is not storable, there exists some static efficiency frontier in the three dimensional space: Consumption, capital production and resource use, leaving aside the labor dimension since the labor supply is assumed to be inelastic and constant through time. This frontier may be described as some function denoted by κ, giving the maximum instantaneous production of capital good which can be obtained from some available capital K and resource use s, assuming that a given consumption rate c has to be achieved. Thus taking into account that no capital is never discarded excepted the unescapable wear and tear attrition, the instantaneous rate of change of the capital stock must be equal to:

$$\dot{K}(t) \quad = \quad \kappa(K(t), s(t), c(t)) - \delta K(t) \quad .$$

Note that $\dot{K}(t)$ may be either positive or negative. Because $\kappa \geq 0$, the RHS of the above equation may be as low as $-\delta K(t)$. Under a free disposal assumption, $\dot{K}(t)$ could be even lower, although it will never happen for trivial efficiency reasons in the present setting.

The second stage is the truly dynamical problem. For a given consumption path $c^*(t)$ to be achieved, the tradeoff at each point of time is between accumulating capital at a higher rate today but at the cost of a higher present use of the resource, allowing to save the resource in the future, versus saving the resource today but at the cost of a lower capital accumulation inducing a higher use of the resource in the future. Using the function κ, this second stage arbitrage problem may be formulated as a problem in which the only command variable is the global resource extraction rate $s(t)$.

3.1 Solving the static optimization problem

Delete the time index and let K, l, s, be the capital, labor and resource extraction rate at time t. The maximum consumption rate which can be expected is attained when all the inputs are allocated to the gelly production sector and the whole gelly production is consumed. Let us denote by $\bar{c}$ this maximum consumption rate: $\bar{c} \equiv g(K, l, s)$.

Assume that some consumption rate c, $c \leq \bar{c}$, has to be produced. The problem is now to allocate K, l, s so as to maximize the capital production rate that is to solve the following static efficiency program (S.E):

$$
\begin{array}{llll}
(S.E) & \max\limits_{(K^k,l^k,g^k,s^k)} & k(K^k, g^k, l^k, s^k) & \\
& \text{s.t} & g(K-K^k, l-l^k, s-s^k) - g^k - c \geq 0 & (3.1) \\
& & K-K^k \geq 0 \quad \text{and} \quad K^k \geq 0 & (3.2) \\
& & l-l^k \geq 0 \quad \text{and} \quad l^k \geq 0 & (3.3) \\
& & s-s^k \geq 0 \quad \text{and} \quad s^k \geq 0 & (3.4) \\
& & g^k \geq 0 & (3.5)
\end{array}
$$

Proposition 1 *Under G.1 and K.1 the solution of the (S.E) program is unique. Furthermore for $c \in (0, \bar{c})$ the constraints (3.2)-(3.5) are not binding so that the first order conditions reduce to:*

$$k_K - k_g g_K = 0 \ , \ k_k - k_g g_l = 0 \quad \textit{and} \quad k_s - k_g g_s = 0 \qquad (3.6)$$

Proof. The unicity of the solution is trivial for $c = \bar{c} : K^k = 0 \ , \ l^k = 0$ and $s^k = 0$, because all the available inputs must be allocated to the gelly production sector. For the other cases the proof is given in Appendix A.1. For $c < \bar{c}$, clearly some capital good can be produced. Because gelly is an essential input we must have $g(K-K^k \ , \ l-l^k \ , \ s-s^k) > 0$ and because anyone of these inputs is essential we must have too $K - K^k > 0 \ , \ l - l^k > 0$ and $s - s^k > 0$. Because capital has to be produced we must have: $K^k > 0 \ , \ l^k > 0, \ , \ s^k > 0$. Hence the constraints (3.2)-(3.5) cannot be tight and we may rewrite the (S.E) problem more simply as the following, corner free problem:

$$\max_{(K^k,l^k,s^k)} k(K^k, l^k, g(K-K^k, l-l^k, s-s^k) - c, s^k)$$

the first order conditions of which are (3.6). ■

Note that the conditions (3.6) are implying that:

$$\frac{g_l}{g_K} = \frac{k_l}{k_K} \quad , \quad \frac{g_s}{g_K} = \frac{k_s}{k_K} \quad \text{and} \quad \frac{g_l}{g_s} = \frac{k_l}{k_s} \quad . \qquad (3.7)$$

As expected the marginal rates of transformation between any pair of inputs used in the both sectors must be equalized.

The conditions (3.6) also imply that:

$$\frac{1}{k_g} = \frac{g_K}{k_K} = \frac{g_l}{k_l} = \frac{g_s}{k_s} \quad . \tag{3.8}$$

Equation (3.8) means that the direct marginal cost of capital in terms of the consumption good or gelly, $1/k_g$, must be equal to any one of its indirect marginal costs, also in terms of the consumption good, obtained by diverting marginally some input (capital, labor or resource) from the gelly production sector towards the capital good production sector.

To conclude there exists some function $\tilde{k}$ the arguments of which are K, l, s and c, with $c \leq \bar{c}$, which gives the maximum production level of the capital good sector for any available global inputs K, l and s, and a consumption rate c having to be secured. Since we assume that l is constant we may drop this argument and define $\gamma(K,s)$ and $\kappa(K,s,c)$ as follows:

$$\gamma(K,s) \equiv \bar{c}(K,l,s) \quad \text{and} \quad \kappa(K,s,c) \equiv \tilde{k}(K,l,s,c)\ ,\ c \leq \gamma(K,s)\ . \tag{3.9}$$

Clearly γ is increasing in each of its arguments and κ is an increasing function of K, of s and a decreasing function of c:

$$\gamma_K \equiv \frac{\partial \gamma}{\partial K} = g_K > 0 \quad \text{and} \quad \gamma_s \equiv \frac{\partial \gamma}{\partial s} = g_s > 0 \quad , \tag{3.10}$$

and by the envelope theorem:

$$(i) \qquad \kappa_K \equiv \frac{\partial \kappa}{\partial K} = k_g g_K > 0 \tag{3.11}$$

$$(ii) \qquad \kappa_s \equiv \frac{\partial \kappa}{\partial s} = k_g g_s > 0 \tag{3.12}$$

$$(iii) \qquad \kappa_c \equiv \frac{\partial \kappa}{\partial c} = -k_g < 0 \quad . \tag{3.13}$$

Furthermore:

$$\lim_{c \uparrow \gamma(K,s)} \kappa(K,s,c) = 0 \quad . \tag{3.14}$$

We have to define a last boundary relationship which will happen to be useful later for characterizing the solution of the second stage problem. Consider some $c \leq \gamma(K,s)$ and assume that no new capital has to be produced. Then we must have:

$$\kappa(K,s,c) \quad = \quad 0 \quad .$$

This equation may be solved for s as a function of K and c. Let us denote by $\underline{s}(K,c)$ the solution, that is the minimum resource extraction necessary to achieve a consumption rate c when the available capital amounts to K. Because $\kappa = 0$, then $\underline{s}(K,c)$ is nothing but the solution of: $c = g(K,l,s)$ where l is the constant labor supply. Thus $\underline{s}$ is a decreasing function of K and an increasing function of c:

$$\underline{s}_K \equiv \frac{\partial \underline{s}}{\partial K} = -\frac{g_K}{g_s} < 0 \quad \text{and} \quad \underline{s}_c \equiv \frac{\partial \underline{s}}{\partial c} = \frac{1}{g_s} > 0 \; ; \tag{3.15}$$

3.2 Solving the dynamical problem

Armed with the κ function we may focus the attention upon the proper dynamical aspect of the problem. Given that $c^*(t)$, $t \in [t_1, t_2]$, has to be achieved, minimizing the cumulated extraction of the resource over $[t_1, t_2]$ may be formulated as the following problem (E) in which the only command variable is the instantaneous rate of resource extraction $s(t)$:

$$(E) \qquad \max_{\{s(t), t\in[t_1,t_2]\}} \; -\int_{t_1}^{t_2} s(t)dt$$

$$\dot{K}(t) = \kappa(K(t), s(t), c^*(t)) - \delta K(t) \;\; , \qquad t \in [t_1, t_2] \tag{3.16}$$

$$K(t_1) = K^*(t_1) \;\text{ and }\; K(t_2) - K^*(t_2) \geq 0 \;\; , \tag{3.17}$$

$$s(t) - \underline{s}(K(t), c^*(t)) \geq 0 \;\; , \qquad t \in [t_1, t_2]. \tag{3.18}$$

Note that if the constraints (3.16) is tight, then $s(t) = \underline{s}(K(t), c^*(t))$, so that $\kappa(K(t), \underline{s}(K(t), c^*(t)), c^*(t)) = 0$, implying that:

$$\dot{K}(t) \;\; = \;\; -\delta K(t) \; .$$

No new capital is produced. The capital stock decreases at its proportional decay rate δ.

Let $\mathcal{L}^E(t)$ be the Lagrangian of the program (E):

$$\mathcal{L}^E(t) \;\; = \;\; -s(t) + \nu^E(t)\left[\kappa(K(t), s(t), c^*(t)) - \delta K(t)\right] + \alpha^E(t)[s(t) - \underline{s}(K(t), c^*(t))] \; .$$

The first order condition is:

$$\frac{\partial \mathcal{L}^E}{\partial s} = 0 \qquad \Longleftrightarrow \qquad \nu^E(t)\kappa_s(t) = 1 - \alpha^E(t) \; , \tag{3.19}$$

together with: $\alpha^E(t) \geq 0 \quad$ and $\quad \alpha^E(t)[s(t) - \underline{s}(K(t), c^*(t))] = 0$.(3.20)

The dynamics of the costate variable $\nu^E(t)$ must satisfy:

$$\dot{\nu}^E(t) = -\frac{\partial \mathcal{L}^E}{\partial K} \iff \dot{\nu}^E(t) = -\nu^E(t)[\kappa_K(t) - \delta] + \alpha^E(t)\underline{s}_K(t) \text{ .(3.21)}$$

Last the transversality condition is:

$$\nu^E(t_2) \geq 0 \quad \text{and} \quad \nu^E(t_2)[K(t_2) - K^*(t_2)] = 0 \ . \tag{3.22}$$

Assume that the solution is an interior solution, i.e. (3.18) is not effective so that $\alpha^E(t) = 0$. Then differentiating (3.19) and using (3.21), we obtain the below relationship (3.23). Next using (3.8) and (3.13), (3.23) may be given the three equivalent arbitrage conditions (3.24)-(3.26) between the use of the resource and the uses of the other inputs, gelly, capital and labor, conditions which must hold along any dynamically efficient path at each point in time.

Proposition 2 *Under* $G.1$ *and* $K.1$*, along any dynamically efficient interior path:*

$$\frac{\dot{\kappa}_s(t)}{\kappa_s(t)} = -\frac{\dot{\nu}^E(t)}{\nu^E(t)} = \kappa_K(t) - \delta \ , \ t \in [t_1, t_2], \tag{3.23}$$

which is equivalent to:

$$\kappa_s = k_g g_s \ : \ (3.23) \implies \frac{\dot{g}_s}{g_s} = -\frac{\dot{k}_g}{k_g} + k_g g_K - \delta \tag{3.24}$$

$$\kappa_s = \frac{k_K}{g_K} g_s \ : \ (3.23) \implies \frac{\dot{g}_s}{g_s} = \frac{\dot{g}_K}{g_K} - \frac{\dot{k}_K}{k_K} + k_K - \delta \tag{3.25}$$

$$\kappa_s = \frac{k_l}{g_l} g_s \ : \ (3.23) \implies \frac{\dot{g}_s}{g_s} = \frac{\dot{g}_l}{g_l} - \frac{\dot{k}_l}{k_l} + k_l \frac{g_K}{g_l} - \delta \tag{3.26}$$

The conditions (3.24)-(3.26) are conditions warranting that all the arbitrage opportunities are locally exhausted. Any tradeoff, either direct or indirect, between some increase of the resource extraction rate and some simultaneous decrease of the investment rate today being balanced by a higher investment rate in the near future and a simultaneous decrease in the extraction rate, while maintaining the consumption level c^*, cannot reduce the cumulative resource extraction.

To give some intuition about the type of arbitrage opportunities these conditions are exhausting, let us detail how, under (3.24), the following intertemporal input substitutions cannot save resource.

Let us consider a sequence of three consecutive time intervals: $\Theta_1 \equiv [t, t+dt[$, $\Theta_2 \equiv [t+dt, t+h[$, $h > dt > 0$, and $\Theta_3 \equiv [t+h, t+h+dt[$ such that (3.24) is satisfied at each instant $\tau \in [t, t+h+dt[$. Denote by $\mathcal{P}^E$ an efficient policy, each component of $\mathcal{P}^E$ being indexed by the superscript E and the same for the derivatives of the functions g and k along the path.

During the first interval Θ_1, assume that the society decides to increase by an amount ds the resource use in the gelly production sector at each time within this interval. Such an increase allows for an increase of gelly production by an amount $dg = g_s^E ds$. In order to stay upon the consumption reference path, the society transfers this gelly production increase to the capital good production sector through an increase of g^k by the same amount. This transfers allows for an increase in the capital stock level by an amount $d_1K > 0$ but at the cost of an extra consumption of the natural resource stock $d_1S < 0$. Assume that this increase in the capital stock is allocated only to the capital good production sector. We get at the end of Θ_1:

$$d_1K^k \simeq k_g^E(t)g_s^E(t)dsdt \quad \text{and} \quad d_1S \simeq -ds\,dt$$

During the second interval Θ_2, the capital stock increase is maintained constant. This allows for a decrease of the use of gelly in the production of the capital good. Since wear and tear has also been increased, the reduction is $dg^k(\tau) = d_1K^k(k_K^E(\tau) - \delta)/k_g^E(\tau)$ at any time $\tau \in \Theta_2$. Having to sustain the consumption reference path, the economy can decrease the gelly production level by reducing the use of the natural resource by an amount: $ds^g(\tau) = dg^k(\tau)/g_s^E(\tau)$. This allows for resource savings at the end of Θ_2 equal to:

$$d_2S \quad = \quad d_1K^k \int_{t+dt}^{t+h} \frac{k_K^E(\tau) - \delta}{k_g^E(\tau)g_s^E(\tau)} d\tau$$

For dt sufficiently small we get the following approximation:

$$d_2S \quad \simeq \quad k_g^E(t)g_s^E(t) \left[\frac{k_K^E(t+dt) - \delta}{k_g^E(t+dt)g_s^E(t+dt)} \right] (h - dt)dsdt$$

During the third interval Θ_3, the economy drives back the capital stock to its reference level by cutting down g^k by an amount: $dq^k = (k_g^E(t)g_s^E(t)/k_g^E(t+h))dsdt$ at each time within this interval. This allows for savings of the natural resource stock by an amount d_3S over Θ_3:

$$d_3S \quad = \quad \frac{k_g^E(t)g_s^E(t)}{k_g^E(t+h)g_s^E(t+h)}ds\,dt$$

Let $dS = d_1S + d_2S + d_3S$ be the amount of resource saved over the reference path during the interval $[t, t+h+dt[$.

$$\begin{aligned} dS \quad \simeq \quad & \left\{-1 + k_g^E(t)g_s^E(t)\left[\frac{k_K^E(t+dt)-\delta}{k_g^E(t+dt)g_s^E(t+dt)}\right](h-dt)\right. \\ & \left. + \frac{k_g^E(t)g_s^E(t)}{k_g^E(t+h)g_s^E(t+h)}\right\} ds\,dt \end{aligned}$$

For h sufficiently small and dt infinitely smaller than h we get the following approximations:

$$\begin{aligned} h - dt \quad & \simeq \quad h \\ k_K^E(t+dt) \simeq k_K^E(t) \quad & \text{and} \quad k_g^E(t+dt)g_s^E(t+dt) \simeq k_g^E(t)g_s^E(t) \\ k_g^E(t+h)g_s^E(t+h) \quad & \simeq \quad k_g^E(t)g_s^E(t) + \dot{\big(k_g^E(t)g_s^E(t)\big)}h \\ & \Longrightarrow \quad \frac{k_g^E(t)g_s^E(t)}{k_g^E(t+h)g_s^E(t+h)} \simeq 1 - \frac{\dot{\big(k_g^E(t)g_s^E(t)\big)}}{\big(k_g^E(t)g_s^E(t)\big)}h \end{aligned}$$

Thus dS is approximatively equal to:

$$dS \quad \simeq \quad \left\{-\frac{\dot{\big(k_g^E(t)g_s^E(t)\big)}}{\big(k_g^E(t)g_s^E(t)\big)} + k_K^E(t) - \delta\right\} hds\,dt$$

that is:

$$dS \quad \simeq \quad \left\{-\frac{\dot{g_s^E}(t)}{g_s^E(t)} - \frac{\dot{k_g^E}(t)}{k_g^E(t)} + k_K^E(t) - \delta\right\} hds\,dt \tag{3.27}$$

Now note that since the economy is assumed to follow the reference path over $[0, t) \cup [t+h+dt, \infty)$ then the resource extraction is not affected during this time interval. Thus if the previous perturbation is to be feasible, we must have:

$$dS = \int_t^{t+h+dt} s(\tau)d\tau - \int_t^{t+h+dt} s^E(\tau)d\tau \leq 0$$

But since $dS < 0$ would imply that the reference path is not efficient, clearly we must have $dS = 0$, hence the term into brackets on the right hand side of (3.27) must be equal to zero, which is nothing but than (3.24). The same kind of reasoning, by increasing s^g and transferring some capital amount dK^g to the capital good production sector would lead to (3.25). Similarly, a perturbation increasing s^g and transferring some labor dl^g to the capital good production sector would result in (3.26).

3.3 Remarks about the Dasgupta and Heal (1974) canonical model

The DH model (1974) is not explicitly framed as a two sectors model. But it can be understood as such a model in which first, the production function of the capital good sector takes a one to one form, second, the working life of capital goods is infinite ($\delta = 0$) and third, the capital accumulation process is perfectly reversible, that is the capital can be instantaneously and freely transformed back into gelly and consumed. The same kind of framework is also found in Mitra (1978) or Dasgupta and Mitra (1983), although in a slightly more general form and in a discrete time model.

Thus the production core of the DH model may be written as:

$$g = g(K^g, l^g, s^g) \quad \text{and} \quad k = g^k$$

Static efficiency trivially implies that $K^g = K$, $l^g = l$, and $s^g = s$. Hence the maximization of k results in:

$$\kappa(K, s, c) = g(K, l, s) - c \quad \text{and} \quad \gamma(K, s) = g(K, l, s)$$

Furthermore because the capital is reversible, the only lower bound to the extraction rate s, given any consumption rate c having to be secured, is trivially $\underline{s}(K, c) = 0$ provided that $K > 0$: It is sufficient not to produce any gelly and consume the capital at disposal so that $s^g = s = 0$. Given that the condition (3.18) is not binding, that is $s(t) \geq 0$ is not tight, since furthermore $\delta = 0$, then (3.23) results in:

$$\frac{\dot{g}_s(t)}{g_s(t)} = g_K(t) .$$

This is nothing but that the well known efficiency condition of the DH model in which the consumption good and the capital good are produced within the same sector, g being the production function of this unique sector.

3.4 Efficiency and Hartwick's rule

Let us show now that the so-called Hartwick's Rule may be deduced from the pure efficiency conditions.

We define a global efficiency problem $(G.E)$ as the following extension of a problem (E) in which:

1. First $t_1 = 0$ and $t_2 = \infty$.
2. Second, the constraint on $K(t)$ is the following constraint (3.28) instead of (3.17):

$$K(0) = K^0 > 0 \text{ given, and } \liminf_{t\uparrow\infty} K(t) \geq 0 \qquad (3.28)$$

3. Third and last, the consumption having to be achieved $c^*(t)$ is defined accordingly over $[0, \infty)$, with the qualification that $c^*(t)$ must be strictly positive over some non degenerate time interval $[t, t')$, $0 \leq t < t'$ to avoid trivialities.

A *global efficiency consumption step problem* $(GE.s)$ is a (GE) problem in which the consumption path $\{c^*(t), t \geq 0\}$ is a step function, that is a set of non degenerate time intervals $[0, t_1)$, ..., $[t_{i-1}, t_i)$, ..., $[t_{n-1}, t_n)$, $t_{i-1} < t_i$, $i = 1, \cdots, n^2$, $n \in \mathcal{N}$, $t_n = \infty$, and a corresponding set of non-negative

[2] Denoting $t = 0$ by t_0.

consumption rates $\{c_i^*, i = 1, \cdots, n\}$, one of which at least is strictly positive, such that $c^*(t) = c_i^*$, $t \in [t_{i-1}, t_i)$, $i = 1, \cdots, n$. A *uniform consumption (GE.s) problem*, $(GE.us)$, is a global problem in which $n = 1$, hence $c^*(t) = c^* > 0$, $t \in [0, \infty)$.

Proposition 3 *Assume that $G.1$ and $K.1$ hold and consider some constant consumption path $c^*(t) = c^* > 0$ which, given $K^0 > 0$, would be feasible were the society be endowed with a finite amount of resource sufficiently high. Let $\{s^*(t), t \geq 0\}$ be some continuous path of resource use such that $\int_0^\infty s^*(t)dt < \infty$. Denote by $K^*(t)$ the solution of (3.16) for $c^*(t) = c^*$, $\{s^*(t), t \geq 0\}$ and $K(0) = K^0$. Assume that for $\{(s^*(t), K^*(t)), t \geq 0\}$, (3.18) is satisfied as a strict inequality that is the solution is an interior solution. If $\{s^*(t), t \geq 0\}$ is solving the $(GE.us)$ problem, then there exists some $\mathcal{C}^1$ function $\{\nu^{E*}(t), t \geq 0\}$, the costate variable of $K^*(t)$, such that:*

$$\nu^{E*}(t)\dot{K}^*(t) \quad = \quad s^*(t) \quad , \quad t \in [0, \infty) \tag{3.29}$$

Note that in this version of the rule, $\nu^{E*}(t)$, the shadow marginal value of the capital stock, is a current price in terms of the resource. Given the objective function of the problem $(G.E)$, $\nu^{E*}(t)$ is the amount of resource which could be marginally saved were the stock of capital $K^*(t)$ be marginally higher at time t. In such a context, the current marginal valuation of the resource is equal to 1 at any time t. Thus what (3.29) is asserting is that the value of the instantaneous change in asset endowment[3] at any time t, at prices $(\nu^{E*}(t), 1)$, that is $\nu^{E*}(t)\dot{K}^*(t) - s^*(t)$, must be nil.

The proof is running as follows. Let $\mathcal{H}(t)$ be the Hamiltonian of a $(GE.us)$ problem:

$$\mathcal{H}(t) \quad = \quad -s(t) + \nu^E(t)\left[\kappa(K(t), s(t); c^*) - \delta K(t)\right] \ .$$

By the dynamic envelope theorem[4], we must have:

$$\frac{d\mathcal{H}(t)}{dt} \quad = \quad \frac{\partial \mathcal{H}(t)}{\partial t}$$

[3]Not to be confused with the instantaneous change of the endowment value which amounts to $\dot{\nu}^{E*}(t)K^*(t) + \nu^{E*}(t)\dot{K}^*(t) - s^*(t)$.

[4]For a standard formulation of the theorem, see for example to Seierstad and Sydsæter, 1987, Chap 2, Note 3, p 61.

Thanks to the fact that $c^*(t)$ is constant through time, $\partial\mathcal{H}/\partial t = 0$ so that $d\mathcal{H}/dt = 0$, implying that:

$$\mathcal{H}(t) = h \iff \nu^E(t)\left[\kappa(K(t), s(t); c^*) - \delta K(t)\right] - s(t) = h$$

where h is some constant. Thus:

$$\nu^E(t)\dot{K}(t) - s(t) \quad = \quad h$$

To prove that $h = 0$, we follow the general strategy developed in Michel (1982) with due care to the fact that here there is no discounting. The idea of the proof, formally developed in Appendix A.2, is to convert the problem $(GE.us)$ into a Bolza problem of the form:

$$\max_{\{s(t), t\in[0,T)\}} \int_0^T (-s(t))dt + R(T)$$

where T is any finite time horizon, $R(T) \equiv \int_T^\infty (-s^*(t))dt$, and $\{s^*(t), t \in [T, \infty)\}$ being an efficient path followed from T onwards starting from an efficient level of the capital stock, $K^*(T)$ at time T. Remark that since $\{s^*(t), t \in [T, \infty)\}$ has been assumed to be efficient and hence feasible, one should have:

$$-R(T) = \int_T^\infty s^*(t)dt < \infty \ .$$

$R(T)$ should be a well defined integral bounded from below. Using the same mild assumptions as imposed by Michel (1982)[5], it is possible to derive the limit properties of an efficient solution letting $T \to \infty$. For a constant consumption path having to be achieved, this will result in $\lim_{T\uparrow\infty} \mathcal{H}(T) = 0$, a generalization over an infinite time horizon of a well known transversality condition for a finite free endpoint T. But since the Hamiltonian must be constant along a solution path of the $(GE.us)$ problem, this in turn implies that $\mathcal{H}(t) = 0$, $t \geq 0$, that is $h = 0$ and the Hartwick's rule must be satisfied.

Let us show now that, as an efficiency signal, the Hartwick rule may work iff the problem is a $(GE.s)$ type problem.

[5]In particular, Michel's proof does not require that the Hamiltonian of the Bolza problem be concave in the vector of state and control variables, an assumption sometimes made to derive transversality conditions in infinite time horizon problems, see Seierstad and Sydsæter, 1987, Chap 3, Theorem 13, p 235 for an example. In the present case, since we want to maximize a linear criterion, concavity would be an issue and our proof should not depend upon such an assumption.

Proposition 4 *Assume that* $G1$ *and* $K.1$ *hold. Let* $\{s^*(t), t \geq 0\}$, $\int_0^\infty s^*(t)dt < \infty$, *be solving a* (GE) *problem for some given consumption path* $\{c^*(t), t \geq 0\}$ *and some initial capital endowment* $K(0) = K^0 > 0$. *Denote by* $\{K^*(t), t \geq 0\}$ *the associated capital path solving (3.16). Assume also that, for* $\{(s^*(t), K^*(t)), t \geq 0\}$, *(3.18) is satisfied as a strict inequality that is the solution is an interior solution. Let* $\{\nu^{E*}(t), t \geq 0\}$ *be the path of the costate variable associated to the capital stock. If the Hartwick rule (3.29) is satisfied at each point of time, then* $\{c^*(t), t \geq 0\}$ *is necessarily a step function that is the problem must be a* $(GE.s)$ *type problem.*

Formal details of the proof are given in Appendix A.3. But the intuition of the proof is straightforward. Over any open time interval, were the consumption profile be a time differentiable function, then the constancy of the Hamiltonian resulting from the Hartwick rule combined with the dynamic envelope theorem would imply that the consumption level should be constant within the interval. Next, if over the interval the consumption path is continuous but not differentiable, it is easily checked that the consumption must be constant hence no kink points can exist. Last considering a possible finite size jump of the consumption level at some time, it is always possible to define a jump in the extraction level such that the Hartwick rule would remain verified at the time of the jump.

Although the Hartwick rule may be an efficiency signal for economies having different types of consumption paths, the structure of such paths is strongly constrained. This is the reason why the counterexamples of Asheim *et al.* (2003) showing that the Hartwick rule may hold even if the consumption level is not constant, are all examples of economies in which the consumption path is a step function.

4 OPTIMALITY

The function κ may be used to formulate the optimality problem (P).

$$
\begin{array}{lll}
(P) & \max_{\{(c(t),s(t)),t\geq 0)\}} \int_0^{\infty} u(c(t))e^{-\rho t}dt & \\
s.t. & \dot{S}(t) = -s(t) \ , \ S(0) = S^0 > 0 \text{ given} \ , \ t \in [0,\infty) & (4.1) \\
 & S(t) \geq 0 \qquad , \ t \in [0,\infty) & (4.2) \\
 & (3.16) \text{ over } [0,\infty) \text{ instead of } [t_1,t_2] \text{ and with } K(0) = K^0 \text{ given} \ , & \\
 & (3.18) \text{ over } [0,\infty) \text{ instead of } [t_1,t_2] \ , & \\
 & c(t) \geq 0 \text{ and } s(t) \geq 0 \qquad , \ t \in [0,\infty) \ . & (4.3)
\end{array}
$$

Under assumption U, $c(t)$ must be positive hence $s(t)$ and $S(t)$ too. Thus we my leave aside the corresponding non negativity constraints and write the current value Lagrangian as follows:

$$
\begin{aligned}
\mathcal{L}^P(t) \ = \ & u(c(t)) - \lambda(t)s(t) + \nu(t)\left[\kappa(K(t),s(t),c(t)) - \delta K(t)\right] \\
& + \alpha(t)[s(t) - \underline{s}(K(t),c(t))] \ .
\end{aligned}
$$

The first order conditions are:

$$\frac{\partial \mathcal{L}^P}{\partial c} = 0 \iff u'(c(t)) + \nu(t)\kappa_c(t) - \alpha(t)\underline{s}_c(t) = 0 \tag{4.4}$$

$$\frac{\partial \mathcal{L}^P}{\partial s} = 0 \iff -\lambda(t) + \nu(t)\kappa_s(t) + \alpha(t) = 0 \tag{4.5}$$

together with: $\quad \alpha(t) \geq 0 \text{ and } \alpha(t)[s(t) - \underline{s}(K(t),c(t))] = 0 \ .$ (4.6)

The dynamics of the costate variables must satisfy:

$$
\begin{aligned}
\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial \mathcal{L}^P}{\partial S} & \iff \dot{\lambda}(t) = \rho\lambda(t) \\
& \iff \lambda(t) = \lambda_0 e^{\rho t} \text{ where } \lambda_0 = \lambda(0) \qquad (4.7) \\
\dot{\nu}(t) = \rho\nu(t) - \frac{\partial \mathcal{L}^P}{\partial K} & \iff \dot{\nu}(t) = \rho\nu(t) - \nu(t)[\kappa_K(t) - \delta] \\
& \qquad + \alpha(t)\underline{s}_K(t) \qquad (4.8)
\end{aligned}
$$

Last the transversality conditions are:

$$\lim_{t\uparrow\infty} e^{-\rho t}\lambda(t)S(t) = \lambda_0 \lim_{t\uparrow\infty} S(t) = 0 \tag{4.9}$$

$$\lim_{t\uparrow\infty} e^{-\rho t}\nu(t)K(t) = 0 \tag{4.10}$$

4.1 Hotelling rule

Assume first that the constraint (3.18) is not effective so that $\alpha(t) = 0$. Then time differentiating (4.4) and substituting for $\dot{\nu}(t)$ as given by (4.8), we obtain:

$$\frac{u''(c(t))c(t)}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} - \rho \;=\; \frac{\dot{\kappa}_c(t)}{\kappa_c(t)} - (\kappa_K(t) - \delta) \tag{4.11}$$

Next, time differentiating (4.5), taking into account both (4.7) and (4.8), results in:

$$\kappa_K(t) - \delta = \frac{\dot{\kappa}_s(t)}{\kappa_s(t)} \ .$$

This is nothing but that the efficiency condition (3.23).

Next making use of (3.13), we get also, dropping the time index:

$$\frac{\kappa_s}{\kappa_c} = -\frac{k_g g_s}{k_g} = -g_s \implies \frac{\dot{\kappa}_s}{\kappa_s} - \frac{\dot{\kappa}_c}{\kappa_c} = \frac{\dot{g}_s}{g_s} \ ,$$

Thus denoting by $\eta(c)$ the absolute value of the elasticity of marginal utility $-u''(c)c/u'(c)$, we conclude:

Proposition 5 *Under $G.1$ and $K.1$, along an interior optimal path:*

$$\eta(c)\frac{\dot{c}}{c} + \rho \;=\; \kappa_K - \delta - \frac{\dot{\kappa}_c}{\kappa_c} = \frac{\dot{\kappa}_s}{\kappa_s} - \frac{\dot{\kappa}_c}{\kappa_c} = \frac{\dot{g}_s}{g_s} \tag{4.12}$$

The last equality, $\eta\dot{c}/c + \rho = \dot{g}_s/g_s$ is the standard formulation of the Hotelling rule as appears in the DH model (1979, p 297) with only one production sector. What Proposition 4 is showing is that in a two sectors model the production function involved in the right side of (4.12) must be the production function of the consumption good sector, that is g.

As pointed out in the above subsection 3.3, in the DH model, $\kappa_c = -1$, hence $\dot{\kappa}_c/\kappa_c = 0$, and because $\delta = 0$, then (4.12) results in:

$$\eta(c)\frac{\dot{c}}{c} + \rho \;=\; g_K \tag{4.13}$$

which is nothing but than the DH optimality condition (10.18)[6]. This is ba-

[6]Dasgupta and Heal (1979), Chapter 10, p 296.

sically the Ramsey-Keynes condition in the standard Ramsey-Solow optimal growth model. The DH model merges a Ramsey model, implying the same form of the arbitrage condition between savings and investment as expressed in (4.13), and a Hotelling model, characterized by an arbitrage condition between using the resource either today or in the future, a condition expressed in (4.5) in the present model. Time differentiating the Hotelling condition and identifying with the Ramsey Keynes condition in the DH model leads to the expression of the Hotelling rule:

$$\eta(c)\frac{\dot{c}}{c} + \rho \quad = \quad g_K = \frac{\dot{g}_s}{g_s}$$

In the present model, the resource is involved both in the production of gelly and in the production of capital good. Furthermore, gelly may be either consumed or used to produce the capital good. Hence the saving versus consumption arbitrage and the intertemporal arbitrage over the use of the resource are connected directly at the production stage. This explains why the efficiency condition (3.23) results directly from the Hotelling condition (4.5) and the dynamics of the costate variables λ and ν. Static efficiency collapses the autonomous effect of consumption $\dot{\kappa}_c/\kappa_c$ in the investment versus consumption arbitrage condition, resulting in the standard version of the Hotelling rule, where only the growth rate of the resource productivity in the sole production of gelly has to be balanced with the growth rate of the discounted marginal utility of gelly consumption.

4.2 National accounts

Under constant returns, that is under $G.2$ and $K.2$, it is also possible to derive an interesting national accounting condition. To simplify the exposition, let us denote by $\pi(t) \equiv e^{-\rho t}u'(c(t))$ the discounted marginal utility level and by $\mu(t) \equiv e^{-\rho t}\nu(t)$ the discounted level of the costate variable $\nu(t)$. Multiplying both sides of (4.4) and (4.5) by $e^{-\rho t}$ and making use of these new notations result in:

$$\pi(t) \equiv u'(c(t))e^{-\rho t} \quad = \quad -\mu(t)\kappa_c(t) = \mu(t)k_g(t) \qquad (4.14)$$

$$\lambda_0 = \mu(t)\kappa_s(t) \quad = \quad \mu(t)k_g(t)g_s(t) = \pi(t)g_s \; . \qquad (4.15)$$

We get also from (4.8):

$$\begin{aligned} \dot{\mu}(t) &= -\mu(t)(\kappa_K - \delta) = -\mu(t)(k_g(t)g_k(t) - \delta) \\ \Longrightarrow & \qquad \delta\mu(t) - \dot{\mu}(t) = \mu(t)k_g(t)g_K(t) = \pi(t)g_K(t) \ . \end{aligned} \tag{4.16}$$

Next by $G.2$ and dropping the time index we get:

$$\pi g = \pi g_K K^g + \pi l^g g_l + \pi s^g g^s \ .$$

Making use of (4.15) and (4.16), the above equation is equivalent to:

$$\pi g \ = \ (\delta\mu - \dot{\mu})K^g + \pi g_l l^g + \lambda_0 s^g \ . \tag{4.17}$$

Under $K.2$, we get also:

$$\pi k \ = \ \pi k_K K^k + \pi k_l l^k + \pi k_s s^k + \pi k_g g^k \ .$$

Making use of the static efficiency conditions (3.6), we obtain:

$$\pi k \ = \ \pi k_g g_K K^k + \pi k_g g_l l^k + \pi k_g g_s s^k + \pi k_g g^k \ .$$

Substituting for πg_K and πg_s their expressions given in (4.15) and (4.16), the above equation is equivalent to:

$$\frac{\pi}{k_g} k \ = \ (\delta\mu - \dot{\mu})K^k + \pi g_l l^g + \lambda_0 s^k + \pi g^k \ . \tag{4.18}$$

Summing up (4.17) and (4.18) while taking into account the full employment conditions results in:

$$\pi c + \frac{\pi}{k_g} k \ = \ (\delta\mu - \dot{\mu})K + \pi g_l l + \lambda_0 s \ . \tag{4.19}$$

Since by (4.14), $\pi/k_g = \mu$ and $k = \dot{K} + \delta K$, (4.19) simplifies to:

$$\pi c \ = \ -(\dot{\mu K}) + \pi g_l l + \lambda_0 s \ . \tag{4.20}$$

Integrating over $[0, \infty)$ and using the transversality conditions (4.9) and (4.10), we get the following national accounting relationship[7]:

[7]Note that our national accounts balance is expressed in net terms, in particular, the provision for wear and tear has been included in the expression of the available product. For a detailed treatment of accounts in gross an net terms, refer to Hartwick (2000) or Aronsson *et al.* (1997).

Proposition 6 *Under G.2, K.2 and U, for any optimal interior path:*

$$\int_0^\infty u'(c(t))c(t)e^{-\rho t}dt \;=\; \nu(0)K^0 + \lambda_0 S^0 + l\int_0^\infty u'(c(t))e^{-\rho t}g_l dt \quad (4.21)$$

The left hand side of (4.21) is the sum of all the future consumption rates $c(t)$ valued at their discounted marginal instantaneous utility $u'(c(t))e^{-\rho t}$. Absent any global economy or diseconomies of scale, the intuition suggests that the value of the optimized net output of the economy could be decomposed into the sum of the values of the components of the economy endowments. This is precisely what (4.21) is proving. Homogeneity of both g and k implies the homogeneity of the global production process. The endowments of the economy are its initial capital stock K^0, its initial stock of resource S^0 and last, the constant flow of labor l, that is the constant flow of a renewable resource. In (4.21) all these endowments are valued at their initial shadow prices: $\nu(0)$ and λ_0 for the capital and resource stocks respectively and for the labor flow its marginal productivity in the consumption good industry weighted by the discounted marginal utility of consumption, $u'(c(t)))e^{-\rho t}g_l(K^g(t), l^g(t), s^g(t))$.

4.3 Generalized Hartwick's rule

In their seminal paper, Dixit, Hammond and Hoel (1980) proved that a generalized version of the Hartwick's rule has to hold along any constant utility optimal path. It is easily checked that such a version of the Hartwick's rule holds also in our model.

The Hamiltonian in present value of the optimality problem (P) is:

$$\mathcal{H}(t) \;=\; u(c(t))e^{-\rho t} + \nu^d(t)\dot{K}(t) - \lambda^d(t)s(t)$$

where $\nu^d(t)$ and $\lambda^d(t)$ denotes the costate variables in discounted value. In the Dixit *et al.* formulation, $\nu^d(t)\dot{K}(t) - \lambda^d(t)s(t)$ is nothing but than the net present value at time t of investments in all the capital goods: the capital stock $K(t)$ and the resource stock $S(t)$. Through the dynamic envelope theorem, and denoting by $\mathcal{H}^*(t)$ the maximized Hamiltonian:

$$\frac{d\mathcal{H}^*(t)}{dt} = \frac{\partial \mathcal{H}^*(t)}{\partial t} = -\rho u(c^*(t))e^{-\rho t} \;,$$

where $c^*(t)$ is the optimal consumption level at time t. Assume a constant optimal utility level u^*. Integrating the above relation over $[t,\infty)$, we obtain:

$$\lim_{\tau\uparrow\infty} \mathcal{H}^*(\tau) - \mathcal{H}^*(t) \;=\; -\int_t^{\infty} \rho u^* e^{-\rho\tau} d\tau = -u^* e^{-\rho t} \ .$$

Michel (1982) proved that in an optimality problem of this kind, we must have: $\lim_{\tau\uparrow\infty} \mathcal{H}^*(\tau) = 0$. This results in:

$$\begin{aligned} \mathcal{H}^*(t) \;&=\; u^* e^{-\rho t} + \Big[\nu^d(t)\dot{K}^*(t) - \lambda^d(t) s^*(t)\Big] = u^* e^{-\rho t} \\ &\Longrightarrow\; \nu^d(t)\dot{K}^*(t) - \lambda^d(t) s^*(t) = 0 \ . \end{aligned}$$

The net present value of investments should be equal to zero if the optimal utility level is constant, that is the Hartwick rule should hold. Here the capital investment $\dot{K}^*(t)$ and the resource use $s^*(t)$ are both valued in terms of cumulative discounted utility, the objective function of the problem (P).

Conversely, consider an optimal path $\{(K^*(t), s^*(t), c^*(t)), t \geq 0\}$ satisfying the Hartwick rule at each time t. Denote by $u^*(t) \equiv u(c^*(t))$ the optimized value of the utility. The corresponding Hamiltonian thus verifies:

$$\mathcal{H}^*(t) \;=\; u^*(t) e^{-\rho t} \ , \ t \in [0,\infty).$$

Applying the dynamic envelope theorem requires that $u^*(t)$ be a continuous and time differentiable function along the optimal consumption trajectory. Assuming time differentiability results thus in:

$$\mathcal{H}^*(t) \;=\; u^*(t) e^{-\rho t} = \int_t^{\infty} \rho u^*(\tau) e^{-\rho\tau} d\tau \ .$$

Differentiating with respect to t gives:

$$\dot{u}^*(t) e^{-\rho t} - \rho u^*(t) e^{-\rho t} \;=\; -\rho u^*(t) e^{-\rho t} \;\Longrightarrow\; \dot{u}^*(t) e^{-\rho t} = 0 \ .$$

Thus the utility level, and hence the consumption level should be constant along an optimal path satisfying the Hartwick rule at each time. This is the main result of Dixit *et al.* (1980). But note that Dixit *et al.* (Theorem 1, p 553) are assuming the smoothness of all time functions along the optimal trajectory, which is an additional assumption which cannot be deduced from their primitive regularity assumptions.

But we can exploit the efficiency property of an optimal path to show that the optimal consumption level, and hence the optimal welfare level, should be constant if the Hartwick rule is verified, without relying on the time differentiability of $u^*(t)$, the optimized utility level.

First note that under (4.7), $\lambda^d(t) = \lambda^d$ a constant. Dividing by λ^d side to side the Hartwick rule and making use of (4.5), we obtain:

$$\nu^d(t)\dot{K}^*(t) = \lambda^d s^*(t) \implies \frac{\nu^d(t)}{\lambda^d}\dot{K}^*(t) = \frac{1}{\kappa_s}\dot{K}^*(t) = s^*(t) \ ,$$

and since an optimal path must be efficient: $\nu^E(t) = \kappa_s^{-1}$. We conclude that the simple form of the Hartwick rule along an efficient path should hold. Applying Proposition 4, the optimal consumption path sustained by the Hartwick rule is a step function. But under the strict concavity of the utility function U, a jump in the consumption level would imply a jump in the marginal utility level in the opposite direction. This opens the door to consumption arbitrage opportunities, contradicting the assumption that the consumption path is optimal. Since $c^*(t)$ has to be a continuous time function under U, we conclude that the optimal consumption level should be constant over time if (4.22), and thus (3.29), have to be satisfied at each point of time. That is $\{(K^*(t), s^*(t)), t \geq 0\}$ should be solution of a $(GE.us)$ problem. We conclude as follows:

Proposition 7 *Under* $G.1$, $K.1$ *and* U, *if along an interior optimal path* $\{(s^*(t), K^*(t)), t \geq 0\}$ *the current utility level is constant over time, then:*

$$\nu^d(t)\dot{K}^*(t) = \lambda^d(t)s^*(t) \quad t \in [0, \infty) \tag{4.22}$$

where $\nu^d(t)$ *and* $\lambda^d(t)$ *are the costate variables of* $K^*(t)$ *and* $S^*(t)$ *respectively, both in terms of discounted utility. Reciprocally assume that (4.22) holds, then the current utility level is constant through time.*

Note that we get the generalized Hartwick's rule without invoking 'transversality' conditions stating the limit of $\nu^d(t)$ as time increases up to infinity. A limit property of the optimized Hamiltonian, which can be shown to be a necessary condition for optimality (see Michel, 1982) with a constant discount rate, is all that is needed to obtain the rule along an optimal constant utility path.

5 CONCLUSION

The Dasgupta and Heal (1974) seminal contribution is the basic framework of numerous analysis of the long run sustainability issue through man made capital substitution to the use of an essential exhaustible resource. We depart from this framework by introducing a complete bisectoral model where the consumption good is produced from labor, man made capital and an exhaustible resource, the capital good being also produced from labor, capital, exhaustible resource and some part of the output from the consumption good sector. This is the minimum disaggregation allowing to isolate some fundamental relationships which are blurred in the Dasgupta and Heal model in which the two sectors are merged together.

We focus more upon efficiency issues rather than over optimality issues, the first ones appearing as more fundamental for the sustainability of an economy submitted to an exhaustible resource depletion constraint. We show that local dynamic efficiency relates basically to the properties of the capital good production function, while optimal properties like the Hotelling rule rely upon the properties of the consumption good production function. Our emphasis upon efficiency considerations proves also to be helpful in clarifying important aspects of Hartwick's rule in resource models. We show that Hartwick's result can be obtained without relying upon continuity and smoothness assumptions, as frequently postulated in the literature.

An important issue we do not consider is the existence of efficient or optimal positive constant consumption paths. It is clear that if the economy cannot sustain a constant consumption level through an efficient management of its scarce resources, it cannot do better than experiencing some declining to zero consumption level in the long run.

In our model, the economy is constrained both by the limited availability of an exhaustible resource and by a limited and constant amount of labor. Most existence results of efficient plans sustaining some constant consumption level have been derived from monosectoral models of substitution between the exhaustible resource and a man capital stock[8], and their counter-

[8]Existence results for monosectoral models with or without labor constraints have been derived in Solow (1974), Cass and Mitra (1991), Pezzey and Withagen (1998) and Asheim

parts in a bisectoral model remain an open question.These points are beyond the scope of the present study but are developed in a companion paper[9].

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APPENDIX

A.1.1 Appendix A.1: Proof of Proposition 1

The proof runs as follows:

1. There exists at least a solution
2. The set of vectors (K^k, l^k, g^k, s^k) satisfying the constraints (3.1)-(3.5) is convex
3. Assuming that the program could have different solutions implies a contradiction

1. Because the objective function is continuous the only point having to be checked is that the set of vectors (K^k, l^k, g^k, s^k), satisfying the constraints (3.1)-(3.5), is compact that is bounded and closed.

   For $K^k = l^k = s^k = 0$ we get the maximum finite value of $g^k, g^k = g(K, l, s) - c$. Thus the set is bounded.

   All the constraints are weak inequalities including continuous functions of K^k, l^k, g^k and s^k, so that the set is closed.

2. Let $(K_i^k, l_i^k, g_i^k, s_i^k), i = 1, 2$ be two vectors satisfying (3.1)-(3.5), and consider any linear convex combination of these vectors:

   $$\alpha(K_1^k, l_1^k, g_1^k, s_1^k) + (1-\alpha)(K_2^k, l_2^k, g_2^k, s_2^k),\ \alpha \in (0, 1).$$

   Clearly (3.2)-(3.5) are satisfied. The only poing having to be checked is (3.1). Because each vector "i" is satisfying (3.1), then:

   $$\alpha g(K - K_1^k, l - l_1^k, s - s_1^k) + (1-\alpha) g(K - K_2^k, l - l_2^k, s - s_2^k)$$
   $$-(\alpha g_1^k + (1-\alpha) g_2^k) - c \geq 0$$

   and next, because g is concave:

   $$g(\alpha[K - K_1^k] + (1-\alpha)[K - K_2^k], \alpha[l - l_1^k] + (1-\alpha)[l - l_2^k], \alpha[s - s_1^k] + (1-\alpha)[s - s_2^k])$$

$$\geq \alpha g(K - K_1^k, l - l_1^k, s - s_1^k) + (1-\alpha) g(K - K_2^k, l - l_2^k, s - s_2^k).$$

From the two above inequalities we conclude that the linear convex combination of the vectors "i" is satisfying (3.1).

Note that we need the concavity of g. Assuming the quasi-concavity we could not write the first above inequality.

3. Assume that $c < \bar{c}$ and let $(K_i^k, l_i^k, g_i^k, s_i^k), i = 1, 2$ be two solutions of the (S.E) program:

$$k(K_1^k, l_1^k, g_1^k, s_1^k) = k(K_2^k, l_2^k, g_2^k, s_2^k) \geq k(K^k, l^k, g^k, s^k) \qquad \text{(A.1.1)}$$

for any (K^k, l^k, g^k, s^k) satisfying (3.1)-(3.5), and

$$(K_1^k, l_1^k, g_1^k, s_1^k) \neq (K_2^k, l_2^k, g_2^k, s_2^k).$$

Because each vector "i" is within the domain and the domain is convex, then any convex linear combination of these vectors is also within the domain defined by (3.1)-(3.5).

Because the two vectors are different, the strict quasi-concavity of k implies that:

$$k(\alpha K_1^k + (1-\alpha)K_2^k, \alpha l_1^k + (1-\alpha) l_2^k, \alpha g_1^k + (1-\alpha) g_2^k, \alpha s_1^k + (1-\alpha) s_2^k)$$
$$> \min\{k(K_i^k, l_i^k, g_i^k, s_i^k), i = 1, 2\} = k(k_i^k, l_i^k, g_i^k, s_i^k), i = 1, 2, \qquad \text{(A.1.2)}$$

the last equality being an immediate implication of (A.1.1).

Thus the vectors "i" are not solving the problem, hence the contradiction.

Note we need the strict quasi-concavity of k for (A.1.2).

A.1.2 Appendix A.2: Proof of Proposition 3

We adapt the proof strategy of Michel (1982) to the problem ($GE.uc$) of Proposition 3 in which, contrary to Michel's assumption, there is no discounting.

Denote by $\{(s^*(t), K^*(t)), t \in [0, \infty)\}$ the solution of the problem ($GE.uc$) of the Proposition 3 defined by $c^*(t) = c^* > 0$, $t \geq 0$. Let us define the new time variable τ as $\tau \equiv t - x$ so that $d\tau/dt = 1$ and $s^*(\tau) = s^*(t-x)$. For any given $T > 0$ and $x \geq 0$, define $R^*(T,x)$ as minus the cumulated extraction over the time interval $[T + x, \infty)$, the time being measured by τ:

$$R^*(T,x) \equiv \int_{T+x}^{\infty} (-s^*(\tau))d\tau$$

Note that by construction $\partial R^*/\partial x = 0$.

Consider the following auxiliary problem (P_T) with the non negative state variables $Y(t)$ and $Z(t)$ and the control variables $r(t)$, $r(t) \in \Re_+$, and $v(t)$, $v(t) \in \Re_{++}$:

$$\begin{aligned} P_T \quad : \quad & \max_{\{(r(t),v(t)),t\in[0,T)\}} \int_0^T v(t)(-r(t))dt + R^*(T, Z(T) - T) \\ s.t. \quad & \dot{Y}(t) = v(t)f(Y(t), r(t)) \;\; Y(0) = K^0 \;\; Y(T) = K^*(T) \\ & \dot{Z}(t) = v(t) \;\; \text{and} \;\; Z(0) = 0 \end{aligned}$$

where $f(Y(t), r(t)) \equiv \kappa(Y(t), r(t), c^*) - \delta Y(t)$. It is proved in Michel (1982) that the states $(Y(t), Z(t)) = (K^*(t), t)$ and the controls $(r(t), v(t)) = (s^*(t), 1)$, for $t \in [0, T)\}$ are solving the auxiliary problem P_T (Michel, 1982, Lemma, p 977).

Let $H_T(t)$ be the Hamiltonian of the auxiliary problem (P_T):

$$H_T(t) \quad = \quad a_T v(t)(-r(t)) + \nu_T(t)v(t)f(Y(t), r(t)) + \vartheta_T(t)v(t)$$

Note that we explicitly introduce the scalar a_T, usually implicitly assumed to be equal to one, into the expression of the Hamiltonian.

As proved by Michel (1982, p 983), the necessary optimality conditions for the problem (P_T) are as follows.

First, there must exist a non negative real number a_T, a real number n_T, and continuous functions of time $\nu_T(t)$ and $\vartheta_T(t)$ such that:

$$\begin{aligned}
(a_T, n_T) &\neq (0,0) && \text{(A.1.3)}\\
\dot{\nu}_T(t) &= -\frac{\partial H_T}{\partial Y} = -\nu_T(t)v(t)\frac{\partial f}{\partial Y} \\
&\Longrightarrow \dot{\nu}_T(t) = -\nu_T(t)\frac{\partial f}{\partial K}(K^*(t), s^*(t))\ ,\ t \in [0,T) && \text{(A.1.4)}\\
\nu_T(T) &= n_T && \text{(A.1.5)}\\
\dot{\vartheta}(t) &= -\frac{\partial H_T}{\partial Z} = 0 && \text{(A.1.6)}\\
\vartheta_T(T) &= a_T \frac{\partial R^*}{\partial x}\frac{\partial x}{\partial Z} = a_T \frac{\partial R^*}{\partial x}(T,0) = 0 && \text{(A.1.7)}
\end{aligned}$$

Second, the Hamiltonian must be maximized with respect to the control variables. Concerning $v(t)$, since the Hamiltonian is linear in $v(t)$, in the case $v(t) = 1 \neq 0$, this is implying that:

$$\nu_T(t) f(Y(t), r(t)) + \vartheta_T(t) = a_T r(t) \quad t \in [0,T) \tag{A.1.8}$$

Concerning $r(t)$, we obtain, for $v(t) = 1$:

$$\nu_T(t)\frac{\partial f}{\partial r}(Y(t), r(t)) = a_T \quad t \in [0,T) \tag{A.1.9}$$

Let us show now that both $a_T \neq 0$ and $\nu_T(0) \neq 0$. Consider the above condition (A.1.9) at time $t = 0$:

$$\nu_T(0)\kappa_s(Y(0), r(0)) = a_T \tag{A.1.10}$$

Under the assumptions $G.1$ and $K.1$, $\kappa_s(Y(0), r(0)) > 0$, hence:

$$\nu_T(0) = 0 \Longrightarrow a_T = 0 \text{ and } a_T = 0 \Longrightarrow \nu_T(0) = 0\ . \tag{A.1.11}$$

Thus:

- Either both $\nu_T(0) = 0$ and $a_T = 0$,
- Or $\nu_T(0) \neq 0$ and $a_T \neq 0$.

Assume that $\nu_T(0) = 0$, then by (A.1.4) and $\partial f/\partial K > 0$:

- Either $\nu_T(t) = 0$, $t \in [0, T)$, implying that first $\nu_T(T) = 0$ hence by (A.1.5) $n_T = 0$, and by (A.1.11) $a_T = 0$ because $\nu_T(0)$, thus $(a_T, n_T) = (0, 0)$ contradicting (A.1.3).

- Or $\nu_T(t) \neq 0$ over some first interval (t_1, t_2), $0 \leq t_1 < t_2 \leq T$ after having been equal to 0 over the interval $[0, t_1)$ (possibly degenerate). Because $\partial f/\partial K > 0$ then by (A.1.4) this is possible iff $\nu_T(t)$ is jumping either upwards or downwards at t_1 which is contradicting the continuity of $\nu_T(t)$ which must be equal to 0 over $[0, t_1)$, hence again a contradiction.

We conclude that $a_T > 0$ and $\nu_T(0) \neq 0$[10].

Multiplying side to side (A.1.4), (A.1.5), (A.1.8), (A.1.9) by a constant $\theta > 0$, while taking into account (A.1.6) and (A.1.7) which imply together that $\vartheta(t) = 0$, $t \in [0, T)$, we get:

$$\begin{aligned}
\theta\dot{\nu}_T(t) &= -\theta\nu_T(t)\frac{\partial f}{\partial K} \\
\theta\nu_T(T) &= \theta n_T \\
\theta\nu_T(t)f(Y(t), r(t)) &= \theta a_T r(t) \\
\theta\nu_T(t)\frac{\partial f}{\partial r} &= \theta a_T
\end{aligned}$$

By letting $a'_T \equiv \theta a_T$ and $\nu'_T(0) \equiv \theta\nu_T(0)$, we can choose a value of θ such that $\|a'_T, \nu'_T(0)\| = 1$ without changing the solution of the problem (P_T). Thus we can renormalize a_T and $\nu_T(0)$ in such a way that $(a_T, \nu_T(0))$ lies into the unit simplex, that is a compact set.

Since $(a_T, \nu_T(0))$ is of unit norm, there exists a sequence $(a_{T_n}, \nu_{T_n}(0))$ such that $lim_{T_n \to \infty}(a_{T_n}, \nu_{T_n}(0)) = (a, \nu^0)$ with $a > 0$ and $\nu^0 > 0$. Since $\lim_{T_n \to \infty} a_{T_n} = a$ and $\lim_{T_n \to \infty} \nu_{T_n}(0) = \nu^0$, we can define $\nu(t) = \lim_{T_n \to \infty} \nu_{T_n}(t)$ and $\vartheta(t) = \lim_{T_n \to \infty} \vartheta_{T_n}(t)$. Remembering that $\{K^*(t), s^*(t)\}_0^T$ should be a solution of

[10] Note that $\nu_T(0) \neq 0$ implies that $\nu_T(0) > 0$ under the assumptions of Proposition 2 according to which the efficient path is an interior path, that is (3.18) is satisfied as a strict inequality.

the problem (P_T), $(\nu(t), \vartheta(t))$ should be a solution of:

$$\begin{aligned}
\dot{\nu}(t) &= -\nu(t)\frac{\partial f}{\partial K}(K^*(t), s^*(t)) \;\; \nu(0) = \nu^0 \\
\dot{\vartheta}(t) &= 0 \\
\vartheta(t) &= a \lim_{x\uparrow\infty} \frac{\partial R^*}{\partial x}(0) = 0
\end{aligned}$$

The asymptotic properties of ϑ show that first $\dot{\vartheta}(t) = 0$, that is $\vartheta(t)$ should be constant, and second $\vartheta(t) = 0$. Since (a, ν^0) is of unit norm, we get also:

$$-s^*(t) + a^{-1}\nu(t)f(k^*(t), s^*(t)) = 0 \implies H^*(t) = -s^*(t) + \nu(t)\dot{K}^*(t) = 0$$

which is nothing but than the Hartwick's rule (3.29).

A.1.3 Appendix A.3: Proof of Proposition 4

If a solution of the problem (GE) satisfies the Hartwick's rule (3.29), then the corresponding Hamiltonian $\mathcal{H}^E(t)$ should be zero at each time t along the solution path and hence be constant. Next note that $K(t)$ should be a continuous function of time along an efficient path.

Consider some time interval $\mathcal{T}^C$ where the consumption path having to be achieved would be a continuous function of time. Then over any time interval $\mathcal{T}_D \subset \mathcal{T}^C$ where the consumption path would also be a time differentiable time function, we can apply the dynamic envelope theorem and get:

$$\frac{d\mathcal{H}^E(t)}{dt} = \frac{\partial \mathcal{H}^E(t)}{\partial t} = \nu^E(t)\frac{\partial \kappa(K(t), s(t); c(t))}{\partial c}\frac{dc(t)}{dt} = 0 \; t \in \mathcal{T}_D \subset \mathcal{T}^C$$

Thus $c(t)$ should be constant for $t \in \mathcal{T}_D$.

Next consider some time $\tau \in \mathcal{T}^C$ such that the function $c(t)$ would be non differentiable at time τ. We cannot apply the dynamic envelope theorem at $t = \tau$. Let $\tau^- \equiv t \in (\tau - \epsilon, \tau)$, $\tau^+ \equiv t \in (\tau, \tau + \epsilon)$, $\epsilon > 0$. Since $c(t)$ is continuous and differentiable for $t \in (\tau - \epsilon, \tau)$, $\dot{K}$ and $\nu^E(t)$ are differentiable and continuous at τ^-. The same applies at τ^+. Since through the Hartwick rule, $\nu^E(t)\dot{K}(t) = s(t)$ at $t = \tau^-$ and $t = \tau^+$, $s(t)$ should also be a continuous and differentiable time function for $t = \tau^-$ or $t = \tau^+$. Thus time differentiating for $t \in (\tau - \epsilon, \tau)$, we get:

$$\begin{aligned}
&\dot{\nu}^E(\tau^-)[\kappa(K(\tau^-), s(\tau^-), c(\tau^-) - \delta K(\tau^-)] \\
&+\nu^E(\tau^-)[\kappa_K(\tau^-)\dot{K}(\tau^-) + \kappa_s(\tau^-)\dot{s}(\tau^-) + \kappa_c(\tau^-)\dot{c}(\tau^-) - \delta\dot{K}(\tau^-)] = \dot{s}(\tau^-)
\end{aligned}$$

And since the path $\{K(t), s(t)\}$ is an interior solution of the problem (GE), the following necessary conditions have to be verified:

$$\begin{aligned}
\dot{\nu}^E(\tau^-) &= -\nu^E(\tau^-)[\kappa_K(\tau^-) - \delta] \\
\nu^E(\tau^-) &= \kappa_s(\tau^-)^{-1}
\end{aligned}$$

which gives:

$$\begin{aligned}
&\dot{\nu}^E(\tau^-)\dot{K}(\tau^-) + \nu^E(\tau^-)\dot{K}(\tau^-)[\kappa_K(\tau^-) - \delta] + \dot{s}(\tau^-)[\nu^E(\tau^-)\kappa_s(\tau^-) - 1] \\
&+\nu^E(\tau^-)\kappa_c(\tau^-)\dot{c}(\tau^-) = 0 \\
\Longrightarrow\ &\nu^E(\tau^-)\kappa_c(\tau^-)\dot{c}(\tau^-) = 0
\end{aligned}$$

which is only possible if $\dot{c}(\tau^-) = 0$. But the same computation may be performed for τ^+ leading to $\dot{c}(\tau^+) = 0$. Thus the $c(t)$ consumption level having to be achieved being a continuous time function at τ with equal lefthand and righthand time derivatives limits should be a differentiable time function at $t = \tau$. This implies that $c(t)$ has to be constant for $t \in \mathcal{T}^C$.

It remains to consider the case of a jump in the consumption path having to be achieved. Note that in such a case the function $f(K, s; c)$ defined by $\dot{K}(t) = \kappa(K, s; c) - \delta K \equiv f(K, s : c)$ is no more a continuous function of (K, s), jumping upwards if c jumps downwards or jumping downwards if c jumps upwards since $\kappa_c < 0$. Thus we are not in the usual setting of the standard optimal control theory where f is currently assumed to be a continuous function. In such a situation, the costate variable $\nu^{E*}(t)$ associated to $K^*(t)$ is a piecewise continuous function of time, with possible jumps at the point of discontinuity of the consumption path[11]. This allows to determine some jump in $s^*(t)$ such that:

$$\nu^{E*}(t)\dot{K}^*(t) = s^*(t) \iff \frac{[\kappa(K^*(t), s^*(t); c) - \delta K^*(t)]}{\kappa_s(K^*(t), s^*(t); c)} = s^*(t)$$

would remained satisfied even at point of discontinuities of the consumption path, showing that the Hartwick rule is not a sufficient condition to exclude jumps in the consumption path having to be achieved.

[11] Seierstad A. and K. Sydsæter, 1987, Chap 2, Note 6, p 87.

Thus a solution path of the problem (GE) and satisfying the Hartwick rule (3.29) can only sustain a sequence of constant consumption levels, that is the consumption trajectory having to be achieved can only be some step function.