Quantum Cat's Dilemma: an example of intransitivity in a quantum game

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Abstract

We study a quantum version of the sequential game illustrating problems connected with making rational decisions. We compare the results that the two models (quantum and classical) yield. In the quantum model intransitivity gains importance significantly. We argue that the quantum model describes our spontaneously shown preferences more precisely than the classical model, as these preferences are often intransitive.

Keywords: quantum strategies; quantum games; intransitivity; sequential game.

1 Introduction

A fundamental scientific theory is marked by its ability to solve the widest possible range of problems. In the 20th century, it was quantum mechanics [1] that became such an effective panacea for the problems that could not be either understood or solved with the use of the traditional methods. Quantum

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mechanics describes the fascinating structure of elements of which the world is composed and explains such phenomena as radioactivity, antimatter, stability of molecular structures, stars evolution etc. The traditional paradigms of perceiving the world are enriched by quantum theory. Quantum-like ideas are used in various fields of research and in this way they contribute to the unification of modern science. Some of the mechanisms characteristic for living nature may find their reflection in quantum theory [2]. Presently, quantum information theory is being built at the meeting point of quantum mechanics and theory of information [3, 4, 5]. The concept of a quantum computer stresses the qualitative limitations of orthodox Turing machines which in the future would probably be replaced with the quantum computers whose computing ability will substantially exceed the possibilities of the present computers [6]. It poses a threat of using quantum technology to jeopardize the contemporary methods used to guarantee the confidentiality of data transfer [7]. It seems that the methods of quantum cryptography that are being presently worked out will remain safe even in the times of the quantum computers [8]. The combination of the research methods of both information and game theories results in emerging of the new mysterious field - quantum game theory, in which the subtle quantum rules characterizing the material world determine ways of controlling and transformation of information [9, 10, 11, 12, 13]. In the quantum game formalism, pure strategies correspond to the vectors of Hilbert space (to be more precise: the projective operators on subspaces determined by these vectors). The mixed strategies are represented by the convex combinations of vectors projected on these directions. In comparison with the sets of the traditional strategies, quantum strategies provide players with much more possibilities which they can use while making the most beneficial decision for themselves. This characteristic feature of quantum game theory is the reason why its results go beyond the traditional boundaries [14]. Plenty of quantum variants of problems analysed by the traditional classical game theory (see [15, 16, 17]) have already been put forward. First attempts at creating quantum economy by applying quantum game theory to selected economic problems have been made too [18]. It is assumed that there exists a market where financial transactions are made with help of quantum computers operating on quantum strategies [18, 19, 20]. It is essential to mention here that game theory in its traditional form has been formulated in the context of economic issues.

The quantum game formalism has already been used to describe the idea of the Evolutionary Stable Strategy(ESS) [21]. Perhaps, further research in this direction will be used to explain a number of phenomena that are now being researched by evolutionary biology.

In our work we concentrate on the quantitative analysis of the quantum version of a very simple game against Nature which was presented and analyzed in [22]. To illustrate the problem, we will use the story about Pitts's experiments with cats, mentioned in the Steinhaus diary [23]. Let us assume (alike as in [22]), that a cat (we will be calling it the *quantum cat*) is offered three types of food (no. 1, no. 2 and no. 3), every time in pairs of two types, whereas the food portions are equally attractive regarding the calories, and each one has unique components that are necessary for the cat's good health. The cat knows (it is accustomed to) the frequency of occurrence of every pair of food and his strategy depends only on this frequency. Let us also assume that the cat cannot consume both offered types of food at the same moment, and that it will never refrain from making the choice.

Nonorthodox quantum description of the decision algorithms provides a possibility to extend the results of Ref. [22]. In the following paragraphs, we compare the quantum and the classical variants of the model we are interested in.

2 Intransitivity

However, before we start analyzing all possible behavioral patterns of quantum cat, it would be advisable to explain what the intransitive order is.

Any relation $\succ$ existing between the elements of a certain set is called *transitive* if $A \succ C$ results from the fact that $A \succ B$ and $B \succ C$ for any three elements A, B, C. If this condition is not fulfilled then the relation will be called *intransitive* (not transitive).
The best known example of intransitivity is the children game "Rock, Scissors, Paper". The relation used to determine which throws defeat which is intransitive–Rock defeats Scissors, and Scissors defeat Paper, but Rock loses to Paper (see quantum analysis of this game [24, 25]). Another interesting example of intransitive order is Condorcet's voting paradox. Consideration regarding this paradox led Arrow in the XX-th century to prove the theorem stating that there is no procedure of successful choice that would meet the democratic assumption [26] (some other problems with intransitive options

can be found in [27, 28]). Intransitive orders still are surprisingly suspicious for many researchers. Economists have long presented a view that people should choose between things they like in a specific, linear order [29]. But what we actually prefer often depends on how the choice is being offered [30, 31]. Mentioned in Steinhaus's diary Pitts notice that a cat facing choice between fish, meat and milk prefers fish to meat, meat to milk, and milk to fish! Pitts's cat, thanks to the above-mentioned food preferences, provided itself with a balanced diet.

Let us have a closer look at the effects of the consideration of the problem that Pitts's was trying to tackle, in the language of quantum game theory.

3 Properties of cat's optimal strategies

There is the following relation between the frequencies ω_k, $k = 0, 1, 2$ of appearance of the particular foods in a diet and the conditional probabilities which we are interested in (see [22]):

$$\omega_k := P(C_k) = \sum_{j=0}^{2} P(C_k|B_j)P(B_j), \ \ k = 0, 1, 2\,, \tag{1}$$

where $P(C_k|B_j)$ indicates the probability of choosing the food of number k, when the offered food pair does not contain the food of number j, $P(B_j) =: q_j$ indicates the frequency of occurrence of pair of food that does not contain food number j. The most valuable way of choosing the food by cat occurs for such six conditional probabilities ($P(C_1|B_0)$, $P(C_2|B_0)$,$P(C_0|B_1)$,$P(C_2|B_1)$, $P(C_0|B_2)$, $P(C_1|B_2)$) which fulfills the following condition:

$$\omega_0 = \omega_1 = \omega_2 = \tfrac{1}{3}. \tag{2}$$

Any six conditional probabilities, that for a fixed triple (q_0, q_1, q_2) fulfill (2) will be called a cat's *optimal strategy*. The system of Eq. (2) has the following matrix form:

$$\begin{pmatrix} P(C_0|B_2) & P(C_0|B_1) & 0 \\ P(C_1|B_2) & 0 & P(C_1|B_0) \\ 0 & P(C_2|B_1) & P(C_2|B_0) \end{pmatrix} \begin{pmatrix} q_2 \\ q_1 \\ q_0 \end{pmatrix} = \tfrac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{3}$$

and its solution:

$$\begin{aligned} q_2 &= \tfrac{1}{d}\left(\frac{P(C_0|B_1)+P(C_1|B_0)}{3} - P(C_0|B_1)P(C_1|B_0)\right), \\ q_1 &= \tfrac{1}{d}\left(\frac{P(C_0|B_2)+P(C_2|B_0)}{3} - P(C_0|B_2)P(C_2|B_0)\right), \\ q_0 &= \tfrac{1}{d}\left(\frac{P(C_1|B_2)+P(C_2|B_1)}{3} - P(C_1|B_2)P(C_2|B_1)\right), \end{aligned} \tag{4}$$

defines a mapping $\mathcal{A}_0 : D_3 \to T_2$ of the three-dimensional cube (D_3) into a triangle (T_2) (two-dimensional simplex, $q_0 + q_1 + q_2 = 1$ and $q_i \geq 0$), where d is the determinant of the matrix of parameters $P(C_j|B_k)$. The barycentric coordinates of a point of this triangle are interpreted as probabilities q_0, q_1 and q_2. Thus we get relation between the optimal cat's strategy and frequencies q_j of appearance of food pairs.

4 Quantum cat

We start with the presentation of formalism which is indispensable for the quantum description of the variant of the game presented in the article [22]. Let us denote three different bases of two-dimensional Hilbert space as $\{\,|1\rangle_0, |2\rangle_0\,\}$, $\{\,|0\rangle_1,\ |2\rangle_1\,\}$, $\{\,|0\rangle_2,\ |1\rangle_2\,\} = \{\,(1,0)^T, (0,1)^T\,\}$. The bases should be such that bases $\{\,|0\rangle_1,\ |2\rangle_1\,\}$, $\{\,|1\rangle_0,\ |2\rangle_0\,\}$ are the image of $\{\,|0\rangle_2,$ $|1\rangle_2\,\}$ under the transformations H and K respectively:[1]

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad K = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

It is worth to mention here that the set of so called conjugated bases, which is presented above, allowed Wiesner (before asymmetric key cryptography was invented !) to begin research into quantum cryptography. These bases play also an important role in universality of quantum market games [33]. Let us denote strategy of choosing the food number k, when the offered food pair not contain the food of number l, as $|k\rangle_l$ ($k, l = 0, 1, 2,\ k \neq l$).
A family $\{|z\rangle\}$ ($z \in \overline{\mathbb{C}}$) of convex vectors:

$$|z\rangle := |0\rangle_2 + z|1\rangle_2 = |0\rangle_1 + \frac{1-z}{1+z}|2\rangle_1 = |1\rangle_0 + \frac{1+iz}{1-iz}|2\rangle_0,$$

[1] H is called Hadamard matrix.

defined by the parameters of the heterogeneous coordinates of the projective space $\mathbb{C}P^1$, represents all quantum cat strategies spanned by the base vectors. The coordinates of the same strategy $|z\rangle$ read (measured) in various bases define quantum cat's preferences toward a food pair represented by the base vectors. Squares of their moduli, after normalization, measure the conditional probability of quantum cat's making decision in choosing a particular product, when the choice is related to the suggested food pair (the choice of the way of measuring a strategy). In this way, quantum cat makes a decision to choose the right food pair with the following probabilities:

$$\begin{aligned} P(C_0|B_2) &= \frac{1}{1+|z|^2}, & P(C_1|B_2) &= \frac{|z|^2}{1+|z|^2}, \\ P(C_0|B_1) &= \frac{1}{1+|\frac{1-z}{1+z}|^2}, & P(C_2|B_1) &= \frac{|\frac{1-z}{1+z}|^2}{1+|\frac{1-z}{1+z}|^2}, \\ P(C_1|B_0) &= \frac{1}{1+|\frac{1+iz}{1-iz}|^2}, & P(C_2|B_0) &= \frac{|\frac{1+iz}{1-iz}|^2}{1+|\frac{1+iz}{1-iz}|^2}. \end{aligned} \tag{5}$$

Strategies $|z\rangle$ can be parameterized by the sphere $S_2 \backsimeq \overline{\mathbb{C}}$ by using stereographic projection which establishes correspondence (bijection) between elements of $\overline{\mathbb{C}}$ and the points of S_2 (the north pole of the sphere corresponds with the point in infinity, $|\infty\rangle := |1\rangle_2$). Eq. (5) lead to the mapping $\mathcal{A}_1 : S_2 \to D_3$ of the strategies defined by the parameters of the sphere points $(x_1, x_2, x_3) \in S_2$ onto the three-dimensional cube of conditional probabilities:

$$\begin{aligned} P(C_0|B_2) &= \frac{1-x_3}{2}, & P(C_1|B_2) &= \frac{1+x_3}{2}, \\ P(C_0|B_1) &= \frac{1+x_1}{2}, & P(C_2|B_1) &= \frac{1-x_1}{2}, \\ P(C_1|B_0) &= \frac{1+x_2}{2}, & P(C_2|B_0) &= \frac{1-x_2}{2}. \end{aligned} \tag{6}$$

Combination of the above projection with (4) results in the projection
$\mathcal{A} : S_2 \to T_2$, $\mathcal{A} := \mathcal{A}_0 \circ \mathcal{A}_1$ of two-dimensional sphere S_2 into a triangle T_2.

The knowledge of $\mathcal{A}$ allows to compare the number (measure) of the sets of the possible strategies of the quantum cat and the classical cat having the characteristics we are interested in.

5 Quantum cat versus Classical cat

In this paragraph, we compare the model described above, in which quantum cat can adopt strategies from any group of strategies described above with the quantitative results of Ref. [22]. In order to present the range of representation $\mathcal{A}$ of our interest, we illustrated it with the values of this representation for 10,000 randomly selected points with respect to constant probability distribution on the sphere S_2. The choice of such a measurement method for quantum cat's strategy is justified by the fact that that constant probability distribution corresponds to the Fubini-Study measure on $\mathbb{C}P^1$ [32] which is the only invariant measure in relation to any change of quantum cat's decision regarding the chosen strategy (the so called quantum tactic). Changes of the quantum cat's strategies therefore do not influence the discussed below model.

5.1 Optimal strategies

Figure 1 presents the areas (in both models) of frequency q_m of appearance of individual choice alternatives between two types of food, for which optimal strategies exist. Let us observe that in the quantum case the area of the sim-

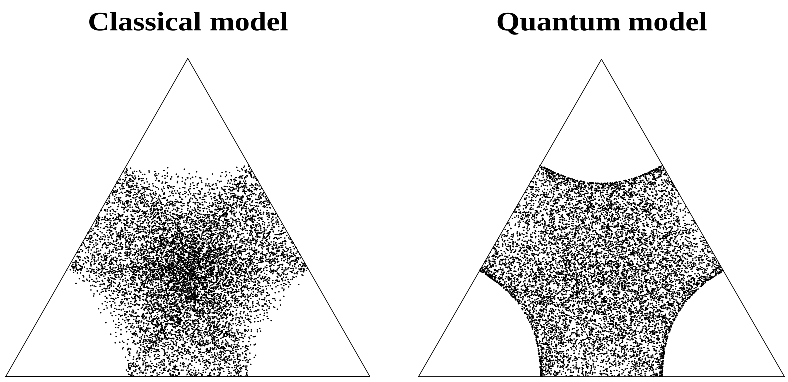

Figure 1: Optimal strategies.

plex corresponding to the optimal strategies has become slightly diminished in relation to the classical model. The difference lies in the disappearance of areas at three boundaries of the regular hexagon which correspond to the arc-bounded surfaces. Assuming the same measure of the possibility of occurrence of determined proportion of all three food pairs, we may say that the number of situations where the optimal strategies can be used in the quantum model makes up about 63% of all possibilities. In the classical variant, the area representing the optimal strategies makes up 67% of the simplex. This difference will be significant for analysis of intransitivity, which will be discussed more precisely in the next paragraph. It is also worth mentioning that in the classical model we deal with sort of condensation of optimal strategies in the central part of the picture in the area of the balanced frequencies of all pairs of food. In the quantum case, they are more evenly spread, although they also appear less frequently towards the sides of the triangle.

5.2 Intransitive orders

In the quantum model, we deal with an intransitive choice if one of the following conditions is fulfilled (see [22]):[2]

- $P(C_2|B_1) = \frac{1-x_1}{2} < \frac{1}{2}$, $P(C_1|B_0) = \frac{1+x_2}{2} < \frac{1}{2}$, $P(C_0|B_2) = \frac{1-x_3}{2} < \frac{1}{2}$.
- $P(C_2|B_1) = \frac{1-x_1}{2} > \frac{1}{2}$, $P(C_1|B_0) = \frac{1+x_2}{2} > \frac{1}{2}$, $P(C_0|B_2) = \frac{1-x_3}{2} > \frac{1}{2}$.

They form two spherical equilateral triangles having three equal $\frac{\pi}{2}$ angles.

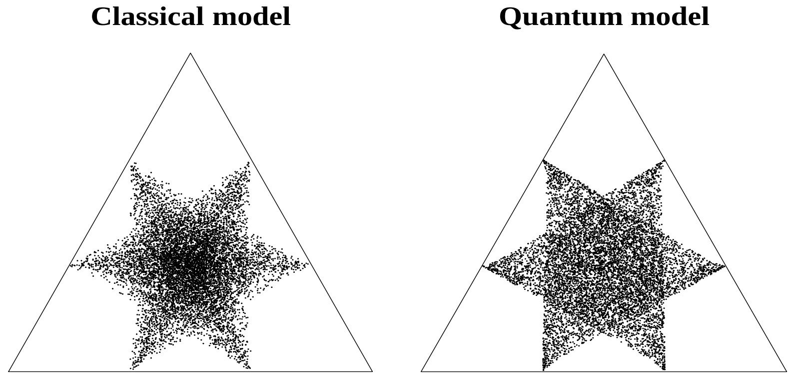

Figure 2: Optimal intransitive strategies.

[2] We have eight orders in which three types of food can be chosen (see the table in [22]). Two of them are intransitive, six are transitive.

It may be seen in Figure 2 in what part of the simplex of parameters (q_0, q_1, q_2) intransitive strategies may be used in both models. They form the six-armed star composed of two triangles[3] in both the quantum and the classical model. As in the previous Figure one can notice that quantum variant is characterized by higher regularity, the star has clearly marked boundaries. In both cases, we have got 33% [4] of conditions allowing to use intransitive optimal strategies in a determined order. There are 44% of conditions allowing to use intransitive strategies with an arbitrary order. However, it is important to remember that in the quantum model, the number of all optimal strategies has decreased in relation to the classical variant. This, when the number of intransitive optimal strategies is equal, means that intransitive orders gain more importance in the quantum model. It is not the only reason leading to such a conclusion (see next paragraph).

5.3 Transitive orders

Let us have a closer look at Figure 3. It presents a simplex area for which there exist transitive optimal strategies in both models. In the classical

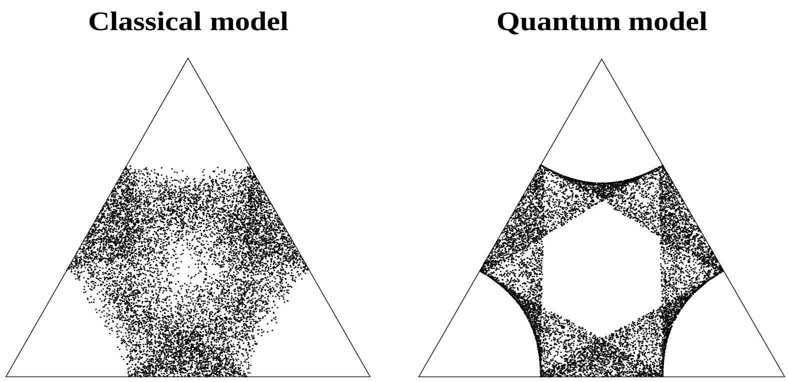

Figure 3: Optimal transitive strategies.

case optimal transitive strategies cover the same area of the simplex as all optimal strategies, however they occur less often in the center of the simplex (near point $q_0 = q_1 = q_2 = \frac{1}{3}$). The quantum version is essentially different

[3]Any of them corresponding to one of two possible intransitive orders.
[4]They are measured by the area of equilateral triangle inscribed into a regular hexagon.

- transitive optimal strategies do not appear within the boundaries of the hexagon in the central part of the picture (thus, there are about 41% of them). Let us observe that this is the area where two different intransitive orders superimpose (22%).[5] Therefore, there cannot be defined for each intransitive order a transitive order whose working effects are identical. Moreover, the transitive strategies appear much less frequently within the arms of the star forming intransitive orders. The above remarks point to the fact that in the quantum model (within the boundaries of the pure strategy) intransitive preferences significantly gain more importance. To make the analysis clear, let us sum up our quantitative discussion by gathering the results round into a table:

Table 1: Comparison of achievability of various types of optimal strategies in both models.

	All	Intransitive	Transitive
Classical model	67 %	44 %	67 %
Quantum model	63 %	44 %	41 %

5.4 Remark about quantum mixed strategies

Any quantum cat's mixed strategy ρ can be identified with a point p inside a ball whose boundary is a set of pure strategies represented by a Bloch sphere S_2. A line passing through a point p and the centre of the ball cuts the sphere in two antipodal points $-\vec{v}$ and $\vec{v}$. The point p divides the segment $[-\vec{v}, \vec{v}]$ in the same ratio as the ratio of weights w_v and w_{-v} in the representation of a mixed strategy ρ as a convex combination of two pure strategies:

$$\rho = w_v |z_v\rangle\langle z_v| + w_{-v} |z_{-v}\rangle\langle z_{-v}|.$$

Two antipodal points $-\vec{v}$ and $\vec{v}$ of the sphere represent pure cat's strategies with the same property (intransitive or transitive).[6] Since formulas (6) are linear, each point lying on the segment $[-\vec{v}, \vec{v}]$ will represent an strategy of

[5] The area of the regular six-armed star is two times bigger than the area of the hexagon inscribed into it.

[6] If coordinates of any vector $\vec{v}$ satisfy one of the conditions of intransitivity (see paragraph 5.2), then coordinates of $-\vec{v}$ satisfy the other one.

the same property as points being the ends of this line.
The randomized model, presented above, in which player operates mixed strategies has the unique property - preferences of mixed strategies are not different from preferences of respective pure strategies lying on the line passing through the middle of the sphere and a point inside the sphere specifying mixed strategy.

6 Conclusion

The aim of this work is to present some methods of quantitative analysis of the, among others, intransitive orders within the boundaries of the quantum game theory. We compared the results that the two models (quantum and classical) yield. The geometrical interpretation presented in this article can turn out to be very helpful in understanding various quantum models in use.

It turns out that the order imposed by the player's rational preferences can be intransitive. The quantum model gives a considerable weight to intransitive orders. They are a constituent part of more of all optimal strategies than in the classical case. Moreover, for some frequencies of appearance of pairs of food, quantum cat is able to achieve optimal results only thanks to the intransitive strategy (it is imposible to specify a transitive optimal strategy !). It is a significant difference in reference to classical cat's situation (we can always find an optimal strategy that determines the transitive order). However, it must be admitted here that it refers only to the simple patterns of cat's behavior.

We presented the quantum game theory model, in which optimal effects can be obtained (in some cases) only by intransitive strategies. Analysis of this kind of models may be important for research on general properties of the games with Nature in the context of quantum information theory. Perhaps, more advanced research into quantum game theory will confirm validity of the intransitive decision algorithms, which are often in contradiction with our intuition. Maybe quantum models describe our spontaneously shown preferences more precisely, as these preferences are often intransitive. More profound analysis of intransitive orders can have importance everywhere where the problem of choice behavior is considered. Thorough analysis of this problem would be of great importance to those who investigate our mind performance or for the construction of thinking machines.

Mathematics have often been inspired by games. This gave rise to the new fields of research (studies of games of chance gave rise to a large branch of mathematic called probability theory). In our everyday lives, we encounter various situations of conflict and cooperation where we have to make particular decisions. Many problems in the fields of economy and political sciences can be expressed in the language of the quantum game theory. In physics, the problem of measurement can be considered as a game against Nature - the observer tries to gain most possible information about the observed object. Other experiments can be modelled in the same way. Therefore, it is vital to carry on research into this new field.

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