Market Equilibria under Procedural Rationality*

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Abstract

We analyze the endogenous price formation mechanism of a pure exchange economy with two assets, riskless and risky. The economy is populated by an arbitrarily large number of traders whose investment choices are described by means of generic smooth functions of past realizations. These choices can be consistent with (but not limited to) the solutions of expected utility maximization problems.

Under the assumption that individual demand for the risky asset is expressed as a fraction of individual wealth, we derive a complete characterization of equilibria. It is shown that irrespectively of the number of agents and of their behavior, all possible equilibria belong to a one-dimensional "Equilibrium Market Curve". This geometric tool helps to illustrate the possibility of different phenomena, as multiple equilibria, and can be used for comparative static analysis. We discuss the relative performances of different strategies and the selection principle governing market dynamics on the basis of the stability analysis of equilibria.

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1 Introduction

There exists a long-standing tradition in theoretical economics to model economic agents as having a strictly limited range of notionally available actions. Maximization of a suitable function (e.g. expected utility) is widely accepted as a reasonable description of individual behavior, as it captures the idea of rationality and profit seeking behind the actions of the "economic man". As early as fifty years ago, however, some writers, most notably Herbert Simon, recognized a strong dissonance between the modeling of human behavior in economics and the description of the same behavior in other social sciences. Indeed Simon (1955) emphasizes that, due to informational and cognitive restrictions, people may not be acting as if they were utility maximizers who are able to perfectly anticipate their own and others' future decisions and reactions. At the same time, however, it is in general true that human beings avoid behaving in a random manner. Rather they tend to follow some deliberate procedures in their decision making process. This broader view on economic behavior led to the concept of *procedural* rationality (Simon, 1976) which still includes, as a special case, the optimizing and perfectly anticipating behavior but which can, at the same time, account for different types of learning.

The assumption of procedurally rational agents implies that the level of heterogeneity in the market is much larger than it is usually assumed. As argued by Kirman (2006) this heterogeneity is probably fundamental for the functioning of market economies. Notice that, in principle, even "substantive rational" agents imbued with perfectly anticipating rationality may differ in terms of their preference structure and, hence, in their implied actions. At the same time, heterogeneity in expectations is reported in several surveys on traders behavior and is the basis for several proposed explanations for the abnormal large trading volume in financial markets and for other observed "anomalies" (e.g. Brock (1997); Hommes (2006) and references therein). Such "rational heterogeneity" is broaden and strengthen by the various violations of axioms of rational choice which have been well documented by a number of different studies in the field of experimental economics[1].

[1]The Handbook of Experimental Economics (Kagel and Roth, 1995) and the Nobel lecture of Daniel

If the evidence supports the idea of procedural rationality of heterogeneous agents, why do the models based on that assumption remain exceptions rather than norm? In his review of the literature on bounded rationality, Conlisk (1996) identifies a number of possible reasons for the dominance of substantive rational behavior in economic modeling. One of them is that such behavior, even if not entirely realistic, seriously restricts the range of possible actions, and, hence, brings discipline into the theory. By acknowledging the need of discipline, this paper seeks to dispel the fear of getting lost in the "wilderness of bounded rationality". In the context of a simple speculative asset market our model demonstrates that (i) market forces and (ii) a natural requirement for consistency between aggregate dynamics and individual actions will lead to quite specific conclusions about the long-run state of the market.

We consider a dynamic model where an arbitrary number of heterogeneous agents trade a riskless bond and a long-lived risky asset. The only restriction imposed on the individual behavior is that the amount of asset demanded by traders is expressed as a fraction of their current wealth. In technical terms, this assumption confines possible agents' behavior to the so-called constant *relative* risk aversion (CRRA) framework. The shares of personal wealth invested in the risky security are chosen, at each period, following individual procedures and on the basis of commonly available information. We model procedural rationality by means of agent-specific *investment functions* which map the information set to the present investment share. The dynamics of the multi-dimensional system describing the evolution of asset price and agents' wealth is derived. Without imposing any constraint about the specific form of the investment functions we are able to completely characterize those equilibria in which aggregate market dynamics is consistent with agents expectations. Equilibrium price return and wealth distribution turn out to be a combined outcome of the agents' adaptive procedures and of the evolutionary selection taking place in the market. Specifically, we show that two types of long-run dynamics are possible. In the first type both securities give the same expected return, and the wealth of all agents grows at the same rate. In the second type one of the

Kahneman (Kahneman, 2003) provides plenty of examples of systematic biases, i.e. individual decisions which would be qualified as irrational from the traditional economic point of view.

securities gives a higher expected return, and one or few "survivors" ultimately possess the total wealth of the economy. We derive the local stability results for all possible steady-states. The conditions are ready to be applied for any specific ecology of traders whose behavior can be accommodated in our framework.

Two distinct streams of theoretical research intersect in our paper. The main source of our inspiration is the growing field of the Heterogeneous Agent Models (HAMs), extensively reviewed in Hommes (2006). The HAM literature considers markets as a feedback system, where agents employ adaptive expectation rules, so that current prices affect expectations about future prices, and, consequently, prices themselves. Modeling stylized behaviors of "fundamentalists" or "trend chasers", the HAMs can explain different "stylized facts" of financial markets, such as excess volatility and repeated patterns of temporary bubbles followed by severe crashes. In our opinion, however, this approach lacks an unifying framework, because expectation rules vary from model to model. By keeping investment functions generic we intend to create such a framework, avoiding, at the same time, an unrealistic level of simplicity in the agents' expectational procedures and considering truly heterogeneous preferences. Furthermore, in the HAMs with evolving population (as e.g. in Brock and Hommes (1998)), agents switch between different forecasting rules on the basis of some performance measure, which is often introduced *ad hoc*. Conversely, the wealth dynamics, explicitly considered in our framework, provide a natural performance measure.[2]

Multi-asset markets populated by several procedurally rational agents are studied in the literature on Evolutionary Finance (EF), initiated by Blume and Easley (1992), and recently

[2]Our paper can be considered as only a first step towards an unifying framework of the HAM literature. An issue is that many HAMs are built in the so-called constant *absolute* risk aversion (CARA) framework, which is outside of the scope of this paper. In the CARA framework agents' demand is independent of their wealth, and thus wealth dynamics do not affect asset pricing. Our framework is, instead, of constant *relative* risk aversion (CRRA), where the agents' investment shares do not depend on their wealth, so that demand for the risky asset increases linearly with wealth. The inclusion of CARA strategies in a CRRA-based framework is a subject of future research. Notice, however, that experiments with human subjects usually reject CARA behavior, supporting decreasing or constant relative risk aversion, see for example Kroll, Levy, and Rapoport (1988) and the discussion of this issue in Levy, Levy, and Solomon (2000).

reviewed in Evstigneev, Hens, and Schenk-Hoppé (2009). Our paper shares a number of distinctive features with the EF literature, such as a descriptive approach to the investment behavior of agents and the central role played by the wealth-driven selection in determining the long-run dynamics of the market. The EF literature focuses on the relative valuation of different risky assets. It is also specifically interested in a search for those strategies which attract most wealth and are evolutionary stable, i.e., are not driven out by alternative behaviors. In contrast to the EF literature, but in the spirit of the HAMs, in this paper we focus on the nature of market instabilities caused by the adaptive behavior of agents and the feedback mechanism existing between realized and expected market conditions. This difference in focus is reflected in our investment functions, which explicitly depend on past realized returns[3]

These distinct streams of economic research, the EF and HAMs, meet in the main result of this paper, the local stability conditions of procedurally consistent equilibria in which aggregate market dynamics is consistent with agents expectations. We find that two conditions are necessary and sufficient for the stability of one or many-survivors equilibria. The first is essentially the requirement of evolutionary stability of the surviving strategies, which is similar to the "survival" criteria adopted in EF models. The second is the dynamic stability of these strategies under the price feedback mechanism, a typical result of the HAMs.

The rest of the paper is organized as follows. In Section 2 we introduce the model, presenting and briefly discussing our assumptions. First, we explicitly write the traders' intertemporal budget constraints. Second, we derive the resulting dynamics in terms of returns and wealth shares. Third, we introduce agent specific investment functions. Finally, we introduce the notion of Procedurally Consistent Equilibrium that will be used as a formal definition

[3]There are other details which distinguish our approach from the EF literature. Namely: a risk-free asset is available in our economy at a given price, we have only one risky asset and we do not let agents consume all the dividend payoffs. In addition, the earlier EF papers by Hens and Schenk-Hoppé (2005) and Amir, Evstigneev, Hens, and Schenk-Hoppé (2005) model assets as short-lived and, consequently, ignore capital gain as a component of agents' wealth accumulation. The last models of the EF literature (see, e.g., Amir, Evstigneev, and Xu (2008), Evstigneev, Hens, and Schenk-Hoppé (2006, 2008)) all deal with long-lived assets, as we do in our framework.

of equilibrium throughout the paper. In Section 3 we present the equilibrium and stability analysis of the system in the simplest case of a single active trader. The geometric locus of all possible equilibria, the Equilibrium Market Curve, is derived, and its use is discussed. Then we derive general stability conditions and we discuss few important special cases which received particular attention from the past literature. Section 4 is devoted to the analysis of the general case in which an arbitrarily large number of traders are present in the market. The implications of our findings concerning the general ability of market forces in selecting the "best" strategy are discussed in Section 4.4 while Section 5 concludes.

2 Model Definition

Consider a simple pure exchange economy where trading activities take place in discrete time. The economy is composed by a riskless asset (bond) giving in each period a constant interest rate $r_f > 0$ and a risky asset (equity) paying a random dividend D_t at the beginning of each period t. The riskless asset is considered the numéraire of economy and its price is fixed to 1. The ex-dividend price P_t of the risky asset is determined at each period, on the basis of the aggregate demand, through market-clearing condition. The resulting intertemporal budget constraint is derived below and the main hypotheses, on the nature of the investment choices and of the fundamental process, are discussed. These hypotheses will allow us to derive the explicit dynamical system governing the evolution of the economy.

2.1 Intertemporal budget constraint

We consider general situation when the economy is populated by a fixed number N of traders[4].

Let $W_{t,n}$ and $x_{t,n}$ stand for the wealth of trader n at time t and for the fraction of his wealth invested into the risky asset. Following Epstein and Zin (1989) and similarly to Amir,

[4]Using the terminology of the heterogeneous agent literature, we consider N *types* of traders (cf. Brock, Hommes, and Wagener (2005)). Notice, however, that all traders possessing the same investment behavior (type) are considered as one single investor. That is, in the terminology of the evolutionary finance literature we deal with N different *strategies* (cf. Hens and Schenk-Hoppé (2005))

Evstigneev, Hens, and Schenk-Hoppé (2005) we assume that total agent's wealth is reinvested and ignore consumption. Thus, after the trading session at time t, agent n possesses $x_{t,n}\,W_{t,n}/P_t$ shares of the risky asset and $(1-x_{t,n})\,W_{t,n}$ shares of the riskless security. In the beginning of time $t+1$ the agent gets (in terms of the numéraire) random dividends D_{t+1} per each share of the risky asset and constant interest rate r_f for all shares of the riskless asset. Therefore, at time $t+1$ the wealth of agent n is given by

$$W_{t+1,n}(P_{t+1}) = (1-x_{t,n})\,W_{t,n}\,(1+r_f) + \frac{x_{t,n}\,W_{t,n}}{P_t}\,(P_{t+1}+D_{t+1}) \quad . \tag{1}$$

Through the capital gain the new wealth depends on the price P_{t+1} of the risky asset, which is fixed so that aggregate demand equals aggregate supply. Assuming a constant supply of the risky asset, whose quantity can then be normalized to 1, the price P_{t+1} is defined as the solution of the equation

$$\sum_{n=1}^{N} x_{t+1,n}\,W_{t+1,n}(P_{t+1}) = P_{t+1} \quad . \tag{2}$$

Simultaneous solution of (1) and (2) provides the new price P_{t+1}. Once the price is fixed, the new portfolios and wealths are determined and economy is ready for the next round.

The dynamics defined by (1) and (2) describe an exogenously growing economy due to the continuous injections of new shares of the riskless asset, whose price remains, under the assumption of totally elastic supply, unchanged. In order to simplify the following analysis, it is convenient to remove this exogenous economic expansion from the dynamics of the model. To this purpose we introduce rescaled variables

$$w_{t,n} = W_{t,n}/(1+r_f)^t\ , \quad p_t = P_t/(1+r_f)^t\ , \quad e_t = D_t/(P_{t-1}\,(1+r_f)) \quad , \tag{3}$$

denoted with lower case names. The quantity e_t represents (to within a factor) the dividend yield. Using new variables, (1) and (2) read

$$\begin{cases} p_{t+1} = \displaystyle\sum_{n=1}^{N} x_{t+1,n}\,w_{t+1,n} \\ w_{t+1,n} = w_{t,n} + w_{t,n}\,x_{t,n}\left(\dfrac{p_{t+1}}{p_t} - 1 + e_{t+1}\right) \quad \forall n \in \{1,\dots,N\} \end{cases} \quad . \tag{4}$$

These equations represent an evolution of state variables $w_{t,n}$ and p_t over time, provided that stochastic process $\{e_t\}$ is given and the set of investment shares $\{x_{t,n}\}$ is specified.[5]

In this paper the agents' investment shares are assumed to be independent of the contemporaneous price and wealth, the assumption which will be formalized in Section 2.3. System (4) implies a simultaneous determination of the equilibrium price p_{t+1} and of the agents' wealths $w_{t+1,n}$, so that the state of the system at time $t+1$ is only implicitly defined. For analytical purposes, one has to derive the explicit equations that govern the system dynamics.

2.2 The dynamical system for wealth shares and price return

Let a_n be an agent specific variable, dependent or independent from time t. We denote with $\langle a \rangle_t$ its *wealth weighted average* on the population of agents at time t, i.e.

$$\langle a \rangle_t = \sum_{n=1}^{N} a_n \, \varphi_{t,n} \; , \quad \text{where} \quad \varphi_{t,n} = \frac{w_{t,n}}{w_t} \quad \text{and} \quad w_t = \sum_{n=1}^{N} w_{t,n}. \tag{5}$$

The transformation of the implicit dynamics (4) into an explicit one is not, in general, possible also because the market price should remain positive over time. On the other hand, the agents are allowed to have negative wealth, which is interpreted as debt in that case. Therefore, $\varphi_{t,n}$ are arbitrary numbers whose sum over all agents is equal to 1 for any period t.

The next result gives the condition for which the dynamical system implicitly defined in (4) can be made explicit without violating the requirement of positiveness of prices.

Proposition 2.1. *Let us assume that initial price p_0 is positive. From equations (4) it is possible to derive a map $\mathbb{R}^N \to \mathbb{R}^N$ that describes the evolution of traders' wealth $w_{t,n}$ with positive prices $p_t \in \mathbb{R}^+ \; \forall t$ provided that*

$$\Big(\langle x_t \rangle_t - \langle x_t \, x_{t+1} \rangle_t \Big) \Big(\langle x_{t+1} \rangle_t - (1 - e_{t+1}) \langle x_t \, x_{t+1} \rangle_t \Big) > 0 \qquad \forall t \quad . \tag{6}$$

If the previous condition is met, the growth rate of (rescaled) price $r_{t+1} = p_{t+1}/p_t - 1$ reads

$$r_{t+1} = \frac{\langle x_{t+1} - x_t \rangle_t + e_{t+1} \, \langle x_t \, x_{t+1} \rangle_t}{\langle x_t \, (1 - x_{t+1}) \rangle_t} \quad , \tag{7}$$

[5] Notice that (4) is equivalent to (1) and (2) and can be simply obtained setting $r_f = 0$.

the growth rates of (rescaled) individual wealth $\rho_{t+1,n} = w_{t+1,n}/w_{t,n} - 1$ are given by

$$\rho_{t+1,n} = x_{t,n}\left(r_{t+1} + e_{t+1}\right) \qquad \forall n \in \{1, \ldots, N\} \quad , \tag{8}$$

and agents' (rescaled) wealth shares $\varphi_{t,n}$ evolve according to

$$\varphi_{t+1,n} = \varphi_{t,n}\,\frac{1 + \left(r_{t+1} + e_{t+1}\right) x_{t,n}}{1 + \left(r_{t+1} + e_{t+1}\right) \left\langle x_t \right\rangle_t} \qquad \forall n \in \{1, \ldots, N\} \quad . \tag{9}$$

Proof. See appendix A. □

The market evolution is *explicitly* described by the system of $N + 1$ equations in (7) and (8), or, equivalently, in (7) and (9). The dynamics of rescaled price p_t can be derived from (7) in a trivial way, but price will remain positive only if condition (6) is satisfied[6]. Finally, using (4), one can easily obtain the evolution of unscaled price P_t.

In (7), analogously to the evolutionary finance literature discussed in the Introduction, agents who have relatively more wealth have a higher impact on the determination of price. Since we consider an infinitely lived asset, the investment decision at time t affects also the determination of price (and return) at time $t + 1$. The wealth dynamics in (8) reveal that individual returns are proportional to the gross return (capital gain or loss plus the dividend yield). Finally, (9) describes the evolution of the relative wealth. As long as higher wealth gains can be considered associated to a higher 'fitness", one can interpret this relation as a *replicator dynamics* (Weibull, 1995), in which the market influence of each agent changes according to his performance *relative* to the average performance.

Following Chiarella and He (2001) and Anufriev, Bottazzi, and Pancotto (2006) we make the following[7]

Assumption 1. The dividend yields e_t are i.i.d. random variables obtained from a common distribution with positive support.

[6] In general, it may be quite difficult to check the validity of this condition at each time step. However, if agents are diversifying their portfolio and do not go short, so that $0 < x_{t,n} < 1$ $\forall t, n$, then inequality (6) is always satisfied (Anufriev, Bottazzi, and Pancotto, 2006).

[7] For different specifications of dividend process inside the same framework see Chiarella, Dieci, and Gardini (2006) and Anufriev and Dindo (2009).

This assumption implies that price and dividends grow at the same rate, as, for instance, fundamental price does in an economy with geometrically growing dividends.[8] Notice however that in our model the price is determined through the market clearing condition and is not necessary fixed on the fundamental level. On the empirical side, a simple inspection of the annual historical data for the Standard&Poor's 500 index suggest that yield can be reasonably described as a bounded positive random variable with roughly stationary behavior.[9]

2.3 Agents' investment functions

We consider a general framework in which every agent decides independently the share of wealth he wants to invest in the risky security. The agent's decision variable is not the amount of shares of the risky asset in the portfolio, but the wealth share invested in the asset, which should be independent of the contemporaneous price and wealth levels. This restriction, which can be referred as a constant relative risk aversion (CRRA) framework, implies that the dynamics derived in Proposition 2.1 are well-defined.

Since in this paper we are mainly concerned with the effect of *speculative* behaviors on the market aggregate performance, we let aside those issues which may occur under asymmetric knowledge of the underlying fundamental process. Thus, we assume that the structure of the yield process defined in Assumption 1 is known to everybody. Consequently, we assume that all agents base their investment decisions at time t exclusively on the public and commonly available information set $\mathcal{I}_{t-1}$ formed by past realized prices. This set can alternatively be defined through the past return realizations as $\mathcal{I}_{t-1} = \{r_{t-1}, r_{t-2}, \dots\}$ and we make the following

Assumption 2. For each agent n there exists a smooth *investment function* f_n which maps the present information set into his or her investment decision

$$x_{t,n} = f_n(\mathcal{I}_{t-1})\,. \tag{10}$$

[8]Inside an economy with infinitely lived assets, Assumption 1 is equivalent to the *i.i.d.* payoff structure considered in Blume and Easley (1992).

[9]See for instance the graph at p.8 of Schiller (2000) and the discussion therein.

The function f_n gives a complete description of the investment decision of the n-th agent who adapts to observed price fluctuations. The knowledge about the yield process is not explicitly inserted in the information set but can be considered embedded in the functional form of f_n.

Assumption 2 is strictly related to the "smooth" learning hypothesis described in Grandmont (1998). It is compatible with a number of different learning processes based on common information, as for instance the Bayesian learning, or, more generally, the adaptive models explored inside the EF literature (Blume and Easley, 1992; Hens and Schenk-Hoppé, 2005). Indeed, the investment choice in (10) can be thought as the result of two separate steps. In the first step agent n, using a set of estimators or "expectation functions" (Grandmont, 1998) $\{g_{n,1}, g_{n,2}, \dots\}$, forms his prediction about the behavior of future prices, $\theta_{n,j} = g_{n,j}(\mathfrak{I}_{t-1})$, where $\theta_{\cdot,j}$ stands for some statistics of the returns distribution at time $t+1$, e.g. the average return, the expected variance or the probability that a given return threshold will be crossed. Then, using a choice function h_n defined in terms of these expectations, the agent computes the fraction of wealth invested in the risky asset $x_{t+1,n} = h_n(\theta_{n,1}, \theta_{n,2}, \dots)$. As a result, the investment function f_n becomes a composition of a set of estimators $\{g_{n,\cdot}\}$ and an individual choice function h_n. The choice function can be derived from some optimization procedure (as the maximization of expected utility under uncertainty) or, more generally, can reflect a satisfying behavior. The expectation functions g's can account for the outcomes of *fundamental* and *technical* valuation, widely used both in trading practices observed in real markets and in the HAMs. The present framework is indeed able to account for a wide spectrum of behavioral assumptions. An example is provided in Appendix B where we briefly discuss how a myopic utility maximization framework, widely adopted by the HAM literature, can be treated in terms of expectation and choice functions. Another example is the model we developed in Anufriev, Bottazzi, and Pancotto (2006) (the predecessor of this paper), in which adaptive agents decide their investment shares on the basis of exponentially weighted moving averages of past returns and their variance. Our Assumption 2 generalizes such behaviour by allowing agents to map the past return history into the future investment choice, using whatever smooth function they like. Using the terminology coined by Herbert Simon (Simon, 1976), in this paper we consider generic *procedurally* rational traders whose investment functions are the collective description of preferences, beliefs and implied actions.

Under Assumption 1, the dynamics in terms of price return, wealth shares and investment shares are described by (7), (9) and (10). In order to analyze a finite-dimensional system we restrict each agent n to base his decision on the past L_n price returns. Without loss of generality we can assume that the "memory span" is the same for all traders and denote it by L. For the following discussion L must be finite, but can be arbitrarily large.

2.4 Procedurally Consistent Equilibria

The "rational expectations" approach (Muth, 1961; Lucas, 1978) postulating that the dynamics generated by the actions of an agent should be consistent with his *a priori* expectations about the dynamics itself, is too restrictive for a framework with heterogeneous, procedurally rational agents. A suitable concept for such framework would be an "adaptive procedural rationality" under which agents' actions generate dynamics which are in turn consistent with these co-evolving actions. In this paper we focus on the emergence of equilibria of this kind, defined as situations in which agents are have no incentive to review their choices. In the setup outlined in Sections 2.1 and 2.2 the agents choose the wealth share x to invest in the risky asset. Consequently we apply the following

Definition 2.1. *Procedurally Consistent Equilibria* (PCE) are the trajectories of the system defined by (7) and (8) (or, equivalently, by (7) and (9)) with fixed investment shares $x_{t,n} = x^*_n$ and stationary wealth distribution $\varphi_{t,n} = \varphi^*_n$ for all n and t.

At the PCE agents are not changing their actions, and, at the same time, the aggregate dynamics is consistent with the procedures agents use to decide their positions in the market. While the consistency requirement seems intuitive, the assumption of constant actions requires some justification. It is, for instance, at odds with complex dynamics emerging in many HAMs, e.g. Day and Huang (1990), Chiarella (1992) or Brock and Hommes (1998), where one observes cyclic or even chaotic motion of individual agents' wealth and portfolios. Indeed, our interest lies not in the global dynamics of a market with a few agents having stylized behaviors, but in the investigation, under general assumptions, of the local properties of the feasible PCE. Moreover, the situation in which an agent changes his position each period in a chaotic manner does not only seems unrealistic, but, in our opinion, can hardly play a role in the generalization of the notion of equilibrium for multi-agent setting. For instance, if a forecasting agent observes persistent mistakes (periodic or quasi-periodic) in his prediction of future market dynamics, he will probably adopt new forecasting procedures. The new procedures would make his investment decision different and, ultimately, perturb the system away from the previous trajectory. As long as a minimal evolutionary pressure is put on the system, so that agents can revise their strategies if they led to incorrect expectations, any equilibrium whose dynamics is not consistent with the actions of agents is likely to have a transitory nature.

In what follows any use of the term "equilibrium" refers to the specific notion of equilibrium introduced by Definition 2.1.

3 Single Agent Case

We start with the analysis of the very special situation in which a single agent operates in the market. The main reason to perform this analysis rests in its relevance for the multi-agent case, as we will see in the next Section. In particular, some type of the generic equilibrium in the setting with N heterogeneous traders requires, as necessary condition for stability, the stability of a suitably defined single agent equilibrium.

This Section starts laying down the dynamics of the single agent economy as a multidimensional dynamical system of difference equations of the first order. All possible steady-states of the system are identified and their stability studied using the associated characteristic polynomial.

3.1 Dynamical system

In the case of one single agent the evolution of wealth shares in (8) is trivial and can be ignored. As a consequence, the whole system can be described with only $L+1$ variables representing the present investment choice and the L past returns. We denote the price return at time $t-l$ as $r_{t,l}$, so that the system reads

$$\begin{cases} x_{t+1} & = & f\big(r_{t,0}, r_{t,1}, \ldots, r_{t,L-1}\big) \\ r_{t+1,0} & = & R\Big(f\big(r_{t,0}, r_{t,1}, \ldots, r_{t,L-1}\big),\ x_t,\ e_t\Big) \\ r_{t+1,1} & = & r_{t,0} \\ \quad\vdots & & \\ r_{t+1,L-1} & = & r_{t,L-2} \quad . \end{cases} \tag{11}$$

The function R in the right hand-side of (7) is defined as

$$R(x', x, e) = \frac{x' - x + e\, x'\, x}{(1-x')\, x} \quad , \tag{12}$$

where x, x' and e denote the *previous period* investment choice, the current (contemporaneous with return) investment choice and the dividend yield, respectively.

The system (11) depends on the noise component e_t which, according to Assumption 1, is an i.i.d. random variable. In the following analysis we substitute the yield realizations $\{e_t\}$ by their mean value $\bar{e}$ considering the *deterministic skeleton* of (11). The resulting system describes, in a sense, the "average" of the stochastic dynamics.[10] Once referred to the deterministic skeleton, the set of Procedurally Consistent Equilibria introduced in Definition 2.1 reduces to a set of fixed points of (11). Indeed, from (12), if the investment choice x is not changing over time, then $x = x'$, and price returns are also constant. The next section investigates whether it is possible to characterize all the fixed points of (11).

[10]This substitution is in general dangerous, as small random shocks could accumulate along a trajectory so that their final effect on the system dynamics becomes huge. This is however not the case when one considers asymptotically stable fixed points, as we will do later. In this case, as long as random shocks are sufficiently small, the dynamics of the stochastic system is bounded in a neighborhood of the fixed point of the associated deterministic skeleton.

3.2 Equilibrium market curve

It turns out that independently of agent's behavior, all possible equilibria belong to a one-dimensional curve, the *Equilibrium Market Curve.* The next definition introduces the locus of equilibria, while the following proposition, which characterizes the equilibria of (11), clarify its role.

Definition 3.1. The *Equilibrium Market Curve* (EMC) is the function $l(r)$ defined as

$$l(r) = \frac{r}{\bar{e} + r} \quad . \tag{13}$$

Let x^* denote the agent's wealth share invested in the risky asset at equilibrium and let r^* be the equilibrium return (equal to the returns for all lags). One has the following

Proposition 3.1. *Let* $\boldsymbol{x^*} = (x^*; r^*, \dots, r^*)$ *be a fixed point of the deterministic skeleton of* (11)*. Then*

(i) the equilibrium return r^* *and the equilibrium investment share* x^* *satisfy*

$$l(r^*) = f(r^*, \dots, r^*) \ , \quad x^* = f(r^*, \dots, r^*) \ . \tag{14}$$

(ii) at the fixed point $\boldsymbol{x^*}$ *prices are positive if either* $x^* < 1$ *or* $x^* \geq 1/(1 - \bar{e})$*.*

(iii) in $\boldsymbol{x^*}$ *the growth rate of agent's wealth is equal to the price return* r^**.*

Proof. See appendix C. □

The first statement in the previous Proposition justifies the introduction of the Equilibrium Market Curve in Definition 3.1. Indeed, according to (14) all fixed points of the dynamics can be found as the intersections of the EMC with the symmetrization of function f, i.e. with the restriction of this function to the one-dimensional subspace defined as $r_0 = r_1 = \cdots = r_{L-1}$. The main reason for such a simple characterization of equilibria is the underlying requirement of consistency between self-fulfilling agent's choice and the resulting dynamics. The EMC is the locus of points where agents' expectation formation mechanism is compatible with the intertemporal relation governing market dynamics.

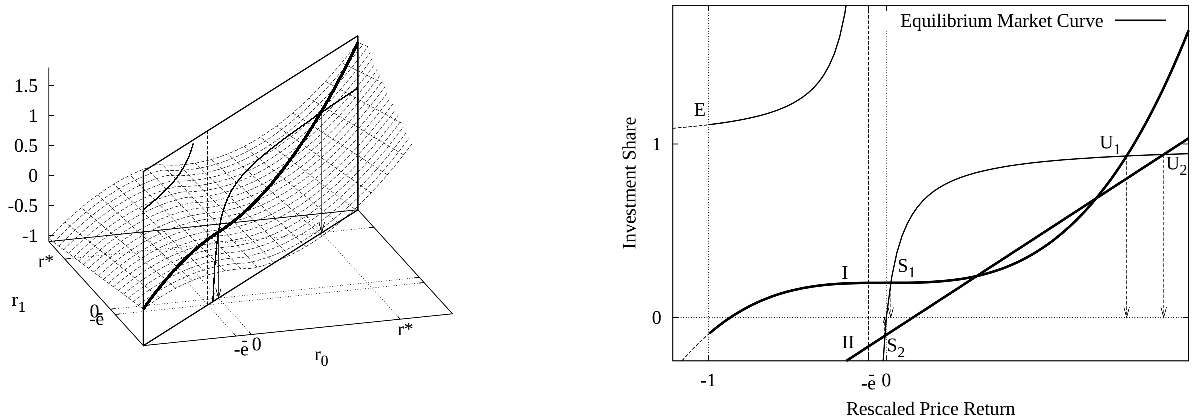

Figure 1: **Left panel:** Investment function based on the last two realized returns and its intersection (thick line) with the symmetric plane. Equilibria are found on this plane as intersections with the EMC. **Right panel:** Equilibria of system (11) are intersections of the EMC with symmetrizations of the agents' investment functions (shown as thick lines and labeled as I and II). There are two equilibria in both cases: S_1 and U_1 in the market with agent I, and S_2 and U_2 in the market with agent II.

Condition (14) is illustrated in Fig. 1. The left panel shows a two-dimensional investment function which depends on the two last realized returns, $f(r_{t-1}, r_{t-2})$. Only the intersection with the diagonal plane defined by the equation $r_{t-1} = r_{t-2}$ is relevant for the question of equilibria location. This plane is represented in the left panel. The hyperbolic curve shown as a thin line represents the EMC (13), while two thick curves depict two investment functions. These curves are symmetrizations of the investment functions ,which in general depend on several variables. The thick curves are the intersections of these multi-dimensional functions with the hyper-plane $r_0 = \cdots = r_{L-1}$. In turn, the intersections of the (symmetrized) investment functions with the EMC are all possible equilibria of the system. The ordinate of the intersection gives the value of equilibrium investment share x^*, while the abscissa gives the equilibrium return r^*.

According to Proposition 3.1*(ii)*, economically meaningful equilibria are characterized by values of the investment share inside the intervals $(-\infty, 1)$ or $[1/(1-\bar{e}), +\infty)$. It amounts to require $r^* > -1$, which implies that part of the EMC on the left from point E is meaningless.

On the remaining part of the Curve one can distinguish between three qualitatively different scenarios.

In the first scenario, when $r^* \in [-1, -\bar{e})$, the return is negative and, hence, rescaled price p_t of the risky asset decreases to 0. The wealth of the agent is positive at any moment of time and is eventually vanishing. The agent possesses the total supply of the risky asset, while his own amount of the numéraire is negative. Thus, the agent has to borrow money in order to keep his relatively high demand for the risky asset. Due to the decrease in the agent's wealth, this demand is insufficient to generate high (not even positive) returns.

In the second scenario, when $r^* \in (-\bar{e}, 0)$, the capital gain on the risky asset is negative and the price of the asset decreases. However, the contribution from the dividend makes the gross return $r^* + \bar{e}$ positive. Furthermore, agent has to have negative wealth (has to be indebted) and, at the same time, has to borrow money in order to keep the demand of the asset positive.[11] The agent possesses the total supply of the risky asset and a negative amount of the numéraire. From Proposition 3.1*(iii)* it follows that the dividend payment allows the agent's wealth to increase to 0. Equilibrium S_2 for agent II in the left panel of Fig. 1 is of such kind.

Finally, in the third scenario, when the rescaled return is positive, the price p_t of the asset increases. Agent has a positive amount of the numéraire and his total wealth is positive and increasing. Such a situation is observed in equilibria S_1, U_1 and U_2.

What can be said about the dynamics of *unscaled* price P_t in all these three scenarios? To answer this question it is important to bear in mind the following relation between the scaled return r_t and return R_t in terms of unscaled price:

$$1 + R_t = (1 + r_t)\,(1 + r_f) \quad .$$

Therefore, in the third scenario, where the rescaled price increases, the unscaled price also increases with a higher rate. Conversely, in the first and second scenarios, even if the rescaled

[11] In general, to guarantee the positiveness of the price at the initial period one has to choose initial wealth appropriately. Since $p_0 = x^* w_0$, for positive x^* the initial (and consequent) agent's wealth is positive, while for negative x^* the wealth is negative.

price is decreasing, the unscaled price may increase due to high enough risk-free interest rate.

To conclude our discussion about equilibrium properties notice that in all possible equilibria there exists a non-zero equity premium, i.e. a difference between the total return of the riskless and the risky asset given by

$$\frac{P_{t+1} - P_t + D_{t+1}}{P_t} - r_f = \frac{\bar{e}\,(1 + r_f)}{1 - x^*} \quad . \tag{15}$$

The equity premium, which is empirically observed in real markets (Mehra and Prescott, 1985), can be explained, within the classical paradigm, as a monetary incentive required by an optimizing risk-averse representative agent to hold the risky asset. In our framework, instead, the risk premium is endogenously generated by the feedback effect from market return to agent's wealth and the reinvestment of the latter. Consequently, the equity premium increases with the dividend yield $(1 + r_f)\bar{e}$ and with the propensity of agent to invest in the risky asset, x^*.

3.3 Stability of single-agent equilibria

The stability conditions are derived from the analysis of the roots of the characteristic polynomial associated with the Jacobian of system (11) computed at equilibrium. The characteristic polynomial does, in general, depend on the behavior of the individual investment function f in an infinitesimal neighborhood of the equilibrium $\boldsymbol{x}^*$. This dependence can be summarized with the help of the following

Definition 3.2. The *stability polynomial* $P(\mu)$ of the investment function f in $\boldsymbol{x}^*$ is

$$P_f(\mu) = \frac{\partial f}{\partial r_0}\,\mu^{L-1} + \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + \frac{\partial f}{\partial r_{L-2}}\,\mu + \frac{\partial f}{\partial r_{L-1}} \ , \tag{16}$$

where all the derivatives are computed in $(r^*, \ldots, r^*)$.

Using the previous definition, the stability conditions can be formulated in terms of the equilibrium return r^*, and of the slope of the EMC at equilibrium

$$l'(r^*) = \frac{\bar{e}}{(\bar{e} + r^*)^2} \ .$$

The following applies

Proposition 3.2. *The fixed point* $\boldsymbol{x^*} = (x^*; r^*, \ldots, r^*)$ *of system* (11) *is (locally) asymptotically stable if all the roots of the polynomial*

$$Q(\mu) = \mu^{L+1} - \frac{P_f(\mu)}{r^* \, l'(r^*)} \Big(\big(1 + r^*\big) \, \mu - 1 \Big) \tag{17}$$

are inside the unit circle. The equilibrium $\boldsymbol{x^*}$ *is unstable if at least one of the roots of* $Q(\mu)$ *lies outside the unit circle.*

Proof. The condition above is a direct consequence of the characteristic polynomial of the Jacobian matrix at equilibrium. See Appendix D for a derivation. □

Once the investment function f is known, the polynomial $P_f(\mu)$ and, in turn, the polynomial $Q(\mu)$ can be explicitly derived. The analysis of the $L+1$ roots of $Q(\mu)$ (usually called *multipliers*) reveals the role of the different parameters in stabilizing a given equilibrium.

3.4 Examples of single-agent system

Since the explicit expression for the roots of (17) cannot, in general, be derived even for relatively simple investment functions, the analytical study of the effect of different parameters is often unfeasible and one has to rely on numerical investigations. Mainly for illustrative purposes we present below three relatively simple cases where analytical results are, to some extent, available. Inspired by models already discussed in the literature, we show how Propositions 3.1 and 3.2 can be applied to obtain rather general results about the effect of different behavioral assumptions. The reader is referred to Appendix E for the derivation of results and for further discussions.

Example 1. Agent with short memory, $L = 1$.

Consider an agent with a memory spanning a single lag, i.e. whose present investment share depends only on the last realized return, $x_{t+1} = f(r_t)$. This is, for instance, the case of an agent who simply predicts the next price return to be equal to the last realized return. In this case, the stability polynomial is simply $P_f = f'(r^*)$. Applying the general result obtained in Propositions E.1 and E.2, one gets

Proposition 3.3. *The fixed point* $\boldsymbol{x}^* = (x^*; r^*)$ *of system* (11) *with* $L = 1$ *is (locally) asymptotically stable if*

$$\frac{f'(r^*)}{l'(r^*)}\,\frac{1}{r^*} < 1\,, \qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \textit{and} \qquad \frac{f'(r^*)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -1 \quad . \tag{18}$$

The fixed point exhibits a Neimark-Sacker, fold *or* flip *bifurcation if the first, second or third inequality in* (18) *turns to equality, respectively.*

The stability region S defined by the three inequalities in (18) is shown as a dark area in the upper left panel of Fig. 2 in coordinates r^* and $f'(r^*)/l'(r^*)$. The second coordinate is the relative slope of the investment function at equilibrium with respect to the slope of the Equilibrium Market Curve. The boundaries of the stability region are labeled as "Neimark-Sacker", "flip" and "fold" depending on the type of bifurcation undertaken by the system when a particular boundary is crossed (e.g. "Neimark-Sacker" curve corresponds to those points where two complex conjugated multipliers cross the unit circle). Notice that if the slope of f at equilibrium is zero, that is, the investment function is locally constant, the equilibrium is always stable. A constant investment function represent an agent whose portfolio choice is insensitive to price variations. An increase in the sensitivity to price, that is of the slope of f, would ultimately lead to system instability. Since for $r > 0$ the slope of the EMC is decreasing, the larger the value of r, the lower is the minimal strength of agent reaction to price fluctuations necessary to destabilize the equilibrium.

As an example of application of Proposition 3.3 consider the investment functions drawn in the left panel of Fig. 1. Assume that the memory span of each agent L is equal to one. In this case, both equilibrium U_1 for agent I and U_2 for agent II are unstable, since the second inequality in (18) is violated. On the contrary, S_1 is (presumably) a stable equilibrium, since the slope of the investment function I in that point is positive and very small. If this slope would increase, the equilibrium S_1 will lose its stability through a Neimark-Sacker bifurcation (c.f. upper left panel of Fig. 2). Conversely, the negative value of the equilibrium return r^* in S_2 implies that an increase of the slope of the investment function II in that point would lead to a flip bifurcation.

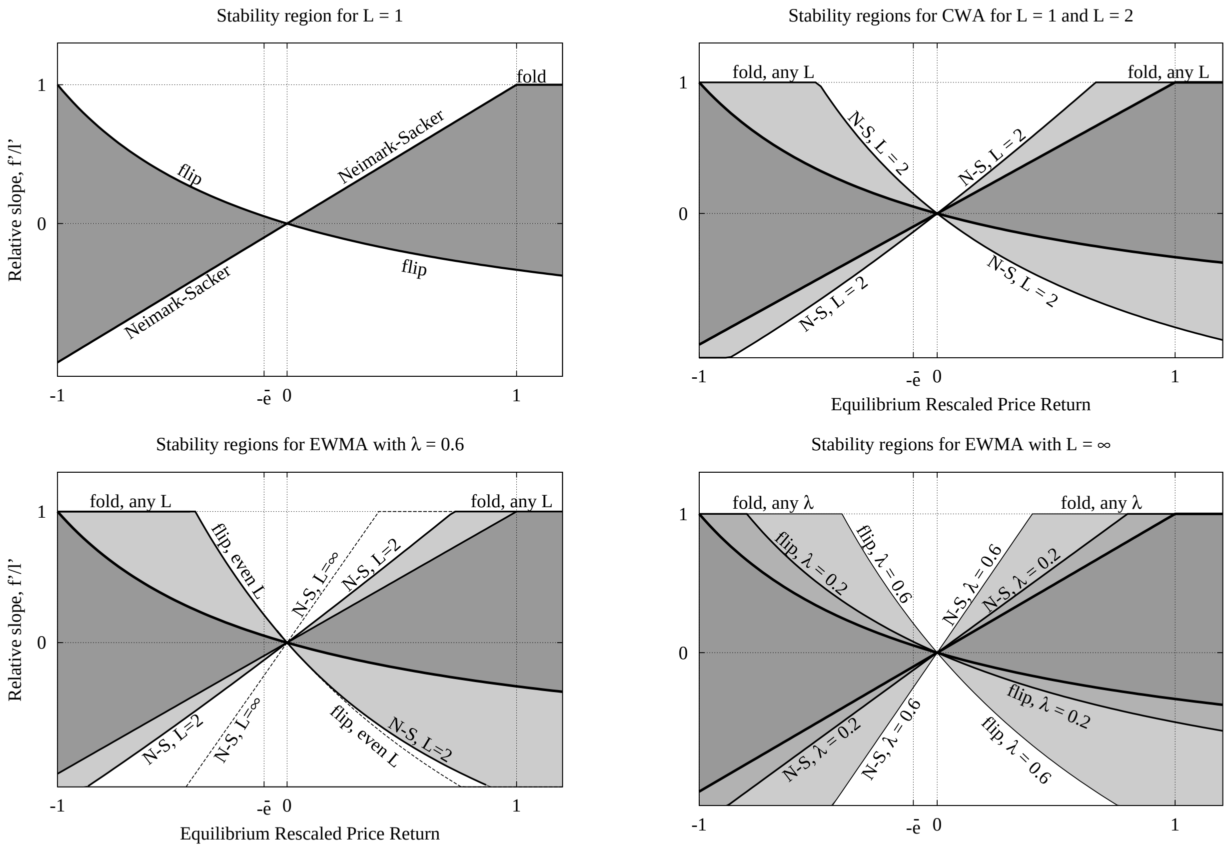

Figure 2: Stability regions and bifurcation types of system (11). The boundaries of the stability region are labeled according to which kind of bifurcation is generated when they are crossed. **Upper Left Panel:** Example 1, $L = 1$. Fixed point is stable if $(r^*, f'/l')$ belongs to the dark gray area. **Upper Right Panel:** Example 2, CWA estimator for different values of L. **Lower Left Panel:** Example 3, EWMA estimator with $\lambda = 0.6$ for different values of L. **Lower Right Panel:** Example 3, EWMA estimator for different values of λ and $L = \infty$.

Example 2. Sample average forecasting rule

Assume that agent invests in the risky asset a fraction of his wealth $x_{t+1} = f(y_t)$ which depends on his estimate about future price return $y_t = \mathrm{E}[r_{t+1}]$. Furthermore, as in e.g. Levy, Levy, and Solomon (2000) and Chiarella and He (2001), assume that the forecast is obtained as a Constant Weighted Average (CWA), i.e. a sample average, of past realized returns

$$y_t = \frac{1}{L}\left(r_t + r_{t-1} + \cdots + r_{t-L+1}\right) \quad . \tag{19}$$

The parameter L defines the length of agent's memory, that is how many past realizations are considered to obtain an estimation about future return. The parameter L clearly acts as a smoothing factor on the agent's behavior: the larger its value, the more the time steps needed for a new trend in returns to be reflected in agent's forecast. We will apply the result of the previous sections to understand how different memory lengths affect the behavior of the market.

Since at equilibrium the forecast coincides with the return, $y_t = r^*$, the stability polynomial reads

$$P_f(\mu) = f'(r^*)\,\frac{1}{L}\,\frac{\mu^L - 1}{\mu - 1} \quad . \tag{20}$$

By plugging this expression in (17) one can in principle compute the $L + 1$ multipliers of the polynomial $Q(\mu)$ for the CWA agent. The region of the parameter space where all these multipliers are inside the unit circle is the stability region of the system. Let's denote it with S_L, since it clearly depends on the memory length L. As before, this region can be represented using the $(r^*, f'/l')$ coordinates system.

We study the effect on the system of different memory lengths by analyzing the dependence of the stability region on the parameter L. When $L = 1$ we are back to the previous Example. The stability region S_1 is shown as a dark gray area in the upper right panel of Fig. 2. Irrespectively on the value of L, the boundary associated with a fold bifurcation is given by the line $f'/l' = 1$. Notice that a flip bifurcation is possible only for odd values of L. The stability region for $L = 2$ can be obtained analytically and is depicted in the upper right panel of Fig. 2 as the union of the dark and light gray areas. Since locally horizontal investment functions always lead to stable equilibria, the points of the horizontal axes lie in the stability region. This region increases with L and for large enough values of the memory parameter, *any* fixed point with $f'/l' < 1$ becomes stable.

Summarizing, a large memory span L has a stabilizing effect on the dynamics of the system. However, if the agent investment function f is too steep, that is if she overreacts to price fluctuations by a too large readjustment of her portfolio position, then the market is unstable, irrespectively of the value of L. Referring again to the EMC plot in Fig. 1, for investment functions based on CWA estimators, equilibria U_1 and U_2 are always unstable. Conversely, both equilibria S_1 and S_2 will become stable for large enough value of L.

Example 3. EWMA forecasting rule

Assume again that agent investment choice $x_{t+1} = f(y_t)$ depends on estimated future price return $y_t = \mathrm{E}[r_{t+1}]$. This time, the forecast is obtained from past realized returns using the following rule

$$y_t = C_L(\lambda)\,\left(r_t + \lambda\, r_{t-1} + \cdots + \lambda^{L-1}\, r_{t-L+1}\right)\ , \tag{21}$$

with the normalization coefficient $C_L(\lambda) = (1-\lambda)/(1-\lambda^L)$. That is, the agent uses an Exponentially Weighted Moving Average (EWMA) of past returns as predictor of future return. The decay factor $\lambda \in [0, 1)$ quantifies the relative weights of recent observations with respect to the older ones. The larger the value of λ, the more weight is assigned to distant observations. When λ is small, the most recent observations have very large relative weights. In the extreme case $\lambda = 0$, only the last available observation is considered.

In the case of a single EWMA-forecaster the stability polynomial reads

$$P_f(\mu) = f'(r^*)\,\frac{1-\lambda}{1-\lambda^L}\,\frac{\mu^L - \lambda^L}{\mu - \lambda}\ . \tag{22}$$

The stability region of the PCE $S_{\lambda,L}$ now depends on both the memory span L and the decay factor λ. Fixing the value of λ, one can immediately obtain analytical results for the two limiting cases: $L = 1$ and $L = \infty$. In the first case, one is back to Example 1. On the other extreme, when $L = \infty$, the following applies

Proposition 3.4. *The fixed point* $\boldsymbol{x}^* = (x^*; r^*)$ *of system* (11) *with* $L = \infty$ *is (locally) asymptotically stable if*

$$\frac{f'(r^*)}{l'(r^*)}\,\frac{1}{r^*} < \frac{1}{1-\lambda}\,,\qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \textit{and} \qquad \frac{f'(r^*)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -\frac{1+\lambda}{1-\lambda}\ . \tag{23}$$

This fixed point exhibits Neimark-Sacker*,* fold *or* flip *bifurcation if the first, second or third inequality in* (23) *turns to equality, respectively.*

Proof. The result can be obtained rigorously through reduction of infinite-dimensional system (11) to the two-dimensional one, using the recursive relation available for the EWMA estimator in case $L = \infty$. See Anufriev, Bottazzi, and Pancotto (2006). In Appendix E we sketch an alternative proof. $\square$

The conditions in (23) can be used to establish the boundaries of the stability region $S_{\lambda,\infty}$. Three examples for different values of λ ($S_{0,\infty}$, $S_{0.2,\infty}$ and $S_{0.6,\infty}$) are depicted on the lower right panel of Fig. 2. For $\lambda = 0$ the stability region $S_{0,\infty}$ is the same as in Example 1, and is shown as a dark gray area. If $\lambda = 0.2$ it expands and becomes the union of the dark and semi-dark gray areas. When $\lambda = 0.6$ the region expands further and contains the light gray areas in addition. As expected, an increase of the decay factor brings stability to the system.

For the intermediate case, when L is greater than one, but finite, analytic results are limited. The case $L = 2$ is analyzed in Appendix E, and the corresponding bifurcation curves are shown, for $\lambda = 0.6$, in the lower left panel of Fig. 2. The stability region $S_{.6,1}$ is shown as a dark gray area. The region $S_{.6,2}$ is the union of the dark and light gray areas. The boundaries for $L = \infty$ are shown as dotted lines.

Differently from the $L = \infty$ case, for equilibria with $r^* > 0$ investment functions with a negative slope generate Neimark-Sacker (and not flip) bifurcations. The boundary leading to the fold bifurcation, on the contrary, remains the same for *any* L and is given by the line $f'/l' = 1$. Finally, the boundary leading to the flip bifurcation depends on whether L is even or odd. For even L, the locus is invariant and coincides with the corresponding boundary for $L = \infty$. When L is odd, the locus depends on L and converges point-wise to the boundary of $S_{\lambda,\infty}$ when $L \to \infty$. This result implies that the expansion of the stability region is not monotone in the memory span.

As Proposition 3.4 guarantees, an increase in the memory span L *ultimately* brings stability to all fixed points belonging to the interior of $S_{\lambda,\infty}$. Since for a given λ this region does not cover the whole parameter space $(r^*, f'/l')$, not all fixed points can be stabilized by a sole increase of L. One can then ask whether unstable fixed points become stable by an appropriate choice of the decay factor λ. The general answer to this question is no, since the condition $f'/l' < 1$

has to be satisfied. However, all fixed points for which this condition holds can be stabilized through an increase of λ. Indeed, from (23) it follows that the region $S_{\lambda,\infty}$ enlarges with λ and for $\lambda \to 1$ contains all equilibria with $f'(r^*)/l'(r^*) < 1$ (c.f. the lower right panel of Fig. 2).

Let us exemplify our findings, referring again to the EMC plot in Fig. 1. For investment functions based on EWMA estimators, equilibria U_1 and U_2 cannot be stabilized neither by increasing the memory span L, nor by increasing the decay factor λ. Conversely, both equilibria S_1 and S_2 will become stable for large enough value of λ and an appropriate L.

4 Economy with Many Agents

This Section extends the previous results to the case of a finite, but arbitrarily large, number of heterogenous agents. Each agent n possesses his own investment function f_n based on a finite number L of past market realizations. We start this section with the derivation of the $2N + L - 1$ dimensional stochastic dynamical system which describes the evolution of the economy. Then we identify all possible equilibria of the associated deterministic skeleton and analyze their stability.

4.1 Dynamical system

In the many agents case, the evolution of agents' wealths is no longer decoupled from the system and, consequently, all equations in (9) are relevant for the dynamics. The first-order dynamical system can be defined in terms of the following $2N + L - 1$ independent variables

$$x_{t,n} \quad \forall n \in \{1, \dots, N\}\,; \qquad \varphi_{t,n} \quad \forall n \in \{1, \dots, N-1\}\,; \qquad r_{t,l} \quad \forall l \in \{0, \dots, L-1\}\,, \tag{24}$$

where $r_{t,l}$ denotes the price return at time $t-l$. Notice that, since they sum to one, only $N-1$ wealth shares are needed and at any time step t it is $\varphi_{t,N} = 1 - \sum_{n=1}^{N-1} \varphi_{t,n}$. The dynamics of the system is provided by the following

Lemma 4.1. *Under Assumptions 1 and 2 the* $2N + L - 1$ *dynamical system defined by* (7) *and* (9) *reads*

$$
\begin{aligned}
&\mathcal{X}: \left[\begin{array}{lcl}
x_{t+1,1} & = & f_1\big(r_{t,0},\dots,r_{t,L-1}\big)\\
\vdots & \vdots & \vdots\\
x_{t+1,N} & = & f_N\big(r_{t,0},\dots,r_{t,L-1}\big)
\end{array}\right.\\
&\mathcal{W}: \left[\begin{array}{lcl}
\varphi_{t+1,1} & = & \Phi_1\Big(x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1};\\
& & \qquad R\big(f_1(r_{t,0},\dots,r_{t,L-1}),\dots,f_N(r_{t,0},\dots,r_{t,L-1});\\
& & \qquad\qquad x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1})\Big)\\
\vdots & \vdots & \vdots\\
\varphi_{t+1,N-1} & = & \Phi_{N-1}\Big(x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1};\\
& & \qquad R\big(f_1(r_{t,0},\dots,r_{t,L-1}),\dots,f_N(r_{t,0},\dots,r_{t,L-1});\\
& & \qquad\qquad x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1})\Big)
\end{array}\right.\\
&\mathcal{R}: \left[\begin{array}{lcl}
r_{t+1,0} & = & R\Big(f_1(r_{t,0},\dots,r_{t,L-1}),\dots,f_N(r_{t,0},\dots,r_{t,L-1});\\
& & \qquad x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1}\Big)\\
r_{t+1,1} & = & r_{t,0}\\
\vdots & \vdots & \vdots\\
r_{t+1,L-1} & = & r_{t,L-2}
\end{array}\right. ,
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
&R\Big(y_1,y_2,\dots,y_N;x_1,x_2,\dots,x_N;\varphi_1,\varphi_2,\dots,\varphi_{N-1};e\Big) =\\
&= \frac{\sum_{n=1}^{N-1}\varphi_n\big(y_n\,(1+e\,x_n)-x_n\big)+\Big(1-\sum_{n=1}^{N-1}\varphi_n\Big)\,\big(y_N\,(1+e\,x_N)-x_N\big)}{\sum_{n=1}^{N-1}\varphi_n\,x_n\,(1-y_n)+(1-\sum_{n=1}^{N-1}\varphi_n)\,x_N\,(1-y_N)}
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
&\Phi_n\big(x_1,x_2,\dots,x_N;\varphi_1,\varphi_2,\dots,\varphi_{N-1};e;R\big) =\\
&\quad = \varphi_n\,\frac{1+x_n\,(R+e)}{1+(R+e)\,\Big(\sum_{m=1}^{N-1}\varphi_m\,x_m+\big(1-\sum_{m=1}^{N-1}\varphi_m\big)\,x_N\Big)} \qquad \forall n\in\{1,\dots,N-1\}\,.
\end{aligned}
\tag{27}
$$

Proof. We ordered the equations to obtain three separated blocks: $\mathcal{X}$, $\mathcal{W}$ and $\mathcal{R}$. The N equations in block $\mathcal{X}$ define the investment choices of agents. Block $\mathcal{W}$ contains $N-1$ equations describing the evolution of the wealth shares. The evolution of price returns is accounted for by the L equations of block $\mathcal{R}$, in ascending order with respect to the time lag.

The block $\mathcal{X}$ is directly obtained from the definition of the investment functions. The first equation of block $\mathcal{R}$ is (7) rewritten in terms of variables (24) using (26) and (5), while the remaining equations are just the result of a "lag" operation. Notice that (26) reduces to (12) in the case of one agent. Finally, the evolution of wealth shares described by block $\mathcal{W}$ is obtained from (9) expanding the notation introduced in (5). Due to the presence of function R in the last expression, all functions Φ_n depend on the same set of variables as R. □

The rest of this Section is devoted to the analysis of the *deterministic skeleton* of (25). We replace the yield realizations $\{e_t\}$ by their mean value $\bar{e}$ and analyze the procedurally consistent equilibria, the is the fixed points, of the resulting deterministic system.

4.2 Determination of equilibria

The characterization of fixed points of system (25) is in many respects similar to the single agent case discussed above. Let $\boldsymbol{x^*} = (x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_{N-1}^*; r^*, \ldots, r^*)$ denotes a fixed point where r^* is the equilibrium return, and x_n^* and φ_n^* stand for the equilibrium value of the investment function and the equilibrium wealth share of agent n, respectively. Let us introduce the following

Definition 4.1. Agent n is said to *survive* in $\boldsymbol{x^*}$ if his equilibrium wealth share is different from zero, $\varphi_n^* \neq 0$. Agent n is said to *dominate* the economy if he is the only survivor, i.e. $\varphi_n^* = 1$.

One can recognize the parallel between our definition above and the frameworks in DeLong, Shleifer, Summers, and Waldmann (1991) and Blume and Easley (1992). We adopt here the deterministic version of the concepts of survival and dominance used in that papers. The following statement characterizes all possible equilibria of system (25).

Proposition 4.1. *Let* $\boldsymbol{x^*}$ *be a PCE of the deterministic skeleton of system* (25). *Then equilibrium investment shares are defined according to*

$$x_n^* = f_n(r^*, \ldots, r^*) \qquad \forall n \in \{1, \ldots, N\} \quad , \tag{28}$$

and three mutually exclusive cases are possible:

(i) ***Single survivor.*** *In* $\boldsymbol{x}^*$ *only one agent survives and, therefore, dominates the economy. Without loss of generality we can assume this agent to be agent* 1 *so that* $\varphi_1^* = 1$ *and all other equilibrium wealth shares are equal to zero.*

The equilibrium return r^* *is determined as the solution of*

$$l(r^*) = f_1(r^*, \dots, r^*) \quad , \tag{29}$$

and is equal to the equilibrium wealth growth rate of the survivor.

(ii) ***Non-generic many survivors.*** *In* $\boldsymbol{x}^*$ *more than one agent survives. Without loss of generality one can assume that the agents with non-zero wealth shares are the first* k *agents (with* $k > 1$*) so that the equilibrium wealth shares satisfy*

$$\varphi_n^* = 0 \quad \forall n > k \qquad \textit{and} \qquad \sum_{n=1}^{k} \varphi_n^* = 1 \quad . \tag{30}$$

The equilibrium return r^* *must simultaneously satisfy the following set of* k *equations*

$$l(r^*) = f_n(r^*, \dots, r^*) \qquad \forall n \in \{1, \dots, k\} \quad , \tag{31}$$

implying that the first k *agents have the same equilibrium investment share* $x^*_{1\diamond k}$*. The wealth growth rates of all survivors are equal to* r^**.*

(iii) ***Generic many survivors.*** *In* $\boldsymbol{x}^*$ *the equilibrium return* $r^* = -\bar{e}$*. The wealth shares of agents satisfy the two following conditions*

$$\sum_{n=1}^{N} x_n^* \varphi_n^* = 0 \qquad \textit{and} \qquad \sum_{n=1}^{N} \varphi_n^* = 1 \quad . \tag{32}$$

The wealth growth rates of all agents are equal to 0*.*

Proof. See appendix F. □

The difference between items *(i)* and *(ii)* is that in the first case, when a single agent survives, Proposition 4.1 defines a precise value for each component (x^*, φ^* and r^*) of the equilibrium $\boldsymbol{x}^*$, so that a single point is uniquely determined. In the second case, on the contrary, there is a residual degree of freedom in the definition of the equilibrium: while the price return r^* and the investment share x^* are uniquely defined, the only requirement on the equilibrium wealth shares of the surviving agents, φ_n^* (for $n \leq k$), is the fulfillment of the second equality in (30). Consequently we have

Corollary 4.1. *Consider the deterministic skeleton of system* (25)*. If it possesses one equilibrium* $\boldsymbol{x}^*$ *with* k *survivors as in Proposition 4.1*(ii)*, it possesses a* $k-1$*-simplex of equilibria with* k*-survivors constituted by all the points obtained from* $\boldsymbol{x}^*$ *through a change in the relative wealths of the survivors. Assuming that the first* k *agents survive as in* (30)*, this set can be written as*

$$\left\{ \left(x_1^*, \ldots, x_N^*; \varphi_1, \ldots, \varphi_k, \underbrace{0, \ldots, 0}_{N-1-k}; \underbrace{r^*, \ldots, r^*}_{L}; \right) \middle| \sum_{j=1}^{k} \varphi_j = 1 \right\}.$$

The difference among the first two cases of Proposition 4.1 does not only regard the geometrical nature of the *locus* of equilibria. Indeed, while in the first case no requirements are imposed on the behavior of the investment function of the different agents, in the second type of equilibria all the investment shares $x_1^*, \ldots, x_k^*$ must *at the same time* be equal to a single value $x_{1\diamond k}^*$. The equilibrium with $k > 1$ survivors exists only in the particular case in which the survivors' investment functions $f_1, \ldots, f_k$ satisfy this constraint. This constraint is not, in general, satisfied by a set of "generic" functions. Indeed if one considers a population of agents whose investment functions are "randomly defined", for instance by picking the values of their parameters from some continuous distribution, the probability of observing any equilibrium with multiple survivors is zero. In other terms, the equilibria with many survivors defined in Proposition 4.1*(ii)* are non-generic.

Multi-agent equilibria defined in Proposition 4.1*(i)* and *(ii)* are both strictly related to "special" single-agent equilibria. As in the single agent case, the growth rate of the total wealth is equal to the equilibrium price return, which is, in turn, equal to the growth rate of survivors' individual wealth. Moreover, the determination of the equilibrium return level r^* for the multi-agent case in (29) or (31) is identical to the case where the agent, or one of the agents, who would survive in the multi-agent equilibrium, is present alone in the market.

Since the relation between equilibrium return level and survivors' investment shares for the many survivors case is equivalent to the single survivor case, one can use the geometrical interpretation of market equilibria presented in Section 3.2 to illustrate how equilibria with many agents are determined. As an example let us consider Fig. 1 again. Suppose that two agents with the investment functions shown in the right panel *simultaneously* operate in the market. According to Proposition 4.1*(i)* all possible equilibria can be found as intersections of one of the functions with the EMC (cf. (29)). In this case there are four possible equilibria. In two of them (S_1 and U_1) agent I survives, so that $\varphi_1^* = 1$. In the other two equilibria (S_2 and U_2) is agent II who survives and $\varphi_1^* = 0$. In each equilibrium, the intersection of the investment function of the surviving agent with the EMC gives both the equilibrium return r^* and the equilibrium investment share of the survivor. The equilibrium investment share of the other agent can be found, accordingly to (28), as the intersection of his own investment function with the vertical line passing through the equilibrium return. Since the two investment functions shown in the left panel of Fig. 1 do not possess common intersections with the EMC, equilibria with more than one survivor are impossible. An example of a set of investment functions which allows for multiple survivors equilibria is reported in the left panel of Fig. 3. The common intersection of different investment functions with the EMC define the manifold of the multiple survivors equilibria.

Let us now turn to the equilibria identified in Proposition 4.1*(iii)*. In these equilibria many agents survive, and their investment and wealth shares are balanced in such a way that the capital gain and the dividend yield offset each other so that the riskless and the risky assets have the same expected return. Indeed according to (15) these equilibria are characterized by a zero equity premium. Since the growth rates of all individual portfolio is the same, agents position on the market does not impact on their survival at equilibrium, so that in general one has N survivors. This is however not a requirement, as equilibria of this type exist even when the wealth shares of a subset of agents go to zero. Differently from the case identified in Proposition 4.1*(ii)*, these are generic equilibria with many survivors. Analogously to that case, however, any zero equity premium equilibrium has additional degrees of freedom corresponding to a change in the relative wealths of the survivors. The following applies

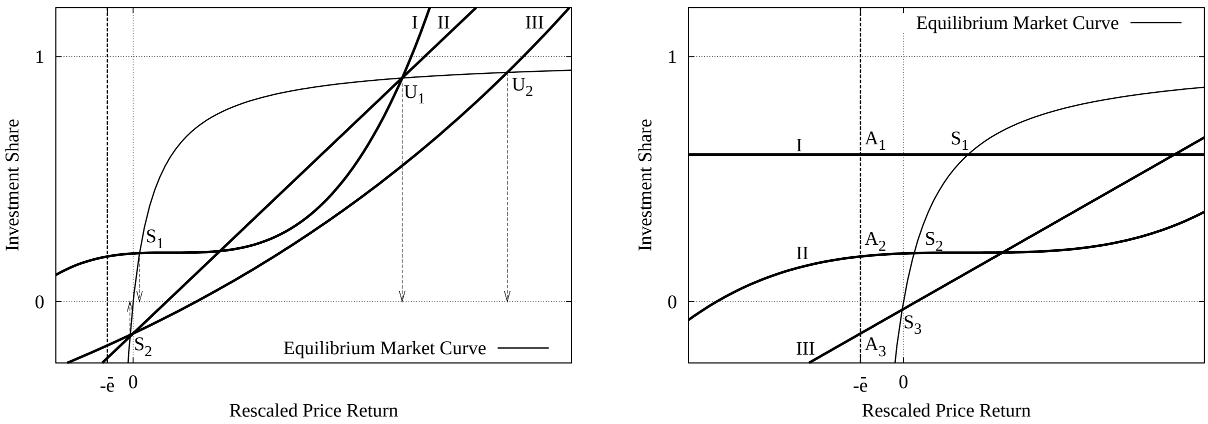

Figure 3: **Left panel**: Multi-agent equilibria. S_2 and U_1 are equilibria with two survivors while s_1 and U_2 are single-survivor equilibria. In S_1 agent I dominates the economy, while in U_1 agents I and II both survive. **Right panel**: No-equity-premium equilibrium with three survivors, with market position A_1, A_2 and A_3, together with three single survivor equilibria, S_1, S_2 and S_3.

Corollary 4.2. *Consider the deterministic skeleton of system* (25)*. If it possesses one equilibrium* $\boldsymbol{x}^*$ *as in Proposition 4.1*(iii)*, it possesses a* $N-2$*-manifold of generic many survivors equilibria constituted by all the points which satisfy conditions* (32)*, namely*

$$\left\{ \left(x_1^*, \ldots, x_N^*; \varphi_1, \ldots, \varphi_{N-1-k}; \underbrace{-\bar{e}, \ldots, -\bar{e}}_{L} \right) \middle| \sum_{j=1}^{N} \varphi_j = 1\,, \sum_{j=1}^{N} \varphi_j x_j^* = 0 \right\}.$$

Geometrically, in the (r, x) plane, the locus of these equilibria is the vertical asymptote of the EMC. Assume the market is populated by three agents having (symmetrized) investment functions as depicted in the right panel of Fig. 3. The ordinates of the points A_1, A_2 and A_3 represents the agents' investment share in the risky asset when (rescaled) price return is equal to $-\bar{e}$, that is when the equity premium is zero. Denote them with $x_1^{-\bar{e}}$, $x_2^{-\bar{e}}$ and $x_3^{-\bar{e}}$ respectively. If there are wealth shares φ_1, φ_2 and φ_3 which, together with the above investment shares, satisfy (32), i.e.

$$\sum_{j=1}^{3} \varphi_j = 1 \quad \text{and} \quad \sum_{j=1}^{3} \varphi_j x_1^{-\bar{e}} = 0\,,$$

then the economy admits equilibria of the "no-equity-premium" type. Notice that since $x_3^{-\bar{e}} < 0$ while $x_{1,2}^{-\bar{e}} > 0$, these equilibria do in fact exists also when all wealth shares are positive.

4.3 Stability of multi-agents equilibria

This Section presents the stability analysis of the equilibria defined in Proposition 4.1. The three Propositions below provide the stability region in the parameter space for the cases enumerated in Proposition 4.1, i.e. for the generic case of a single survivor, for the non-generic case of many survivors and for the generic case with many survivors and without equity premium. The derivation of these Propositions requires quite cumbersome algebraic manipulations and we refer the reader to Appendix G for the intermediate lemmas and final proofs.

For the generic case of a single survivor equilibrium we have the following

Proposition 4.2. *Let $\boldsymbol{x}^*$ be a fixed point of* (25) *associated with a single survivor PCE. Without loss of generality we can assume that the survivor is the first agent. Let $P_{f_1}(\mu)$ denote the $(L-1)$-dimensional stability polynomial associated with the investment function of the survivor. The equilibrium $\boldsymbol{x}^*$ is (locally) asymptotically stable if the two following conditions are met:*

1) *all roots of the polynomial*

$$Q_1(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\,l'(r^*)}\,P_{f_1}(\mu)\;, \tag{33}$$

are inside the unit circle.

2) *the equilibrium investment shares of the non-surviving agents satisfy*

$$-2 - r^* < x_n^*\left(r^* + \bar{e}\right) < r^*\;, \quad 1 < n \leq N\;. \tag{34}$$

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of the polynomial in (33) *is outside the unit circle or if at least one of the inequalities in* (34) *holds with the opposite (strict) sign.*

In particular, the system exhibits a fold bifurcation if one of the $N-1$ right-hand inequalities in (34) *becomes an equality and a flip bifurcation if one of the $N-1$ left-hand inequalities becomes an equality.*

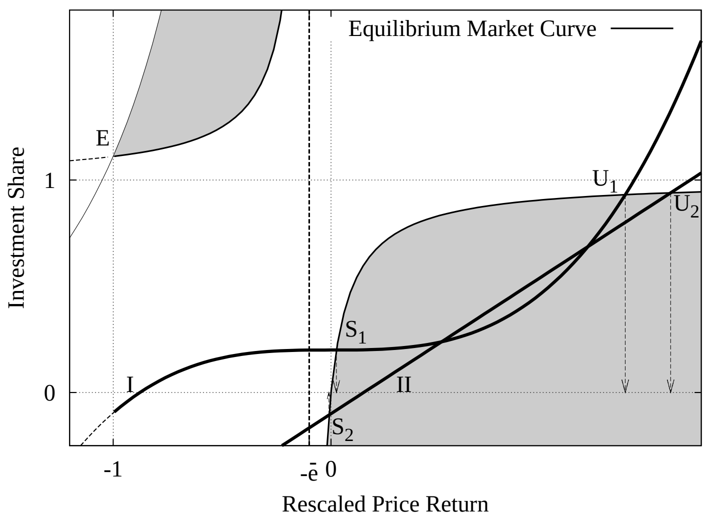

Figure 4: Four equilibria in the market with two agents. The gray region is where condition (34) is satisfied.

The stability condition for a single survivor PCE in the multi-agent case is twofold. First, equilibrium should be “self-consistent”, i.e. it should remain stable even if any non-surviving agent would be removed from the economy. This intuitive result strictly follows from the comparison between $Q_1(\mu)$ in (33) and $Q(\mu)$ in (17). This is however not enough. A further requirement comes from the inequalities in (34). In particular, according to the left-hand inequality, the wealth growth rate of those agents who do not survive in the stable equilibrium should be strictly lower than the wealth growth rate of the survivors r^*. Thus, in those equilibria where $r^* > -\bar{e}$, the surviving agent must be the most aggressive (risk prone) trader and invest in the risky asset the highest, among all traders, share of wealth. Conversely, in those equilibria where $r^* < -\bar{e}$, the survivor has to be the least aggressive (most risk averse) trader.

The EMC plot can be used to geometrically illustrate the previous Proposition. In Fig. 4 we draw again the two investment functions used in the examples of Section 3. Let us now suppose that they are both present in the market at the same time. The region where the additional condition (34) is satisfied is reported in gray. In Section 4.2 we found four possible equilibria: S_1, S_2, U_1 and U_2. Proposition 4.2 states that, first, the dynamics cannot be attracted by an equilibrium which was unstable in the respective single-agent case. And, second, it cannot be attracted by an equilibrium in which non-surviving agents investment shares at equilibrium belong to the white region. As we have seen in Section 4.2, if an agent uses EWMA or CWA forecast the points U_1 and U_2 will be unstable (cf. examples 2 and 3 in Section 3.3). Therefore, they will be unstable also in the multi-agents case. From item **2)** of Proposition 4.2 it follows that S_1 is the only stable equilibrium of the two agents system. Notice, indeed, that in the abscissa of S_1, i.e. for the equilibrium return, the linear investment function of the non-surviving agent II passes below the investment function of the surviving agent and belongs to the gray area. On the contrary, in the abscissa of S_2, the investment function of the non-surviving agent I is higher and does not belong to the gray area. Consequently, this equilibrium is unstable. A similar situation occurs in the right panel of Fig. 3. Among the three single survivor equilibria S_1, S_2 and S_3, only the first is stable.

Let us move now to consider the non-generic case, when k different agents survive in the equilibrium. The following applies

Proposition 4.3. *Let $\boldsymbol{x}^*$ be a fixed point of* (25) *associated with a k survivors PCE, as defined by* (28)*,* (30) *and* (31)*.*

The fixed point $\boldsymbol{x}^*$ is never hyperbolic and, consequently, never (locally) asymptotically stable. Its non-hyperbolic submanifold is the $k-1$-dimensional manifold defined in Corollary 4.1.

Let $P_{f_n}(\mu)$ be the stability polynomial associated with the investment function f_n. The equilibrium $\boldsymbol{x}^*$ is (locally) stable if the two following conditions are met:

1) *all the roots of the polynomial*

$$Q_{1\diamond k}(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\, l'(r^*)} \sum_{n=1}^{k} \varphi_n^*\, P_{f_n}(\mu) \quad , \tag{35}$$

are inside the unit circle.

2) *the equilibrium investment shares of the non-surviving agents satisfy*

$$-2 - r^* < x_n^*\,(r^* + \bar{e}) < r^* \,, \quad k < n \leq N \,. \tag{36}$$

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of the polynomial in (35) *is outside the unit circle or if at least one of the inequalities in* (36) *holds with the opposite (strict) sign.*

The non-hyperbolic nature of the equilibria with many survivors turns out to be a direct consequence of their non-unique specifications. The motion of the system along the $k-1$ dimensional subspace consisting of the continuum of equilibria defined in Corollary 4.1 leaves the aggregate properties of the system invariant so that all these equilibria can be considered equivalent. Proposition 4.3 also provides the stability conditions for perturbations in the hyperplane orthogonal to the non-hyperbolic manifold formed by equivalent equilibria. The polynomial $Q_{1\diamond k}(\mu)$ is quite similar to the corresponding polynomial in Proposition 4.2, except that one has to weight the stability polynomial of the different investment functions $P_{f_k}(\mu)$ with the weights corresponding to the relative wealth of survivors. At the same time, the constraint on the investment shares (36) is identical to the one obtained in (34). Similarly to the case with a single survivor, in those equilibria where $r^* > -\bar{e}$ all surviving agents must be more aggressive buyer of risky asset than those who do not survive, and *vice versa* for $r^* < -\bar{e}$.

Finally, let us analyze the generic many survivors equilibria with $r^* = -\bar{e}$. Without loss of generality, we assume that the survivors are the first $k \leq N$ agents. The following result characterizes the stability of such equilibria

Proposition 4.4. *Let $\boldsymbol{x}^*$ be a fixed point of system* (25) *belonging to a $N-2$-dimensional manifold of k-survivors equilibria defined by* (28) *and* (32).

If $N \geq 3$, the fixed point $\boldsymbol{x}^*$ is non-hyperbolic and, consequently, is not (locally) asymptotically stable. The equilibrium $\boldsymbol{x}^*$ is (locally) stable if all the roots of the following polynomial are inside the unit circle

$$\mu^{L+1} + \frac{\mu - 1}{\langle x^2 \rangle} \sum_{j=1}^{k} \varphi_j^* \, P_{f_j}(\mu) \quad , \tag{37}$$

where $P_{f_j}(\mu)$ is the stability polynomial of the investment function f_j computed in $(-\bar{e}, \ldots, -\bar{e})$, and $\langle x^2 \rangle = \sum_{n=1}^{k} \varphi_n^* \, {x_n^*}^2$.

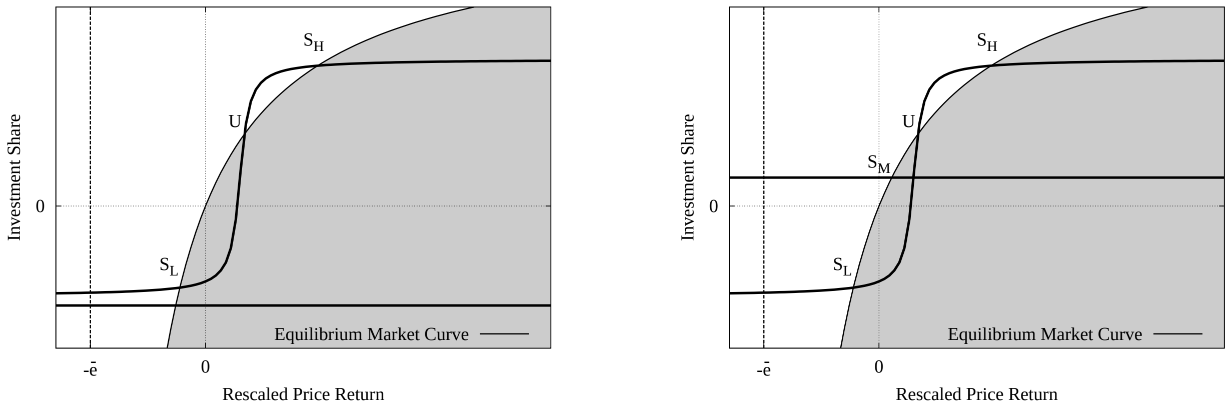

Figure 5: **Left Panel**: Non-linear investment function leading to multiple equilibria. S_H and S_L are stable while U is unstable. A second linear investment function always lays below the first. **Right Panel**: The linear investment function is raised above S_L. As a result, market now possesses a new stable equilibria, S_M, while S_L looses its stability. In the stable equilibria S_H and S_M two different agents survive.

The equilibrium $\boldsymbol{x}^*$ *is unstable if at least one of the roots of polynomial in* (37) *is outside the unit circle.*

As in the case of Proposition 4.3, the "no-equity-premium" equilibria can be non-hyperbolic, due to the possibility to reallocate wealth between agents without affecting the aggregate properties of the dynamics. Once the functional form of the investment functions is known, one can derive the expression of the polynomial Q and obtain the appropriate stability conditions from the analysis of its roots, analogously to what has been done in the examples of Section 3.3. In particular, when all the investment functions possess a zero slope in $-\bar{e}$, the equilibria (if they exist) are always stable. Consider the right panel of Fig. 3. Since the investment functions of agent I and II are flat in A_1 and A_2 respectively, if the wealth share of agent III is small enough, then the no-equity-premium equilibrium is stable.

4.4 Optimal selection and multiple stable equilibria

In this Section, using the geometric interpretation based on the EMC, we discuss the implications of Proposition 4.2 about the asymptotic behavior of the model and its global properties.

For definiteness, we will mainly confine the discussion to the generic case of equilibria with a single survivor, but many aspects of it do apply in general.

The first implication concerns the aggregate dynamics of the economy. Let us consider a stable many-agent equilibrium $\boldsymbol{x}^*$. Let us suppose that r^* is the equilibrium return in $\boldsymbol{x}^*$ and that the first agent survives. Then his wealth return is equal to r^* and this is also the asymptotic growth rate of the *total* wealth. Then, we can interpret the second requirement of Proposition 4.2 as saying that, in the dynamic resulting from the competition of different trading strategies, the surviving agent is the one who allows the economy to have the highest possible rate of growth. Indeed, if any other agent $n \neq 1$ survived, the economy would have grown at a rate $x_n^* \, (r^* + \bar{e})$, which is lower than r^* according to (34). This result constitutes a sort of *optimal selection principle* since it clearly states the market endogenous selection toward the best aggregate outcome. Indeed, the market selection mechanism seems to act in accordance with the asymptotic behavior of the system. In equilibria with $r^* > -\bar{e}$ the aggregate wealth is growing. In this case, and according to the optimal selection principle, the surviving agent is the most aggressive investor, so as to guarantee the highest possible growth rate to the whole economy. Conversely, in equilibria with $r^* < -\bar{e}$, when the wealth of the economy decreases, the survivor is the least aggressive traders, so that the economy shrinks at the lowest speed.

Notice, however, that this selection criterion does not apply across the whole set of equilibria, but only inside the subset formed by equilibria associated with stable fixed points in the single agent case (c.f. (35)). For instance, with the investment functions shown in Fig. 4, the dynamics will never end up in U_2, even if this is the equilibrium with the highest possible return. Furthermore, the variety of possible investment functions implies that the optimal selection principle has a local character. Indeed, due to the "lock-in" effect generated by the boundedness of attraction domains, it is possible that the market does not select, among all *stable* equilibria, the one which provides the highest returns. An illustrative example is provided by the "step-wise" investment function shown in the left panel of Fig. 5. This function possesses two stable equilibria, S_L and S_H. Now suppose that an agent using this function competes on the market with a second agent, more risk averse, who always invest smaller shares of wealth in the risky asset. For instance assume that the second agent uses the constant investment function shown in the same plot. In this situation, the two equilibria of the nonlinear function remain stable and the riskier agent will ultimately dominate the market. However, initial conditions completely determine which one of the two equilibria will be eventually selected by the market.

Staring from the situation in the left panel of Fig. 5, imagine to shift the constant function upward, so as to cross the step-wise function between S_L and S_H. The resulting situation is represented in the right panel of Fig. 5. This change of strategy by the second agent will affect the economy in different ways, depending on what was the prevailing equilibrium before the shift. If the system was in S_H, no effect is observed and the dominant agent remains the one with the sigmoid investment function. If instead the economy was in S_L, this equilibrium becomes unstable and the system will tend to move away from it. The ensuing dynamic could ultimately leads to the dominance of the second agent, so that the system ends up in the equilibrium S_M. This simple example shows that the possibility for a strategy to become (or remain) dominant does not depend exclusively on the strategy itself, but also, and in an essential way, on the conditions prevailing in the market when the strategy is actually implemented and on the global behavior of all other strategies. This result does not only apply to single survivor equilibria. Indeed another possible source of multiple equilibria is given by the existence of generic many survivors fixed points identified in Proposition 4.1*(iii)*. For example, in the situation depicted in the right panel of Fig. 3, there is a stable equilibrium where agent I survives alone (point S_1) and a manifold of generic equilibria where all three agents survive. In the latter case, if the wealth share of agent I is large enough, according to Proposition 4.4, all these equilibria are stable and, depending on the initial condition, the market can display both single survivor or many survivors equilibria.

5 Conclusion

This paper introduces novel results concerning the characterization and stability of equilibria in speculative pure exchange economies with heterogeneous CRRA traders. The framework is relatively general in terms of agents' behaviors and differs from most of the Heterogeneous Agents Models in two important respects. First, we analyze the aggregate dynamics and asymptotic behavior of the market when an *arbitrary large number* of traders participate to the trading activity. Second, we do not restrict in any way the procedure used by agents in order to build their forecast about future prices, nor the way in which agents can use this forecast to obtain their present asset demand: any smooth investment function mapping the information set to the present investment choice can be present in the model.

We confine our analysis to those equilibria in which aggregate market dynamics is consistent with the procedures adopted by agents to form expectations about future price movement. We derive a complete characterization of Procedurally Consistent Equilibria (PCE) and a description of their stability conditions in terms of few parameters derived from traders investment strategies. In particular, we show that a simple function, the "Equilibrium Market Curve", can be used to obtain a geometric characterization of the location of all possible equilibria. Furthermore, some results about stability conditions can also be inferred from the same EMC.

We find that, irrespectively of the number of agents operating in the market and of the structure of their demand functions, only three types of PCE are possible:

1. isolated fixed points, where a single agent asymptotically possesses the entire wealth of the economy,
2. non-generic equilibria, associated with continuous manifolds of fixed points, where many agents possess a finite shares of the total wealth,
3. generic equilibria with many survivors, consistent with no-arbitrage pricing.

The notion of PCE allows us to join together the EF and HAMs approaches. Indeed the stability conditions for the first two types of PCE (see Propositions 4.2 and 4.3) incorporates two parts. The first part shows that the stability of market dynamics depends on agents' reactions to past prices, as common to HAMs. This result relates to Grandmont (1998) who puts forward the general theory of individual learning under expectation feedback in *homogeneous* environments. The second part highlights the role played by the wealth-driven selection, and is essentially the requirement of local evolutionary stability common among the EF models. Notice that this part is not present in the stability conditions of the PCE of the third type (see Proposition 4.4), where the no-arbitrage condition makes the wealth selection dynamics irrelevant.

Our general results provide a simple and clear description of the principles governing the asymptotic market dynamics resulting from the competition of different trading strategies. The optimizing agents may dominate non-optimizing agents but may also be dominated by them. In general, the ultimate result of competition between agents depends on the whole market ecology. As we have shown, the EMC is a handy tool to discuss such phenomena as absence of equilibrium, presence of multiple equilibria, and also for comparative statics exercises. From the "geometric" analysis made possible by the EMC (c.f. the stability analysis in Section 4.3) the following two "impossibility theorems" follow in an obvious way. First, there exists no "best" strategy, independently of what "best" means exactly, since any possible market equilibrium can be destabilized by some investment function. Second, it is impossible to build a dominance order relation inside the space of trading strategies, since two strategies may generate multiple stable equilibria with *different* survivors, so that the outcome will depend on system initial conditions or on noise.

The main advantage of the present model rests in its ability to provide a relatively simple framework in which the interplay between agents' decision processes and market forces can be studied with reasonable generality. In his famous argument in favor of a rational approach in economics, Friedman (1953) appeals to the evolutionary ideas of natural selection to justify optimization. Friedman's argument states that since market forces tend to weed out non-rational agents, the modeler can assume that the agents behave *as if* they are maximizers. A number of recent studies has shown that Friedman's argument is not generally valid (e.g. DeLong, Shleifer, Summers, and Waldmann (1990b), Blume and Easley (1992), Brock and Hommes (1998)). Our results support this view showing that the survivors in the market are determined not only by their own strategies but also by the complete behavioral ecology. Alternatively, one can consider the optimizing agents' behavior as an outcome of the learning process. In this case, however, due to possible presence of multiple equilibria or instability, there is no reason to assume *a priori* that such learning process will converge to the corresponding fixed point.

The present analysis can be extended in many directions. First of all, one may raise the question of the robustness of the results with respect to Assumption 1 about constant dividend yield. Model in Anufriev and Dindo (2009) investigates the case of growing dividend yield and shows that some results (as presence of equity premium in equilibria and possibility to represent the equilibria by the EMC) remain essentially the same. Second, in the limits of our framework, one can wonder about other possible dynamics. For instance, we have shown that there is a theoretical possibility of not having any equilibrium at all. The dynamics in this case remain unknown. We also did not investigate the dynamics after bifurcation, which is the key question in many heterogeneous agent models. Probably numerical methods can be effectively applied to study these questions and also clarify the role of initial conditions and the determinants of the relative size of the basins of attraction for multiple equilibria scenarios. Third, our general CRRA-framework led us, in Proposition 2.1, to write the system in terms of returns and wealth shares. There are many behavioral specifications which were not analyzed here and are still consistent with our framework. These specifications range from the evaluation of the "fundamental" value of the asset, possibly obtained from a private source of information, to a strategic behavior that tries to keep in consideration the reaction of other market participants to the revealed individual choices.

APPENDIX

A Proof of Proposition 2.1

Plugging the expression for $w_{t+1,n}$ from the second equation in system (4) into the right-hand side of the first equation of the same system, and assuming that $p_t > 0$ and, consistently with (6), $p_t \neq \sum x_{t+1,n}\, x_{t,n}\, w_{t,n}$, one gets

$$p_{t+1} = \left(1 - \frac{1}{p_t}\sum_{n=1}^{N} x_{t+1,n}\, x_{t,n}\, w_{t,n}\right)^{-1} \left(\sum_{n=1}^{N} x_{t+1,n}\, w_{t,n} + \left(e_{t+1} - 1\right)\sum_{n=1}^{N} x_{t+1,n}\, w_{t,n}\, x_{t,n}\right) =$$

$$= p_t\, \frac{\sum_n x_{t+1,n}\, w_{t,n} + (e_{t+1} - 1)\sum_n x_{t+1,n}\, w_{t,n}\, x_{t,n}}{\sum_n x_{t,n}\, w_{t,n} - \sum_n x_{t+1,n}\, x_{t,n}\, w_{t,n}} =$$

$$= p_t\, \frac{\left\langle x_{t+1}\right\rangle_t - \left\langle x_t\, x_{t+1}\right\rangle_t + e_{t+1}\left\langle x_t\, x_{t+1}\right\rangle_t}{\left\langle x_t\right\rangle_t - \left\langle x_t\, x_{t+1}\right\rangle_t} \quad ,$$

where we used the first equation of (4) rewritten for time t to get the second equality. Condition (6) is obtained imposing $p_{t+1} > 0$, and the dynamics of price return in (7) are immediately derived. From the second equation of (4) it follows that

$$w_{t+1,n} = w_{t,n}\left(1 + x_{t,n}\left(r_{t+1} + e_{t+1}\right)\right) \qquad \forall n \in \{1, \ldots, N\} \quad , \tag{38}$$

leading to (8). To obtain the wealth share dynamics, divide both sides of (38) by w_{t+1} to have

$$\varphi_{t+1,n} = \frac{w_{t,n}}{w_{t+1}}\left(1 + x_{t,n}\left(r_{t+1} + e_{t+1}\right)\right) =$$

$$= \frac{w_{t,n}}{\sum_m w_{t,m} + (r_{t+1} + e_{t+1})\sum_m x_{t,m} w_{t,m}}\left(1 + x_{t,n}\left(r_{t+1} + e_{t+1}\right)\right) =$$

$$= \frac{\varphi_{t,n}}{1 + (r_{t+1} + e_{t+1})\sum_m x_{t,m}\varphi_{t,m}}\left(1 + x_{t,n}\left(r_{t+1} + e_{t+1}\right)\right) \quad ,$$

where (38) has been used to get the second line and we divided both numerator and denominator by the total wealth at time t to get the third.

B Example of agents behavior

Suppressing any agent-specific index, we start with the evolution of the agent's (rescaled) wealth given by (4)

$$w_{t+1} = w_t + x_t w_t\left(r_{t+1} + e_{t+1}\right).$$

The future wealth depends on the total return $z_{t+1} = r_{t+1} + e_{t+1}$ at time t, which is unknown . The decision to invest a fraction of wealth x_t yields the portfolio return

$$\rho_{t+1} = w_{t+1}/w_t - 1 = x_t\left(r_{t+1} + e_{t+1}\right).$$

Assume that agent's utility depends on his portfolio return and has a mean-variance expression with a constant risk aversion coefficient $\beta > 0$. The problem

$$\max_{x_t} \left\{ \mathrm{E}_{t-1}[\rho_{t+1}] - \frac{\beta}{2}\, \mathrm{V}_{t-1}[\rho_{t+1}] \right\} \tag{39}$$

requires that an agent forms expectations $\mathrm{E}_{t-1}[z_{t+1}]$ and $\mathrm{V}_{t-1}[z_{t+1}]$ about the first two moments of future total return on the basis of the information set $\mathfrak{I}_{t-1}$ available at the beginning of period t. Solving the optimization problem (39) one gets

$$x_t^* = \frac{1}{\beta}\, \frac{\mathrm{E}_{t-1}[z_{t+1}]}{\mathrm{V}_{t-1}[z_{t+1}]} \,. \tag{40}$$

Referring to Section 2.3 the right hand-side represents a family of choice' functions

$$h_\beta(\theta_1, \theta_2) = \frac{1}{\beta}\, \frac{\theta_1}{\theta_2} \,,$$

parameterized by the risk aversion coefficient β. These functions depends on two statistics of the future returns distribution, $\theta_1 = g_1(z_{t+1}) = \mathrm{E}_{t-1}[z_{t+1}]$ and $\theta_2 = g_2(z_{t+1}) = \mathrm{V}_{t-1}[z_{t+1}]$. The behavior of the agent is ultimately determined by the expectation functions g_1 and g_2 used to obtain these statistics.

One possible direction, inspired by the HAMs literature, is to distinguish different types of traders based on stylized behavior. For example, suppose that "fundamentalists" do not take past observations into account, but expect an exogenous risk premium (i.e., excess return) $g_1^F \equiv \delta$ and no excess volatility, i.e., $g_2^F \equiv \sigma_e^2$, where σ_e^2 is the variance of the yield process. As a result of substitution of these expectation functions in the choice function (40), we obtain the investment function of fundamentalists

$$h_\beta^F = \frac{\delta}{\beta\, \sigma_e^2} \,,$$

which is constant. Conversely, suppose that "chartists" try to learn the level of return from the past observations, so that $g_1^C(\mathfrak{I}_{t-1}) = a + b z_{t-1}$, while they still have constant estimation for the variance, $g_2^C \equiv \sigma_e^2$. As a result, we obtain for the chartists the investment function

$$h_\beta^C = \frac{a + b(r_{t-1} + e_{t-1})}{\beta\, \sigma_e^2} \,,$$

which is a linear function of the previous total return. Within our framework one can study the dynamics of a market populated by these two types of traders and address such questions as the role of risk-aversion β, risk premium δ, or extrapolation coefficient b for the equilibrium and stability of the system. The reader is referred to Anufriev (2008) for a study of these questions in the case of arbitrary linear investment functions. The choice function (40) with special linear predictors was previously considered in Chiarella and He (2001).

The discussion above represents but a simple example of an agent investment function. Many generalization are possible and have been explored in the past. Instead of a mean-variance utility the agent could maximize the expected utility (EU) of future wealth. Several functional specifications are suggested in the literature, most notably the *power* utility function for which, however, closed form solutions are in general not available. Or one can replace the expectation operator with more sophisticated alternative, like the *maximin* criterion or the *minimax-regret* criterion discussed in Brock and Manski (2008). Both these criteria will give choice functions which depend on a small set of statistics which are in turn specified through the definition of appropriate expectation functions. Concerning the latter, one can adopt a classical Bayesian approach or, more common in the HAMs literature, use adaptive learning estimators derived from statistical analysis, like moving windows estimators of central tendency. Different learning rules of different degree of sophistication have been suggested in the literature. See Colucci and Valori (2006) for an overview. Since all of them can be represented as a function of past data, they all satisfy to the requirements of our framework. A typical question in the adaptive learning literature is whether multiple equilibria are possible and whether they can be approached through the learning mechanism. One can easily study these questions within our framework, see Section 3.4. What is more, one can do it in the presence of heterogeneous agents' behavior.

C Proof of Proposition 3.1

Plugging the equilibrium values of the variables in the first equation of (11), one gets $x^* = f(r^*, \ldots, r^*)$. Now using $R(x^*, x^*, e) = e\,x^*/(1 - x^*)$ one can invert the second equation to get (14). Item *(ii)* follows directly from condition (6) written at equilibrium. Finally, from (8) using the previous relations one has $\rho^* = x^*(r^* + \bar{e}) = l(r^*)(r^* + \bar{e}) = r^*$.

D Proof of Proposition 3.2

We start with a simple result which will be useful in what follows.

Lemma D.1.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_n \\ 1 & -\mu & 0 & \ldots & 0 & 0 \\ 0 & 1 & -\mu & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\mu & 0 \\ 0 & 0 & 0 & \ldots & 1 & -\mu \end{vmatrix} = (-1)^{n+1} \sum_{k=1}^{n} x_k\, \mu^{n-k} \quad , \qquad (41)$$

Proof. Consider this determinant as a sum of elements from the first row multiplied on the corresponding minor. The minor of element x_k, whose corresponding sign is $(-1)^{k+1}$, is a block-diagonal matrix consisting of two blocks. The upper-left block is an upper-diagonal matrix with 1's on the diagonal. The lower-right block is a lower-diagonal matrix with $-\mu$'s on the diagonal. The determinant of this minor is equal to $(-\mu)^{n-k}$ and the relation above immediately follows. $\square$

The $(L+1) \times (L+1)$ Jacobian matrix $\boldsymbol{J}$ of system (11) reads

$$\left\| \begin{array}{ccccccc} 0 & \frac{\partial f}{\partial r_0} & \frac{\partial f}{\partial r_1} & \frac{\partial f}{\partial r_2} & \cdots & \frac{\partial f}{\partial r_{L-2}} & \frac{\partial f}{\partial r_{L-1}} \\ R^x & R^f \frac{\partial f}{\partial r_0} & R^f \frac{\partial f}{\partial r_1} & R^f \frac{\partial f}{\partial r_2} & \cdots & R^f \frac{\partial f}{\partial r_{L-2}} & R^f \frac{\partial f}{\partial r_{L-1}} \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{array} \right\| , \tag{42}$$

where according to (12)

$$R^x = \frac{\partial R(x^*, x^*)}{\partial x} = -\frac{1}{x^*\,(1-x^*)} \qquad R^f = \frac{\partial R(x^*, x^*)}{\partial x'} = \frac{1+r^*}{x^*\,(1-x^*)} \ . \tag{43}$$

Let $\boldsymbol{I}$ be the $(L+1) \times (L+1)$ identity matrix. Expanding the determinant of $\boldsymbol{J} - \mu\,\boldsymbol{I}$ by the elements of the first column and using Lemma D.1 one has

$$\begin{aligned} \det\big(\boldsymbol{J} - \mu\,\boldsymbol{I}\big) = (-\mu)\,(-1)^{L-1} \left(\left(R^f\,\frac{\partial f}{\partial r_0} - \mu \right) \mu^{L-1} + R^f\,\frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + R^f\,\frac{\partial f}{\partial r_{L-1}} \right) - \\ - R^x\,(-1)^{L-1} \left(\frac{\partial f}{\partial r_0}\,\mu^{L-1} + \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + \frac{\partial f}{\partial r_{L-2}}\,\mu + \frac{\partial f}{\partial r_{L-1}} \right) = \\ = (-1)^{L-1} \left(\mu^{L+1} - \left(\mu R^f + R^x \right) \sum_{k=0}^{L-1} \frac{\partial f}{\partial r_k}\,\mu^{L-1-k} \right) \end{aligned} ,$$

which, using (43) and the definition of stability polynomial in (16) reduces to

$$P_J(\mu) = (-1)^{L-1} \left(\mu^{L+1} - \frac{(1+r^*)\mu - 1}{x^*(1-x^*)} P_f(\mu) \right) \ . \tag{44}$$

Remembering that $l'(r^*) = x^*(1-x^*)/r^*$ it is immediate to see that (44) is proportional to (17).

E Analysis of single agent examples

In what follows we will make use of few basic results of bifurcation analysis in low dimensional discrete dynamical systems. We briefly summarize them here to help the unacquainted reader. For the extensive theory of the stability and bifurcation analysis of the autonomous dynamical systems see e.g. Guckenheimer and Holmes (1983), Medio and Lines (2001) and Kuznetsov (2004).

A fixed point of a discrete dynamical system is asymptotically stable if all the eigenvalues of the Jacobian matrix computed in the fixed point are inside the complex unit circle. For two-dimensional dynamical systems this amount to

Proposition E.1 (Sufficient conditions for the local stability)**.** *The fixed point of a two-dimensional discrete dynamical system is locally asymptotically stable if the following conditions are satisfied*

$$d < 1 \quad , \qquad t < 1 + d \quad , \qquad t > -1 - d \quad . \tag{45}$$

where t and d are, respectively, the trace and the determinant of the Jacobian matrix computed in the fixed point.

The fixed point is unstable if at least one of the eigenvalues lies outside the unit circle. The situation in which the change of one or more parameters of the system leads to the cross of the unit circle by one (or two complex conjugated) eigenvalues is called a *bifurcation.* Three types of generic bifurcations are usually considered, depending on where the unit circle is crossed. A bifurcation is called *fold*, *flip* or *Neimark-Sacker*, if an eigenvalues crosses the unit circle in 1, -1 or outside the real axis, respectively.For two-dimensional dynamical system this amount to

Proposition E.2. *The fixed point of a two-dimensional discrete dynamical system looses its stability when one of the inequalities in conditions* (45) *is changing its sign. The system exhibits a Neimark-Sacker bifurcation if $d = 1$, a fold bifurcation if $t = 1 + d$, and a flip bifurcation if $t = -1 - d$.*

E.1 Stability for the system with short memory forecast

In this case the second degree polynomial associated with the constant stability polynomial reads

$$Q(\mu) = \mu^2 - \frac{f'(r^*)}{l'(r^*)\, r^*} \Big(\left(1 + r^*\right) \mu - 1 \Big) \,,$$

and Proposition 3.3 follows straight-forwardly from the application of the two Propositions above.

E.2 Stability for the system with Constant Weighted forecast

The characteristic polynomial for this case can be written as follows:

$$Q(\mu) = \mu^{L-1} \left(\mu^2 - \frac{1}{L} \frac{1 - (1/\mu)^L}{1 - 1/\mu} \frac{Y}{X} \Big((1 + X)\mu - 1 \Big) \right) \,, \tag{46}$$

where, as before, variables $X = r^*$ and $Y = f'(r^*)/l'(r^*)$ correspond to the abscissa and ordinate of the stability region diagrams in Fig. 2.

Bifurcations Loci

Unfortunately, for polynomials of degree higher than 2, conditions analogous to (45) are unavailable. One can, however, characterize the *loci* of different types of bifurcations. By locus of, say, the flip bifurcation we mean the set of those pairs (X, Y) under which one of the roots of (49) is equal to -1. Notice that, in general, not any point on these loci corresponds to the associated bifurcation, since a bifurcation happens only if all other roots are inside the unit circle. With this remark in mind, we formulate

Lemma E.1. *Consider polynomial* (46) *for arbitrary L. The locus of* fold *bifurcations is defined by $Y = 1$. The locus of* flip *bifurcations does not exist for even L and is provided by the following expression for odd L:*

$$Y = -\frac{L\,X}{2+X} \quad .$$

Proof. The computations are pretty straight-forward: one has to let $\mu = 1$ and $\mu = -1$ in (46). □

The locus of *Neimark-Sacker* bifurcation is found by plugging $\mu = e^{i\psi}$, where ψ is arbitrary angle and i is the imaginary unit, into equation $Q(\mu) = 0$. For high L the corresponding curve may have high dimension, so we confine ourselves to the case $L = 2$.

Lemma E.2. *Consider polynomial* (46) *for $L = 2$. The locus of Neimark-Sacker bifurcations is given by*

$$Y^2\,(2+X) - 2\,X^2 Y - 4\,X^2 = 0 \quad , \tag{47}$$

subject to constraint $16\,X^2 - (2+X)^2 Y^2 > 0$.

Proof. One has to solve the equation

$$2Xe^{3i\psi} - (1+X)Ye^{2i\psi} + XYe^{i\psi} + Y = 0 \quad . \tag{48}$$

Using Euler formula and setting the real and imaginary parts of the right-hand side to zero, one gets a system of two equations. The equation for the imaginary part implies

$$\sin\psi\Big(2X(3-4\sin^2\psi) - 2(1+X)Y\cos\psi + Y - (1+X)Y\Big) = 0 \quad .$$

Since we are interested in pure complex roots of the characteristic polynomial we assume that $\sin\psi \neq 0$. Thus, the expression in the big parenthesis is equal to 0. Then, we substitute the resulting expression for $3 - 4\sin^2\psi$ in the equation of the real parts of (48) and, using relation $\cos 3\psi = \cos\psi(1 - 4\sin^2\psi)$, obtain

$$\cos\psi = \frac{Y\,(1+X) + Y}{4\,X} \quad .$$

Plugging this expression into the equations for the real and imaginary parts of (48), we get the following system

$$\begin{aligned}(2+X)\,Y\,\Big(-2Y^2 + 2\,(2+Y)X^2 - X\,Y^2\Big) &= 0\\ \sqrt{16\,X^2 - (2+X)^2\,Y^2}\,\Big(-2Y^2 + 2\,(2+Y)\,X^2 - X\,Y^2\Big) &= 0\,.\end{aligned}$$

In any solution of this system, the common expression in the parenthesis must be equal to zero, which gives (47). Additional constraint guarantees that the squared root in the second equation is real. $\square$

In the upper right panel of Fig. 2 the curve (47) is reported together with the loci derived in Lemma E.1.

Limiting case, $L \to \infty$

The behavior of the locus of flip bifurcations when the memory parameter increases may suggest that any point of the system for which $f'/l' < 1$ can be stabilized for large enough value of L. Indeed, let us consider the region outside of the unit circle (including the circle itself), fix $\mu = \mu_0$ and let $L \to \infty$. Since $|\mu_0| \geq 1$, it is $1/\mu_0^L \to 0$ and (46) reduces to

$$\frac{\mu_0^L}{\mu_0 - 1}\left(\mu_0^2 - \mu_0(1 + o(\frac{1}{L})\right)$$

which does not admit zeros outside the unit circle irrespectively of the value of X and Y.

E.3 Stability for the system with EWMA forecast

Using stability polynomial (22), the characteristic polynomial for this case can be written as follows:

$$Q(\mu) = \mu^{L-1}\left(\mu^2 - \frac{1-\lambda}{1-\lambda^L}\frac{1-(\lambda/\mu)^L}{1-\lambda/\mu}\frac{Y}{X}\Big((1+X)\mu - 1\Big)\right) \,, \tag{49}$$

where $X = r^*$ and $Y = f'(r^*)/l'(r^*)$. We proceed through the same steps as in the previous case.

Bifurcations Loci

First we characterize the *loci* of fold and flip bifurcations.

Lemma E.3. *Consider polynomial* (49) *for arbitrary* λ *and* L. *The locus of* fold *bifurcations is given by* $Y = 1$, *while the locus of* flip *bifurcations is provided as follows:*

$$1 + Y\frac{2+X}{X}\frac{1-\lambda}{1+\lambda}\frac{1+\lambda^L}{1-\lambda^L} = 0 \quad \text{for } L \text{ odd} \qquad \text{and} \qquad 1 + Y\frac{2+X}{X}\frac{1-\lambda}{1+\lambda} = 0 \quad \text{for } L \text{ even}.$$

Proof. The direct substitution of $\mu = 1$ and $\mu = -1$ to (49) leads to the conclusion. $\square$

For the locus of *Neimark-Sacker* bifurcations we consider the case $L = 2$ and prove the following

Lemma E.4. *Consider polynomial* (49) *for* $L = 2$ *and arbitrary* λ. *The locus of Neimark-Sacker bifurcations is given by the following curve of the second order*

$$Y^2\,\lambda\,\big(X + 1 + \lambda\big) + Y\,X\,(1+\lambda)\,\big(1 - \lambda - \lambda X\big) - X^2\,(1+\lambda)^2 = 0 \quad , \tag{50}$$

subject to constraint $4(1+\lambda^2)X^2 - (1+\lambda+X)^2Y^2 > 0$.

Proof. The proof is completely analogous to that of Lemma E.2. $\square$

The curve (50) is depicted in the lower left panel of Fig. 2 together with the loci derived in Lemma E.3. The construction of the stability region is completed applying the property of continuity of the roots of a polynomial on its coefficients.

Limiting case, $L \to \infty$

Finally, we sketch here a heuristic proof of Proposition 3.4 Consider a value μ_0 outside the unit circle, $|\mu_0| \geq 1$. When $L \to \infty$, since $\lambda < 1$, it is $(\lambda/\mu_0)^L \to 0$ so that (49) reduces to

$$\frac{\mu_0^{L-1}}{1-\lambda/\mu_0}\left(\mu_0^2 - \lambda\,\mu_0 - (1-\lambda)\,\frac{Y}{X}\,((1+X)\mu_0 - 1) = 0\right) .$$

Since $|\mu_0| > \lambda$ for any value of λ, the fraction is finite and bounded away from zero. By applying the conditions in Proposition E.1 to the last equation one sees that if (23) are satisfied, its roots are inside the unit circle.

F Proof of Proposition 4.1

¿From block $\mathcal{X}$ one immediately has (28). From the first row of block $\mathcal{R}$ it is

$$r^* = \bar{e}\,\frac{\sum_{n=1}^{N-1}\varphi_n^* {x_n^*}^2 + \big(1-\sum_{n=1}^{N-1}\varphi_n^*\big){x_N^*}^2}{\sum_{n=1}^{N-1}\varphi_n^*\,x_n^*\,(1-x_n^*) + \big(1-\sum_{n=1}^{N-1}\varphi_n^*\big)\,x_N^*\,(1-x_N^*)} \,. \tag{51}$$

Let us, first, assume that $r^* + \bar{e} \neq 0$. Then, from block $\mathcal{W}$ using (27) one obtains

$$\varphi_n^* = 0 \quad \text{or} \quad \sum_{m=1}^{N-1}\varphi_m^*\,x_m^* + \left(1-\sum_{m=1}^{N-1}\varphi_m^*\right)x_N^* = x_n^* \qquad \forall n \in \{1,\ldots,N-1\}\,. \tag{52}$$

Together with (51) this equation admits two types of solutions, depending on whether one or many equilibrium wealth shares are different from zero.

If only one wealth share is zero, one can assume $\varphi_1^* = 1$. In this case (52) is satisfied for all agents. From (51) one has $x_1^* = r^*/(\bar{e}+r^*)$ which together with (28) leads to (29).

If, instead, many agents survive one can assume (30). In this case, the second equality of (52) must be satisfied for any $n \leq k$. Since its left-hand side does not depend on n, a $x_{1\diamond k}^*$ must exist such that $x_1^* = \cdots = x_k^* = x_{1\diamond k}^*$. Substituting $\varphi_n^* = 0$ for $n > k$ and $x_n^* = x_{1\diamond k}^*$ for $n \leq k$ in (51) one gets $x_{1\diamond k}^* = r^*/(\bar{e}+r^*)$. The equilibrium return r^* is implicitly defined combining this last relation with (28) for $n \leq k$.

Consider finally the case when $r^* + \bar{e} = 0$. Then all equations in block $\mathcal{W}$ are satisfied, while (51) straightforwardly leads to (32).

The equilibrium wealth growth rates of the survivors are immediately obtained from (8) and (28).

G Proofs of Propositions of Section 4.3

Before proving Propositions 4.2, 4.3 and 4.4 we need some preliminary results. The Jacobian matrix of the deterministic skeleton of system (25) is a $\big(2N+L-1\big)\times\big(2N+L-1\big)$ matrix. Using the block structure introduced in Section 4.1 it is separated in 9 blocks

$$\boldsymbol{J} = \begin{Vmatrix} \frac{\partial\mathcal{X}}{\partial\mathcal{X}} & \frac{\partial\mathcal{X}}{\partial\mathcal{W}} & \frac{\partial\mathcal{X}}{\partial\mathcal{R}} \\ \\ \frac{\partial\mathcal{W}}{\partial\mathcal{X}} & \frac{\partial\mathcal{W}}{\partial\mathcal{W}} & \frac{\partial\mathcal{W}}{\partial\mathcal{R}} \\ \\ \frac{\partial\mathcal{R}}{\partial\mathcal{X}} & \frac{\partial\mathcal{R}}{\partial\mathcal{W}} & \frac{\partial\mathcal{R}}{\partial\mathcal{R}} \end{Vmatrix} , \tag{53}$$

The block $\partial\mathcal{X}/\partial\mathcal{X}$ is a $N\times N$ matrix containing the partial derivatives of the agents' present investment choices with respect to the agents' past investment choices. According to (10) the investment choice of any agent does not explicitly depend on the investment choices in the previous period, therefore

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{X}}\right]_{n,m} = \frac{\partial f_n}{\partial x_m} = 0\,, \qquad 1\le n,m\le N \quad ,$$

and this block is a zero matrix.

The block $\partial\mathcal{X}/\partial\mathcal{W}$ is a $N\times(N-1)$ matrix containing the partial derivatives of the agents' investment choices with respect to the agents' wealth shares. According to (10) this is a zero matrix and

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{W}}\right]_{n,m} = \frac{\partial f_n}{\partial\varphi_m} = 0\,, \qquad 1\le n\le N\,, \quad 1\le m\le N-1 \quad .$$

The block $\partial\mathcal{X}/\partial\mathcal{R}$ is a $N\times L$ matrix containing the partial derivatives of the agents' investment choices with respect to the past returns. We use the following notation

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{R}}\right]_{n,l} = \frac{\partial f_n}{\partial r_{l-1}} = f_n^{r_{l-1}}\,, \qquad 1\le n\le N\,, \quad 1\le l\le L \quad .$$

The definitions of the other blocks will make use of the functions R and Φ_n which have been defined in (26) and (27), respectively. Function R depends on agents' contemporaneous investment choices given by the investment functions f_m, on agents' previous investment choices $x_{t,m}$ and on agents' wealth shares $\varphi_{t,m}$. Let denote the corresponding derivatives as R^{f_m}, R^{x_m} and R^{φ_m}. The function Φ_n depends on agents' previous investment choices $x_{t,m}$, on agents' wealth shares $\varphi_{t,m}$ and on the value of return given by function R. The corresponding derivatives are denoted as $\Phi_n^{x_m}$, $\Phi_n^{\varphi_m}$ and Φ_n^R.

The block $\partial\mathcal{W}/\partial\mathcal{X}$ is a $(N-1)\times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' investment choices. It is

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} = \frac{\partial\varphi_n}{\partial x_m} = \Phi_n^{x_m} + \Phi_n^R\cdot R^{x_m}\,, \qquad 1\le n\le N-1\,, \quad 1\le m\le N \quad . \tag{54}$$

The block $\partial\mathcal{W}/\partial\mathcal{W}$ is a $(N-1)\times(N-1)$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' wealth shares. It is

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right]_{n,m} = \frac{\partial\varphi_n}{\partial\varphi_m} = \Phi_n^{\varphi_m} + \Phi_n^R\cdot R^{\varphi_m}\,, \qquad 1\le n,m\le N-1 \quad . \tag{55}$$

The block $\partial\mathcal{W}/\partial\mathcal{R}$ is a $(N-1)\times L$ matrix containing the partial derivatives of the agents' wealth share with respect to lagged returns. It reads

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{R}}\right]_{n,l} = \frac{\partial\varphi_n}{\partial r_{l-1}} = \Phi_n^R \cdot \sum_{m=1}^{N} R^{f_m} f_m^{r_{l-1}}\ , \quad 1 \le n \le N-1\ , \quad 1 \le l \le L \quad . \tag{56}$$

The block $\partial\mathcal{R}/\partial\mathcal{X}$ is the $L \times N$ matrix containing the partial derivatives of the lagged returns with respect to the agents' investment choices. Its structure is simple, since only the first line can contain non-zero elements. It reads

$$\left[\frac{\partial\mathcal{R}}{\partial\mathcal{X}}\right]_{l,n} = \begin{cases} R^{x_n} & l = 1 \\ 0 & \text{otherwise} \end{cases} \quad , \qquad 1 \le l \le L\ , \quad 1 \le n \le N \quad .$$

The block $\partial\mathcal{R}/\partial\mathcal{W}$ is the $L \times (N-1)$ matrix containing the partial derivatives of the lagged returns with respect to agents' wealth shares. It also has $L-1$ zero rows and reads

$$\left[\frac{\partial\mathcal{R}}{\partial\mathcal{W}}\right]_{l,n} = \begin{cases} R^{\varphi_n} & l = 1 \\ 0 & \text{otherwise} \end{cases} \quad , \qquad 1 \le l \le L\ , \quad 1 \le n \le N-1 \quad .$$

The block $\partial\mathcal{R}/\partial\mathcal{R}$ is the $L \times L$ matrix containing the partial derivatives of the lagged returns with respect to themselves.

$$\left[\frac{\partial\mathcal{R}}{\partial\mathcal{R}}\right] = \begin{Vmatrix} R^{r_0} & R^{r_1} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{Vmatrix} \quad ,$$

where $R^{r_l} = \sum_{m=1}^{N} R^{f_m} f_m^{r_l}$.

With the previous definitions and differentiating the correspondent functions, one obtains

Lemma G.1. *Let $\boldsymbol{x}^*$ be an equilibrium of system* (25)*. The corresponding Jacobian matrix, $\boldsymbol{J}(\boldsymbol{x}^*)$, has the following structure, where the actual values of non-zero elements vary depending on whether there exists an* *equity premium in $\boldsymbol{x}^*$.*

$$
\left\|
\begin{array}{cccccc|cccccc|cccc}
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & f_1^{r_0} & \cdots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & f_N^{r_0} & \cdots & f_N^{r_{L-2}} & f_N^{r_{L-1}} \\
\hline
\star & \cdots & \star & 0 & \cdots & 0 & \star & \cdots & \star & \star & \cdots & \star & \star & \cdots & \star & \blacksquare \\
[& \varPhi^x &] & \vdots & \ddots & \vdots & [& \varPhi^{\varphi}_{1,k} &] & [& \varPhi^{\varphi}_{k+1,N} &] & [& \varPhi^r &] & \vdots \\
\star & \cdots & \star & 0 & \cdots & 0 & \star & \cdots & \star & \star & \cdots & \star & \star & \cdots & \star & \blacksquare \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \bigstar & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \bigstar & 0 & \cdots & 0 & 0 \\
\hline
R^{x_1} & \cdots & R^{x_k} & 0 & \cdots & 0 & R^{\varphi_1} & \cdots & R^{\varphi_k} & R^{\varphi_{k+1}} & \cdots & R^{\varphi_{N-1}} & R^{r_0} & \cdots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0
\end{array}
\right\|
$$

Namely, the "varying" elements belong to the 4 blocks filled by $\star$'s and denoted as $[\varPhi^x]$, $[\varPhi^{\varphi}_{1,k}]$, $[\varPhi^{\varphi}_{k+1,N}]$ and $[\varPhi^r]$. In particular, all the elements of the latter block are zeros in the equilibrium with equity premium. Furthermore, in such equilibrium the elements labeled as $\blacksquare$'s are also 0's. Finally, the elements labeled as $\bigstar$'s on the diagonal of the central block are all equal to 1 in the no-equity-premium equilibria.

Proof. For the computation of the elements in the second big row-block, it will be useful to establish the values of relevant derivatives in equilibrium. In general it is

$$
\varPhi^{x_m}_n = \varphi^*_n \frac{(\delta_{n,m} - \varphi^*_m)(r^* + \bar{e})}{1 + (r^* + \bar{e})\left\langle x^* \right\rangle} \quad , \qquad R^{x_m} = \varphi^*_m \frac{x^*_m(r^* + \bar{e}) - 1 - r^*}{\left\langle x^*(1 - x^*) \right\rangle} \quad ,
$$

$$
\varPhi^{R}_n = \varphi^*_n \frac{x^*_n - \left\langle x^* \right\rangle}{1 + (r^* + \bar{e})\left\langle x^* \right\rangle} \quad , \qquad R^{f_m} = \varphi^*_m \frac{1 + x^*_m(r^* + \bar{e})}{\left\langle x^*(1 - x^*) \right\rangle} \quad ,
$$

$$
\varPhi^{\varphi_m}_n = \frac{\delta_{n,m}\big(1 + x^*_n(r^* + \bar{e})\big) - \varphi^*_n(r^* + \bar{e})(x^*_m - x^*_N)}{1 + (r^* + \bar{e})\left\langle x^* \right\rangle} \quad ,
$$

$$
R^{\varphi_m} = \frac{(\bar{e} + r^*)({x^*_m}^2 - {x^*_N}^2) - r^*(x^*_m - x^*_N)}{\left\langle x^*(1 - x^*) \right\rangle} \quad .
$$

Consider the equilibrium with $r^* \neq -\bar{e}$, i.e. the one described in Proposition 4.1*(i)* and *(ii)*. Since in such an equilibrium all survivors invest the same wealth share in the risky asset, one has $\varPhi^R_n = 0$ for any agent n, so that

$$
\begin{aligned}
\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} &= \begin{cases} /m,n\le k/ & = \varphi_n^*\left(\delta_{n,m}-\varphi_m^*\right)(\bar e+r^*)/(1+r^*) \\ /\text{otherwise}/ & =0 \end{cases} \\
\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right]_{n,m} &= \begin{cases} /n>k\ ,\quad n\neq m/ & =0 \\ /\text{otherwise}/ & =\Big(\delta_{n,m}\big(1+x_n^*(r^*+\bar e)\big)-\varphi_n^*(r^*+\bar e)(x_m^*-x_N^*)\Big)/(1+r^*) \end{cases} \\
\left[\frac{\partial\mathcal{W}}{\partial\mathcal{R}}\right]_{n,m} &= 0\ ,\quad \forall n,m\quad .
\end{aligned}
\tag{57}
$$

On the other hand, in an equilibrium with $r^*=-\bar e$, i.e. the one described in Proposition 4.1*(iii)*, it is $\Phi_n^{x_m}=0$, $\Phi_n^{\varphi_m}=\delta_{n,m}$ and $\Phi_n^R=\varphi_n^*x_n^*$ for all possible n and m. Thus, one has

$$
\begin{aligned}
\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} &= \begin{cases} /\text{for } m,n\le k/ & =(1-\bar e)\,\varphi_n^*\,\varphi_m^*\,x_n^*/\langle x^2\rangle \\ /\text{otherwise}/ & =0 \end{cases} \quad ; \\
\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right]_{n,m} &= \begin{cases} /\text{for } n\le k/ & =\delta_{n,m}-\bar e\,\varphi_n^*\,x_n^*(x_m^*-x_N^*)/\langle x^2\rangle \\ /\text{otherwise}/ & =\delta_{n,m} \end{cases} \quad ; \\
\left[\frac{\partial\mathcal{W}}{\partial\mathcal{R}}\right]_{n,m} &= \begin{cases} /\text{for } n\le k/ & =-\varphi_n^*x_n^*\sum_{m=1}^N\varphi_m^*f_m^{r_{l-1}}/\langle x^2\rangle \\ /\text{otherwise}/ & =0 \end{cases} \quad .
\end{aligned}
\tag{58}
$$

□

Lemma G.2. *Consider equilibrium $\boldsymbol{x^*}$ with $r^*\neq-\bar e$. The characteristic polynomial P_J of the matrix $\boldsymbol{J}(\boldsymbol{x^*})$ reads*

$$
P_J(\mu)=(-1)^{N+L}\,\mu^{N-1}\,(1-\mu)^{k-1}\prod_{j=k+1}^{N}\left(\frac{1+x_j^*\,(r^*+\bar e)}{1+r^*}-\mu\right)
\left(\mu^{L+1}-\frac{(1+r^*)\mu-1}{x_{1\diamond k}^*(1-x_{1\diamond k}^*)}\sum_{j=1}^{k}\varphi_j^*\,P_{f_j}(\mu)\right)
\tag{59}
$$

where P_{f_n} is the stability polynomial associated to the n-th investment function as defined in (16)*.*

Proof. The following proof is constructive: we will identify in succession the factors appearing in (59). At each step, a set of eigenvalues is found and the problem is reduced to the analysis of the residual matrix obtained removing the rows and columns associated with the relative eigenspace.

Consider the Jacobian matrix in Lemma G.1. The last $N-k$ columns of the left blocks contain only zero entries so that the matrix possesses eigenvalue 0 with (at least) multiplicity $N-k$. Moreover, in each of the last $N-1-k$ rows in the central blocks the only non-zero entries belong to the main diagonal. Consequently, $\Phi_j^{\varphi_j}$ for $k+1\le j\le N-1$ are eigenvalues of the matrix, with multiplicity (at least) one. A first contribution to the characteristic polynomial is then determined as

$$(-\mu)^{N-k} \prod_{j=k+1}^{N-1} (\Phi_j^{\varphi_j} - \mu) = (-\mu)^{N-k} \prod_{j=k+1}^{N-1} \left(\frac{1 + x_j^*(r^* + \bar{e})}{1 + r^*} - \mu \right) \tag{60}$$

where we used (57) to compute $\Phi_j^{\varphi_j}$ at equilibrium.

In order to find the remaining part of the characteristic polynomial we eliminate the rows and columns associated to the previous eigenvalues to obtain

$$\boldsymbol{L} = \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 0 & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\ \hline \Phi_1^{x_1} & \dots & \Phi_1^{x_k} & \Phi_1^{\varphi_1} & \dots & \Phi_1^{\varphi_k} & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Phi_k^{x_1} & \dots & \Phi_k^{x_k} & \Phi_k^{\varphi_1} & \dots & \Phi_k^{\varphi_k} & 0 & \dots & 0 & 0 \\ \hline R^{x_1} & \dots & R^{x_k} & R^{\varphi_1} & \dots & R^{\varphi_k} & R^{r_0} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 \end{array} \right\| \tag{61}$$

This quadratic matrix has $2k + L$ rows when $k < N$. If $k = N$, representation (61) is, strictly speaking, not correct. Indeed, there exist only $N - 1$ wealth shares φ's in the original system, therefore the central block of the matrix has maximal dimension $(N-1) \times (N-1)$. Therefore, in this case, the correct matrix has dimension $(2N + L - 1) \times (2N + L - 1)$ and can be obtained from (61) through the removal of the last row and the last column in the central blocks. In computing the characteristic polynomial we consider separately the case when $k < N$ and when $k = N$.

If $k < N$ from (57) it follows that for $n, m \le k$ it is

$$\Phi_n^{\varphi_m} = \begin{cases} 1 - \varphi_n^* \, v & \text{if } n = m \\ -\varphi_n^* \, v & \text{otherwise} \end{cases}, \qquad \text{where} \qquad v = \left(x_{1\diamond k}^* - x_N^*\right) \frac{\bar{e} + r^*}{1 + r^*} \,. \tag{62}$$

Moreover, since all survivors invest the same share $x_{1\diamond k}$, it follows that for $m \le k$

$$R^{\varphi_m} = v \, b \,, \qquad \text{where} \qquad b = x_N^* \frac{1 + r^*}{x_{1\diamond k}^* \, (1 - x_{1\diamond k}^*)} \quad . \tag{63}$$

The central column block of the determinant can be represented as $\| \; v \, \boldsymbol{b} + \boldsymbol{b}_1 \; | \; \dots \; | \; v \, \boldsymbol{b} + \boldsymbol{b}_k \; \|$, where the following column vectors have been introduced

$$
\begin{aligned}
\boldsymbol{b} &= \left\| \; 0 \; \ldots \; 0 \;\middle|\; -\varphi_1^* \; \ldots \; -\varphi_k^* \;\middle|\; b \;\; 0 \; \ldots \; 0 \; \right\| \quad , \\
\boldsymbol{b}_1 &= \left\| \; 0 \; \ldots \; 0 \;\middle|\; 1-\mu \; \ldots \; 0 \;\middle|\; 0 \;\; 0 \; \ldots \; 0 \; \right\| \quad , \\
&\;\;\vdots \\
\boldsymbol{b}_k &= \left\| \; 0 \; \ldots \; 0 \;\middle|\; 0 \; \ldots \; 1-\mu \;\middle|\; 0 \;\; 0 \; \ldots \; 0 \; \right\| \quad .
\end{aligned}
$$

We consider each of the columns in the central block as a sum of two terms and, applying the multilinear property of the determinant, end up with a sum of 2^k determinants. Many of them are zeros, since they contain two or more columns proportional to vector $\boldsymbol{b}$. There are only $k+1$ non-zero elements in the expansion. One of them has the following structure of the central column block: $\left\| \, \boldsymbol{b}_1 \mid \ldots \quad \ldots \quad \ldots \mid \boldsymbol{b}_k \, \right\|$, while k others possess similar structure in the central column block, with column $v\,\boldsymbol{b}$ on the ν'th place instead of $\boldsymbol{b}_\nu$, i.e. for all $\nu \in \{1, \ldots, k\}$ the blocks look like $\left\| \, \boldsymbol{b}_1 \mid \ldots \mid v\,\boldsymbol{b} \mid \ldots \mid \boldsymbol{b}_k \, \right\|$.

The central matrix in the one obtained from the former block is diagonal and, therefore, its determinant is equal to $(1-\mu)^k \det \boldsymbol{M}(k)$, where

$$
\boldsymbol{M}(k) = \left\| \begin{array}{ccc|cccc}
-\mu & \ldots & 0 & f_1^{r_0} & \ldots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & -\mu & f_k^{r_0} & \ldots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\
\hline
R^{x_1} & \ldots & R^{x_k} & R^{r_0} - \mu & \ldots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \ldots & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 0 & \ldots & 1 & -\mu
\end{array} \right\| \tag{64}
$$

The other k determinants can be simplified in analogous way, so that

$$
\det(\boldsymbol{L} - \mu \boldsymbol{I}) = (1-\mu)^k \det \boldsymbol{M}(k) + (1-\mu)^{k-1} \sum_{\nu=1}^{k} \det \boldsymbol{M}_\nu(k) \quad , \tag{65}
$$

where for all $\nu \in \{1, \ldots, k\}$ we define the following matrix

$$
\boldsymbol{M}_\nu(k) = \left\| \begin{array}{ccc|c|cccc}
-\mu & \dots & 0 & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & -\mu & 0 & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\
\hline
\Phi_\nu^{x_1} & \dots & \Phi_\nu^{x_k} & -v\,\varphi_\nu^* & 0 & \dots & 0 & 0 \\
\hline
R^{x_1} & \dots & R^{x_k} & v\,b & R^{r_0} - \mu & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 1 & 0
\end{array} \right\| . \tag{66}
$$

We expand the last matrix on the minors of the elements of the central column. For this purpose we introduce for each $\nu \in \{1, \ldots, k\}$ the matrix

$$
\boldsymbol{N}_\nu(k) = \left\| \begin{array}{ccc|ccccc}
-\mu & \dots & 0 & f_1^{r_0} & f_1^{r_1} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & -\mu & f_k^{r_0} & f_k^{r_1} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\
\hline
\Phi_\nu^{x_1} & \dots & \Phi_\nu^{x_k} & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu
\end{array} \right\| ,
$$

so that

$$
\det \boldsymbol{M}_\nu(k) = v\big(-\varphi_\nu^* \det \boldsymbol{M}(k) - b \det \boldsymbol{N}_\nu(k) \big) \quad . \tag{67}
$$

The determinant of $\boldsymbol{M}(k)$ can be computed in a recursive way. Consider the expansion by the minors of the elements in the first column. The minor of the first element $-\mu$ is a matrix with a structure similar to $\boldsymbol{M}(k)$. Denote it as $\boldsymbol{M}(k-1)$. The minor associated with R^{x_1} has a left upper block with $k-1$ entries equal to $-\mu$ below the main diagonal. This block generates a contribution μ^{k-1} to the determinant and once its columns and rows are eliminated, one remains with a matrix of type (41). Applying Lemma D.1 one then has

$$
\det \boldsymbol{M}(k) = (-\mu)\, \det \boldsymbol{M}(k-1) + (-1)^k R^{x_1} \mu^{k-1} \, (-1)^{L-1} \, P_{f_1}(\mu) \,,
$$

where P_{f_1} is the stability polynomial associated with the first investment function. Applying recursively the relation above, the dimension of the determinant is progressively reduced. At the end the lower right block of the original matrix remains, which is again a matrix similar to (41). Applying once more Lemma D.1 one has the following

$$\det \boldsymbol{M}(k) = (-1)^{L-1+k}\,\mu^{k-1} \sum_{j=1}^{k} R^{x_j}\, P_{f_j}(\mu) + (-1)^{L-1+k}\,\mu^{k} \left(\sum_{j=0}^{L-1} R^{r_j}\,\mu^{L-1-j} - \mu^L \right) \quad . \tag{68}$$

The determinant of $\boldsymbol{N}_\nu(k)$ can be computed using a similar strategy. The only difference is that in the last recursive step one of the matrix has zero determinant. Therefore, we have:

$$\det \boldsymbol{N}_\nu(k) = (-\mu)\,\det \boldsymbol{N}_\nu(k-1) + (-1)^k \Phi_\nu^{x_1}\,\mu^{k-1}\,(-1)^{L-1}\,P_{f_1}(\mu) = (-1)^{L-1+k}\,\mu^{k-1} \sum_{j=1}^{k} \Phi_\nu^{x_j}\,P_{f_j}(\mu) \quad ,$$

which, taking into account (67), implies

$$\sum_{\nu=1}^{k} \det \boldsymbol{M}_\nu(k) = v \left(-\det \boldsymbol{M}(k) - b \sum_{\nu=1}^{k} \det \boldsymbol{N}_\nu(k) \right) =$$

$$= -v\,\det \boldsymbol{M}(k) + v\,b\,(-1)^{L+k}\,\mu^{k-1} \sum_{\nu=1}^{k}\sum_{j=1}^{k} \Phi_\nu^{x_j}\,P_{f_j}(\mu) = -v\,\det \boldsymbol{M}(k) \quad .$$

(The last equality above follows directly from expression for $\Phi_\nu^{x_j}$.) Substituting the last relation into (65) and using the expression for $\det \boldsymbol{M}(k)$ in (68) the last contribution to the characteristic polynomial reads

$$\det(\boldsymbol{L} - \mu \boldsymbol{I}) = (1-\mu)^{k-1}\,(1-\mu-v)\,\det \boldsymbol{M}(k) =$$

$$= (-1)^{L-1+k}\,\mu^{k-1}\,(1-\mu)^{k-1} \left(\frac{1 + x_N^*(r^* + \bar{e})}{1+r^*} - \mu \right) \left(\frac{(1+r^*)\mu - 1}{x_{1\diamond k}^*(1 - x_{1\diamond k}^*)} \sum_{j=1}^{k} \varphi_j^*\,P_{f_j}(\mu) - \mu^{L+1} \right) ,$$

where the derivatives of R are computed in accordance with the results of Lemma G.1.

If $k = N$, i.e. all agents survive, then all investment shares are the same. In this case, from (57), all elements in the central column block of matrix (61) are zeros apart from the 1's on the diagonal in the central matrix. This contributes to the characteristic polynomial a factor $(1-\mu)^{N-1}$. In this case the remaining part is the determinant of $\boldsymbol{M}$ and the derived expression is consistent with the equation above.

The product of the last expression and (60) gives (59), which completes the proof. □

Lemma G.3. *Consider no-equity-premium equilibrium $\boldsymbol{x^*}$ with $r^* = -\bar{e}$. The characteristic polynomial P_J of the matrix $\boldsymbol{J}(\boldsymbol{x^*})$ reads*

$$P_J(\mu) = (-1)^{L+N}\,(1-\mu)^{N-2}\,\mu^{N-1}\,\big(\mu + \bar{e} - 1\big) \left(\mu^{L+1} + \frac{\mu - 1}{\langle x^2 \rangle} \sum_{j=1}^{k} \varphi_j^*\,P_{f_j}(\mu) \right) \tag{69}$$

Proof. Since the proof is analogous to the proof of Lemma G.2. some details are omitted. In particular, we confine the analysis to the case $k < N$. From the Jacobian matrix in Lemma G.1 it follows that the only non-zero entries in the $N-1-k$ last rows belonging to the central row block belong to the main diagonal of $[\partial\mathcal{W}/\partial\mathcal{W}]$ and are equal to 1. In addition, the last $N-k$ columns of the leftmost blocks contain only zero entries. Thus the first factor in the characteristic polynomial reads

$$(-\mu)^{N-k}\,(1-\mu)^{N-1-k} \quad , \tag{70}$$

and the associated rows and columns can be eliminated to obtain the new matrix

$$
\boldsymbol{L} = \left\| \begin{array}{ccc|ccc|cccc}
0 & \dots & 0 & 0 & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & f_N^{r_0} & \dots & f_N^{r_{L-2}} & f_N^{r_{L-1}} \\
\hline
\Phi_1^R R^{x_1} & \dots & \Phi_1^R R^{x_k} & 1+\Phi_1^R R^{\varphi_1} & \dots & \Phi_1^R R^{\varphi_k} & \Phi_1^{r_0} & \dots & \Phi_1^{r_{L-2}} & \Phi_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_k^R R^{x_1} & \dots & \Phi_k^R R^{x_k} & \Phi_k^R R^{\varphi_1} & \dots & 1+\Phi_k^R R^{\varphi_k} & \Phi_k^{r_0} & \dots & \Phi_k^{r_{L-2}} & \Phi_k^{r_{L-1}} \\
\hline
R^{x_1} & \dots & R^{x_k} & R^{\varphi_1} & \dots & R^{\varphi_k} & R^{r_0} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0
\end{array} \right\| \tag{71}
$$

To compute $\det(\boldsymbol{L}-\boldsymbol{I})$ where $\boldsymbol{L}$ stands for the identity matrix, we apply the multilinear property of the determinant to the central block of columns. In order to implement this idea, we introduce the following column vectors of length $2k+L$:

$$
\begin{aligned}
\boldsymbol{d} &= \left\| \; 0 \; \dots \; 0 \; \middle| \; \Phi_1^R \; \dots \; \Phi_k^R \; \middle| \; 1 \; 0 \; \dots \; 0 \; \right\| \; , \\
\boldsymbol{d}_1 &= \left\| \; 0 \; \dots \; 0 \; \middle| \; 1-\mu \; \dots \; 0 \; \middle| \; 0 \; 0 \; \dots \; 0 \; \right\| \; , \\
&\;\vdots \\
\boldsymbol{d}_k &= \left\| \; 0 \; \dots \; 0 \; \middle| \; 0 \; \dots \; 1-\mu \; \middle| \; 0 \; 0 \; \dots \; 0 \; \right\| \; .
\end{aligned}
$$

The central column block can be represented as $\| \, R^{\varphi_1} \boldsymbol{d} + \boldsymbol{d}_1 \mid \dots \mid R^{\varphi_k} \boldsymbol{d} + \boldsymbol{d}_k \, \|$. To compute $\det(\boldsymbol{L}-\boldsymbol{I})$ we consider each of the columns in the central block as a sum of two terms and end up with a sum of 2^k determinants. Notice, however, that many of them are zeros, since they contain two or more columns proportional to vector $\boldsymbol{d}$. There are only $k+1$ non-zero elements in the expansion. The determinant of the matrix with the structure $\| \, \boldsymbol{d}_1 \mid \dots \quad \dots \quad \dots \mid \boldsymbol{d}_k \, \|$ in the central part is equal to $(1-\mu)^k \det \boldsymbol{N}(k)$, where $\boldsymbol{N}(k)$ is identical to the matrix $\boldsymbol{M}(k)$ defined in (64). (We slightly changed notation in order to stress that the partial derivatives R^{x_j} and R^{f_j} used in these two matrices have different values in different equilibria.) Using (68) together with (58) it is immediate to see that

$$
\det \boldsymbol{N}(k) = (-1)^{L+k} \, \mu^{k-1} \left(\mu^{L+1} + \frac{\mu + \bar{e} - 1}{\langle x^2 \rangle} \sum_{j=1}^{k} \varphi_j^* \, P_{f_j}(\mu) \right) \quad . \tag{72}
$$

The other non-zero elements possess a similar structure in the central column block, with column $R^{\varphi_\nu} \boldsymbol{d}$ on the ν'th place instead of $\boldsymbol{d}_\nu$ for all $\nu \in \{1, \dots, k\}$. Their determinants can be represented as $(1-$

$\mu)^{k-1} \det \boldsymbol{N}_\nu(k)$, where for all $\nu \in \{1, \dots, k\}$ we define the matrix

$$\boldsymbol{N}_\nu(k) = \left\| \begin{array}{ccc|c|cccc} -\mu & \dots & 0 & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & 0 & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\ \hline \varPhi_\nu^R R^{x_1} & \dots & \varPhi_\nu^R R^{x_k} & R^{\varphi_\nu} \varPhi_v^R & \varPhi_\nu^{r_0} & \dots & \varPhi_\nu^{r_{L-2}} & \varPhi_\nu^{r_{L-1}} \\ \hline R^{x_1} & \dots & R^{x_k} & R^{\varphi_\nu} & R^{r_0} - \mu & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{array} \right\| .$$

This matrix can be simplified, since its central row is (almost) proportional to the next row (the first row in the bottom block). Applying the multilinear property of the determinant, and computing the determinant of the resulting matrix we get

$$\det \boldsymbol{N}_\nu(k) = (-\mu)^{L+k} R^{\varphi_\nu} \varPhi_v^R \quad .$$

Using the corresponding expressions from Lemma G.1, one can check that $\sum_{\nu=1}^k \varPhi_\nu^R R^{\varphi_\nu} = -\bar{e}$. Therefore,

$$\begin{aligned} \det(\boldsymbol{L} - \mu \boldsymbol{I}) &= (1-\mu)^k \det \boldsymbol{N}(k) + (1-\mu)^{k-1} \sum_{\nu=1}^k \det \boldsymbol{N}_\nu(k) = \\ &= (1-\mu)^k \det \boldsymbol{N}(k) - (1-\mu)^{k-1} (-\mu)^{L+k} \bar{e} = \\ &= (1-\mu)^{k-1} (-1)^{L+k} \mu^{k-1} \left((1-\mu)\mu^{L+1} + (1-\mu) \frac{\mu + \bar{e} - 1}{\langle x^2 \rangle} \sum_{j=1}^k \varphi_j^* \, P_{f_j}(\mu) - \mu^{L+1} \bar{e} \right) = \\ &= (1-\mu)^{k-1} (-1)^{L+k} \mu^{k-1} \, (\mu + \bar{e} - 1) \left(\mu^{L+1} + \frac{\mu - 1}{\langle x^2 \rangle} \sum_{j=1}^k \varphi_j^* \, P_{f_j}(\mu) \right) \end{aligned}$$

Combining now the last expression with (70) we get polynomial (69). □

Using the characteristic polynomial of the Jacobian matrix in the corresponding equilibrium, it is straightforward to derive the equilibrium stability conditions of Section 4.3.

The case of one survivor: Proof of Proposition 4.2

If $k = 1$ the characteristic polynomial (59) reduces to

$$P_J(\mu) = (-1)^{N+L} \, \mu^{N-1} \prod_{j=2}^N \left(\frac{1 + x_j^* \, (r^* + \bar{e})}{1 + r^*} - \mu \right) \left(\mu^{L+1} - \frac{(1 + r^*)\mu - 1}{x_1^*(1 - x_1^*)} \, P_{f_1}(\mu) \right) .$$

¿From the expression of the derivative of the EMC at equilibrium $l'(r^*)$ one can see that the last factor corresponds to the polynomial Q_1 in (33). The conditions in (34) are derived from the requirement

$$\left|\frac{1+x_j^*\,(r^*+\bar{e})}{1+r^*}\right|<1 \qquad j\geq 2\ ,$$

and the Proposition is proved.

The case of many survivors: Proof of Proposition 4.3

In the case of $k>1$ survivors the characteristic polynomial in (59) possesses a unit root with multiplicity $k-1$. Consequently, the fixed point is non-hyperbolic.

To find the eigenspace associated with the eigenvalue 1 we subtract from the initial Jacobian matrix (53) computed at equilibrium the identity matrix of the corresponding dimension and analyze the kernel of the resulting $\boldsymbol{J}-\boldsymbol{I}$ matrix. This can be done through the analysis of the kernel of the matrix obtained by the substitution of the identity matrix from matrix $\boldsymbol{L}$ given in (61). Let us consider the $k<N$ and the $k=N$ cases separately.

When $k<N$, as we showed in the proof of Lemma G.2, in the matrix obtained as a result of subtraction of an identity matrix from (61), the central $k-1$ columns are identical, see (62) and (63). Therefore, the kernel of the matrix $\boldsymbol{J}-\boldsymbol{I}$ can be generated by a basis containing the following $k-1$ vectors

$$\boldsymbol{u}_n=\Big(\underbrace{0,\ldots,0}_{N};\underbrace{0,\ldots,0}_{n-1},1,\underbrace{0,\ldots,0}_{k-n-1},-1;\underbrace{0,\ldots,0}_{N-1-k};\underbrace{0,\ldots,0}_{L}\Big)\ ,\quad 1\leq n\leq k-1\quad . \tag{73}$$

Notice that the direction of vector $\boldsymbol{u}_n$ corresponds to a change in the relative wealths of the n-th and k-th survivor.

If, instead, $k=N$, then the last $k-1$ columns in the resulting (from (61)) matrix are zero vectors, and then the kernel of the matrix $\boldsymbol{J}-\boldsymbol{I}$ can be generated with the $N-1$ vectors of the canonical basis

$$\boldsymbol{v}_n=\Big(\underbrace{0,\ldots,0}_{N};\underbrace{0,\ldots,0}_{n-1},1,\underbrace{0,\ldots,0}_{N-n-1};\underbrace{0,\ldots,0}_{L}\Big)\ ,\quad 1\leq n\leq N-1\quad . \tag{74}$$

whose direction corresponds to a change in the relative wealths of the n-th and N-th survivors.

If the system is perturbed away from equilibrium $\boldsymbol{x}^*$ along the directions defined in (73) or (74), a new fixed point is reached. Then, the system is stable, but not asymptotically stable, with respect to these perturbations.

Moreover, since the eigenspaces identified above do not depend on the system parameters, it is immediate to realize that they do constitute not only the tangent spaces to the corresponding non-hyperbolic manifolds, but the manifolds themselves.

The polynomial (35) is the last factor in (59), while conditions (36) are obtained by imposing

$$\left|\frac{1+x_j^*\,(r^*+\bar{e})}{1+r^*}\right|<1 \qquad j>k+1\ ,$$

which completes the proof.

The case of "no-equity-premium" equilibria. Proof of Proposition 4.4

Independently of the number of survivors, the characteristic polynomial in (69) possesses a unit root with multiplicity $N-2$. Consequently, the fixed point $\boldsymbol{x}^*$ is never hyperbolic, when $N \geq 3$. It is easy to see that in this case all equilibria belong to the manifold of dimension $N-2$ and that this is exactly the non-hyperbolic manifold of $\boldsymbol{x}^*$. For the stability of equilibrium $\boldsymbol{x}^*$ with respect to the perturbations in the directions *orthogonal* to this manifold, it is sufficient to have all other eigenvalues inside the unit circle. If this condition is satisfied, then the equilibrium $\boldsymbol{x}^*$ is stable, but not asymptotically stable. Since $\bar{e} > 0$, this sufficient condition can be expressed through the roots of the last term in (69), which is exactly the polynomial (37).

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