Nonlocal onset of instability in an asset pricing model with heterogeneous agents

A. Gaunersdorfer†, C.H. Hommes‡ and F.O.O. Wagener‡

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Abstract

Empirical time series of financial market data, like day-to-day stock returns, exhibit the phenomenon that although usually tomorrow's price is unpredictable, the absolute value of the price change is correlated with the magnitude of past price changes; though the corresponding correlation coefficients are not very large, they are significantly different from zero. This phenomenon is known as 'volatility clustering' in the financial liturature. In this note a micro-economic model of volatility clustering, introduced by Gaunersdorfer and Hommes[7], will be analysed. The deterministic skeleton of the model has a Chenciner bifurcation, and hence periodic points and invariant quasi-periodic circles coexisting with the 'fundamental' equilibrium. Adding noise in form of stochastic supply shocks, volatility clustering is generated by the system jumping between the bases of attraction of the fundamental equilibrium (low volatility), and that of the non-fundamental attractor (high volatility).

1 The model

We build a variant of the heterogeneous adaptive beliefs model of Brock and Hommes [2], see also [4, 5], investigated by Gaunersdorfer and Hommes [7] and Gaunersdorfer *et al.* [8]. There are economic agents which trade one kind of risky asset (a stock) and one risk-less asset (a bond) on a financial market. Assets are assumed to be infinitely lived and perfectly divisible; moreover, short-selling is allowed (*i.e.* it is possible to buy negative quantities of them).

Information. The risk-less assets live one period; they are bought at a fixed price 1 per unit, and pay an amount $R = 1+r > 0$ in the next period. Risky asset are traded at the market price p_t, and pay each period a dividend y_t: one risky asset bought at time t at price p_t yields dividend y_{t+1} and can be sold at price p_{t+1} next period. In this model, the dividends $\{y_t\}$ are assumed to be independently and identically distributed (iid). The information set $\mathcal{F}_t$ is the σ-algebra generated by the random variables $\{p_t, p_{t-1}, \cdots, y_t, y_{t-1}, \cdots\}$; a function f is measurable with respect to $\mathcal{F}_t$, if it can be written as a Borel function of (a finite number of) the variables $(p_t, p_{t-1}, \cdots, y_t, y_{t-1}, \cdots)$.

Beliefs. Different agents are assumed to form beliefs about the realisations of the random quantities in the next period. Here, only a finite number of beliefs are considered; see [1, 3, 6] for more general setups. Belief type h is characterised by an operator $\mathbb{B}_{ht} = \mathbb{B}_h(. \mid \mathcal{F}_t)$, which associates to every random variable $X \in \mathbb{R}$ another random variable $\mathbb{B}_{ht}X = \mathbb{B}_h(X \mid \mathcal{F}_t)$ that is measurable with respect to $\mathcal{F}_t$. The quantity $\mathbb{B}_{ht}X$ is interpreted as the 'belief of h at time t about X'. This operator is assumed to have the property that if X is measurable with respect to $\mathcal{F}_t$, then $\mathbb{B}_{ht}X = X$; that is, everything that is known is believed as well. Note

that the beliefs are conditioned on the current price p_t, even if though this price is unknown when agents send their demand functions to the market.

Belief operators are typical for boundedly rational agents; a fully rational agent is characterised that his belief is equal to the expectation operator. However, in practice agents have seldomly sufficient information and time to work out mathematical expectations of random variables.

Demand function. Buying one share of the risky asset yields in the next period an excess return ξ_{t+1} of

$$\xi_{t+1} = p_{t+1} + y_{t+1} - Rp_t.$$

An agent tries to maximise his expected utility from trading, which is set equal to his risk-adjusted expected profit from buying z shares

$$u_t = \mathbb{B}_{ht} z_{ht} \xi_{t+1} - \frac{a}{2} \mathbb{V}\mathrm{ar}_{ht} z_{ht} \xi_{t+1}. \tag{1}$$

Here $\mathbb{V}\mathrm{ar}_{ht} X = \mathbb{B}_{ht}(X - \mathbb{B}_{ht} X)^2$. As his trading decision has to be based on the information available at time t, it will be assumed that z is measurable with respect to $\mathcal{F}_t$. Moreover, to bring down the heterogeneity between the trader types down to a minimum, it is assumed that

$$\mathbb{B}_{ht} y_{t+1} = \bar{y}, \qquad \mathbb{V}\mathrm{ar}_{ht} \xi_{t+1} = \sigma^2, \qquad \text{for all } h.$$

In other words, agents' beliefs disagree only on next periods' prices. Utility is then maximised by

$$z_{ht} = \frac{1}{a\sigma^2} \mathbb{B}_{ht} \xi_{t+1}.$$

Market equilibrium. Normalised total demand of risky assets equals the weighted sum $\sum_h n_{ht} z_{ht}$, where n_{ht} is the fraction of agents of type h in the market at time t. Total (outside) supply being equal to z^s, market equilibrium is expressed by $z^s = \sum_h n_{ht} z_{ht}$, which yields

$$Rp_t = \sum_h n_{ht} \mathbb{B}_{ht} p_{t+1} + \bar{y} - a\sigma^2 z_t^s. \tag{2}$$

Supply will be assumed to be a random iid variable with mean 0; the 'deterministic skeleton' of the model is obtained by setting z_t^s equal to 0 throughout. Equation (2) determines the market price p_t.

Type dynamics. In order to obtain an evolution law, the dependence of n_{ht} on past prices has to be specified. For this, the realised utility U_{ht} is computed, which is a measure of the 'fitness' of type h (cf. equation (1) for u_t):

$$U_{ht} = z_{ht-2} \xi_{t-1} - \frac{a\sigma^2}{2} z_{ht-2}^2 = -\frac{1}{2a\sigma^2} \left(p_{t-1} - \mathbb{B}_{ht-2} p_{t-1} \right)^2 + C_t,$$

where C_t contains all terms that are the same for all h. The fraction n_{ht} of agents choosing strategy type h is proportional to $e^{\beta U_{ht}} w_h / Z$, where w_h is a fixed weight factor[3], and where the constant of proportionality Z equals $\sum_h e^{\beta U_{ht}} w_h$. The parameter $\beta \geq 0$ is called 'intensity of choice': it expresses the competitiveness of the market. If β is high, traders are very keen on picking a strategy that performs well; in contrast, if β is zero, differences in fitness are not taken into account, and traders pick a strategy at random.

Fundamental solution. In the special case that all agents are fully rational, the market equilibrium equation simplifies to

$$Rp_t = p_{t+1} + \bar{y} - a\sigma^2 z^s.$$

Assuming that the outside supply of shares $z^s = 0$, this equation has a unique bounded solution $p_t = \bar{p} = \bar{y}/(R-1)$, which is called the rational fundamental price. All prices will from now on be replaced by the deviations $x_t = p_t - \bar{p}$ from the fundamental price.

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Belief specifications. The model is closed by specifying the belief operators $\mathbb{B}_{ht}$. For simplicity, attention is restricted to a two type case

$$\mathbb{B}_{1t}x_{t+1} = vx_{t-1}, \qquad \mathbb{B}_{2t}x_{t+1} = x_{t-1} + g(x_{t-1} - x_{t-2}),$$

with parameters $0 \leq v < 1$, $g > 0$. Type 1 traders are 'adaptive fundamentalists', that is, they believe that prices will gradually decline towards the fundamental price; type 2 traders are 'technical analysts': they believe that the price difference between last period's and next period's price is a function of the latest observed price difference.

As it stands, the model will not yield bounded dynamics: if at a certain time n_{2t} is large, and $g > 1$, the technical analysts dominate the market, and the price dynamics will diverge to infinity. History shows that every price bubble bursts; it is therefore reasonable to demand that technical trading rules are not popular if the price is too far from the fundamental. This is modelled by changing the fraction of technical analysts by a factor that decreases with the square of the deviation of the latest realised price from the fundamental:

$$n_{2t} = e^{-\frac{x_{t-1}^2}{\alpha}} \frac{e^{\beta U_{2t}} w_2}{e^{\beta U_{1t}} w_1 + e^{\beta U_{2t}} w_2} = \frac{e^{-\frac{x_{t-1}^2}{\alpha}}}{e^{\beta(U_{1t} - U_{2t})} \frac{w_1}{w_2} + 1}, \qquad n_{1t} = 1 - n_{2t}.$$

The full system is now given by the evolution equation

$$x_t = \varphi(x_{t-1}, \cdots, x_{t-4}) = \big(n_{1t}vx_{t-1} + n_{2t}(x_{t-1} + g(x_{t-1} - x_{t-2}))\big)/R.$$

The associated dynamical system Φ is given by

$$\Phi(x) = (\varphi(x_1, \cdots, x_4), x_1, x_2, x_3).$$

Note that Φ is an endomorphism.

2 Dynamical properties

Stability of origin. The fundamental equilibrium $x = 0$ is the only fixed point of Φ. Noting that there $dn_{ht} = 0$, the linearised evolution reads as

$$R\, dx_t = (n_1 v + n_2(1 + g))\, dx_{t-1} - gn_2\, dx_{t-2};$$

here $n_1 = w_1/(w_1 + w_2)$ *etc.* This is solved by linear combinations of s^t, where s is a root of $Rs^2 - (n_1 v + n_2(1 + g))s + g = 0$. Note that if $|s| < 1$, then the fixed point is hyperbolically attracting. Since $0 < v < 1$ and $g > 0$, the origin loses its stability in a Hopf bifurcation at $g_H = R/n_2$. Moreover, if $g = 0$, then

$$|x_t| = |R^{-1}(n_{1t}v + n_{2t})||x_{t-1}| < \frac{1}{R}|x_{t-1}|,$$

and the origin is a globally stable hyperbolic attractor.

Nonlocal bifurcations. Numerical simulations show however that for values of g smaller than g_H, there are already attractors, mostly invariant quasi-periodic circles, that are generated by a saddle-node bifurcation, see figure 1. For g small, the fixed point $x = 0$ is globally attracting. As g increases, another attractor is generated by a saddle-node bifurcation of periodic points of invariant circles. The bifurcation points in figure 1 are determined by a Monte Carlo method: for a given parameter value, ten random initial values of the system are taken, and their attractor is determined by iteration. If this attractor is not the point $x = 0$, the fundamental equilibrium is not globally stable. Note that the region of coexisting attractors grows as β increases: if agents act on small utility differences, the market is destabilised more easily.
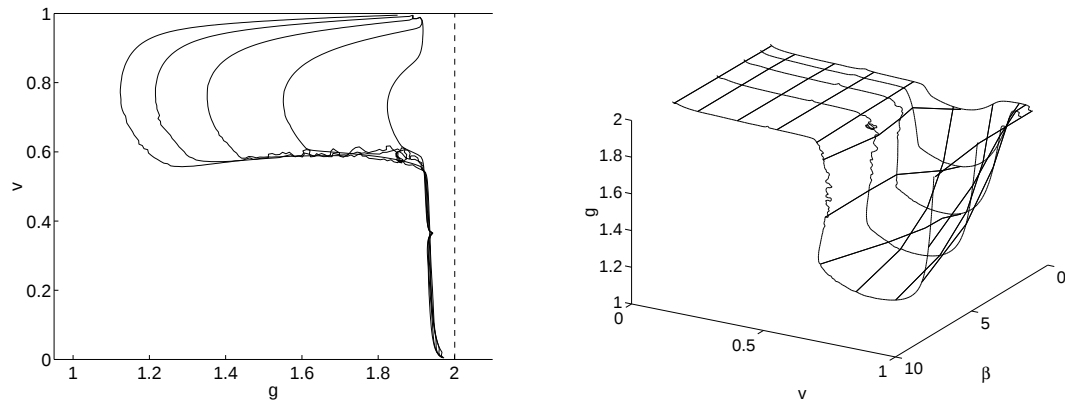
Figure 1: Nonlocal bifurcations before the fundamental equilibrium loses stability: bifurcation curves for several β in a (g, v)-diagram (left), and the three dimensional bifurcation surface (right).
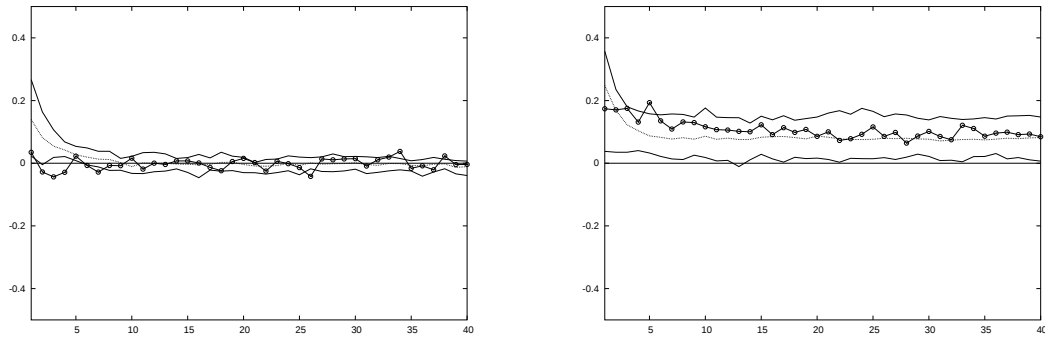
Figure 2: Comparison of autocorrelation coefficients of returns (left) and absolute returns (right) of daily data of the S&P 500 index 1980–2000 (line with circles), and 95%-confidence intervals obtained from 20 model simulation runs ($v = 0.9999$, $g = 1.8$, $\beta = 10$, $R = 1.0004$, $w_1 = w_2 = 1$, $\bar{y} = 0.034$).

3   Statistical properties

After adding random supply shocks, the model is pitted against day-to-day return data of the S&P 500 index in the period 1980–2000. The test-criterion chosen is equality of autocorrelations of returns and absolute returns, which is, despite its deficiencies, a popular choice by economists, since it represents a well-documented qualitative property of financial time-series. The distribution of the simulated data is obtained by performing 20 model simulation runs: with these, 95% confidence intervals are obtained. The results are given in figure 2. On the basis of these results, agreement between the model and the data cannot be rejected.

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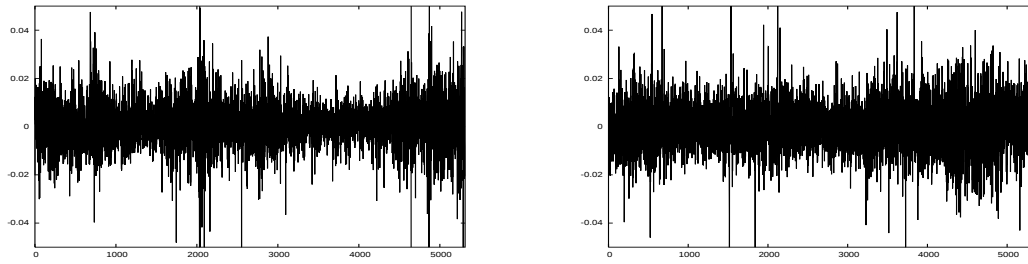
Figure 3: Time series of the S&P 500 daily index returns (left) and a simulation (right). Parameters as in figure 2.

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