

# BOUNDED RATIONALITY AND LEARNING IN COMPLEX MARKETS\*

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**Abstract.** This chapter reviews some work on bounded rationality, expectation formation and learning in complex markets, using the familiar demand-supply cobweb model. We emphasize two stories of bounded rationality, one story of *adaptive learning* and another story of *evolutionary selection*. According to the adaptive learning story agents are identical, and can be represented by an “average agent”, who adapts his behavior trying to learn an optimal rule within a class of simple (e.g. linear) rules. The second story is concerned with heterogeneous, interacting agents and evolutionary selection of different forecasting rules. Agents can choose between costly sophisticated forecasting strategies, such as rational expectations, and freely available simple strategies, such as naive expectations, based upon their past performance. We also confront both stories to laboratory experiments on expectation formation. At the end of the chapter, we integrate both stories and consider an economy with evolutionary selection between a costly sophisticated adaptive learning rule and a cheap simple forecasting rule such as naive expectations.

**Keywords:** complex systems, behavioral economics, evolutionary selection, nonlinear dynamics.

*JEL classification:* B4, C0, C6, D84, E3, G1, G12

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# 1 Introduction

There are two opposing views concerning the expectations hypothesis in economics and finance. According to the traditional, neoclassical view, propagated by Muth (1961) and Lucas (1972) agents form rational expectations (RE) without any systematic forecasting mistakes. In the rational framework it is often assumed that agents have full knowledge of their economic environment, and use all available information from economic theory to compute rational forecast. Moreover, typically it is assumed that *all* agents are fully rational, leading to the representative rational agent benchmark. Friedman (1953) provided an early argument in support of the representative rational agent framework, namely that irrational agents would be driven out of the market, since rational agents earn higher profits or utility. Stated differently, evolutionary selection prevents irrational behaviour and the economy may be described *as if* all agents are perfectly rational.

Simon (1957) already criticized this view, arguing that deliberation and information gathering costs should be taken into account. More recently, work on *bounded rationality* in the 1990s, surveyed e.g. in Sargent (1993) and Conlisk (1996)), has challenged the traditional view, emphasizing that the extreme assumptions concerning perfect knowledge of the economy and infinite computing capacities are highly unrealistic and in sharp contrast with observed behavior in laboratory experiments with human subjects (e.g. Tversky and Kahnemann (1974)). In macroeconomics, much work has been done on *adaptive learning*, as surveyed e.g. in Evans and Honkapohja (2001). A key underlying assumption is that agents do *not* know the underlying “law of motion” of the economy, but instead use time series observations to form expectations based upon their own “perceived law of motion”, trying to learn the model parameters as more observations become available. Much of this literature has focussed on the stability of rational expectations equilibria (REE) and equilibrium selection, in an attempt to justify rationality by adaptive learning.

Stimulated by work at the Santa Fe Institute, the view that markets are *complex evolving systems* has gained popularity, see e.g. the Santa Fe conference proceedings Anderson et al. (1988) and Arthur et al. (1997a), the collection of papers in Rosser (2004) and the recent survey in Arthur (2006). When the economy is viewed as a *complex system* with many interacting agents, it seems

hard to justify perfect structural knowledge about the economy and fully rational expectations, since knowledge about the beliefs of all other agents would be required. A large population of boundedly rational heterogeneous agents, using different forecasting rules ranging from simple to sophisticated, seems much more natural and in line with human behavior. A problem of *bounded rationality* however is that there are *many degrees of freedom*, and which model of bounded rationality is an accurate description of learning behaviour at the individual level?

In this chapter we review some work on bounded rationality, expectation formation and learning in complex markets. We will use the familiar demand-supply cobweb model, exactly the same framework employed by Muth (1961), in his seminal paper introducing rational expectations. We emphasize two stories of bounded rationality, one story of *adaptive learning* and another story of *evolutionary selection*; at the end of the chapter we combine both stories. An important point of departure for both stories is that agents do not understand the world in its full complexity, but have some simple perception of this complex world and use relatively simple decision heuristics or forecasting rules. According to the first adaptive learning story agents are identical, and can be represented by an “average agent”, who adapts his behavior trying to learn an optimal rule within a class of simple rules. An example is the consistent expectations equilibrium proposed by Hommes and Sorger (1998), where agents try to learn the best linear rule, minimizing forecasting errors, in an unknown nonlinear economy. The optimal linear rule fits the observable sample mean and sample autocorrelation structure of the nonlinear economy. The second story is concerned with heterogeneous, interacting agents and evolutionary selection of different forecasting rules. Heterogeneous agent models are becoming increasingly popular in finance, where a distinction between fundamentalists and chartist trading strategies can be made; see e.g. Hommes (2006) and LeBaron (2006) for extensive surveys. Here, we consider the *adaptive belief systems* proposed by Brock and Hommes (1997, 1998), where agents can choose between a costly sophisticated forecasting strategy, such as rational expectations, and a freely available simple strategy, such as naive expectations. At the end of the chapter, we will integrate both stories and consider an economy with evolutionary selection between a costly sophisticated adaptive learning rule and a cheap simple forecasting rule such as naive expectations.

There is a lot of theoretical work on expectations formation and learning when agents are boundedly rational, but surprisingly little experimental work on expectations and learning of human subjects has been done. A controlled laboratory environment is well suited to investigate how individuals form expectations and learn from experience, and how the market aggregates individual forecasting strategies. Recently, Hommes et al. (2007) conducted experiments on expectation formation within a cobweb framework. We confront theoretical work on expectation formation and learning with the observed “stylized facts” in these laboratory experiments.

The chapter is organized as follows. Section 2 discusses cobweb dynamics under various expectations rules, such as naive, rational and adaptive expectations. Section 3 focuses on laboratory experiments with human subjects on expectation formation. In Section 4 we discuss adaptive learning, in particular the notion of consistent expectations equilibrium (CEE) and sample autocorrelation (SAC-)learning. Section 5 focuses on heterogeneity and evolutionary competition between different forecasting rules and ends with an example where adaptive learning and evolutionary selection are combined. Finally, Section 6 briefly discusses a future perspective.

## 2 The Cobweb Model

The classical cobweb model is a partial equilibrium model describing commodity price fluctuations of a non-storable good, such as corn or hogs, that takes one time period to produce. It is one of the simplest benchmark models in economic dynamics and can be found in many standard textbooks (e.g. Nicholson (1995, pp.590-594)). Producers must form price expectations one period ahead and derive their optimal production decision from expected profit maximization. Given producers' price forecast  $p_t^e$ , optimal supply is given by

$$S(p_t^e) = \operatorname{argmax}_{q_t} \{p_t^e q_t - c(q_t)\} = (c')^{-1}(p_t^e). \quad (1)$$

The cost function  $c(\cdot)$  is assumed to be strictly convex so that the second order condition for profit maximization is satisfied. The marginal cost function is then invertible and supply is strictly increasing in expected price. The simplest case arises when the cost function is quadratic,  $c(q) =$

$q^2/(2s)$ , yielding a linear supply curve

$$S(p^e) = sp^e, \quad s > 0. \quad (2)$$

In general a strictly convex cost curve leads to a nonlinear, increasing, supply curve. As an example, we will consider an increasing, S-shaped supply curve

$$S(p^e) = b + \arctan(\lambda p^e), \quad \lambda > 0, b > \pi/2, \quad (3)$$

where the parameter  $\lambda$  tunes the nonlinearity of the supply curve and  $b > \pi/2$  is a parameter tuning the production level ensuring that production is always non-negative<sup>1</sup>.

Consumer demand  $D$  depends upon the current market price  $p_t$ . The demand curve  $D$  can be derived from consumer utility maximization, but for our purposes it is not necessary to specify these preferences explicitly. Throughout the chapter we will simply work with a linearly decreasing demand curve

$$D(p_t) = a - dp_t + \epsilon_t, \quad a, d > 0, \quad (4)$$

where  $-d$  is the slope of the demand curve,  $a$  determines the demand level and  $\epsilon_t$  is an independently and identically distributed (IID) stochastic series representing exogenous random demand shocks. If beliefs are *homogeneous*, i.e., all producers have identical price expectations  $p_t^e$ , market clearing implies

$$D(p_t) = S(p_t^e) \quad (5)$$

yielding the realized market price

$$p_t = D^{-1}(S(p_t^e)) = \frac{a + \epsilon_t - S(p_t^e)}{d}. \quad (6)$$

With an increasing supply curve and a decreasing demand curve, there can only be one price, denoted by  $p = p^*$ , where demand and supply intersect. The price dynamics in (6) thus depends upon the demand and supply curves, as well as on the assumed expectations hypothesis. How do producers form price expectations? We first consider the benchmarks of naive, rational and adaptive expectations.

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<sup>1</sup>An S-shaped supply curve can e.g. be derived from a fourth or higher order polynomial convex cost curve  $c(q) = \frac{1}{d+1}(q-1)^{d+1} + q$ , where  $d$  is an odd integer, e.g.  $d = 3$ . Optimal supply then becomes  $q = S(p^e) = (p^e - 1)^{\frac{1}{d}} + 1$ .

**2.1 Naive expectations** Before the rational expectations revolution it was common practice to use simple forecasting rules. The simplest case studied in the thirties, e.g. by Ezekiel (1938), assumes that producers have naive expectations, that is, their prediction equals the last observed price  $p_t^e = p_{t-1}$ . Under naive expectations, the price dynamics (6) becomes

$$p_t = D^{-1}(S(p_{t-1})). \quad (7)$$

According to the well known cobweb theorem (see e.g. Ezekiel (1938)), there are essentially two possibilities for the price dynamics under naive expectations, depending upon the ratio of marginal supply and marginal demand at the steady state  $p^*$ . When  $-1 < S'(p^*)/D'(p^*) < 0$  the steady state  $p^*$  is (locally) stable, and prices converge to the steady state. If on the other hand  $S'(p^*)/D'(p^*) < -1$  the steady state  $p^*$  is (locally) unstable, and prices diverge from the steady state. In the case of a nonlinear, bounded supply curve as in (3), if the steady state is unstable, prices will converge to a stable 2-cycle, with regular up and down oscillations, as illustrated in Figure 2 in the next Section.

**2.2 Rational expectations** It has been argued that simple forecasting rules such as naive expectations, lead to *systematic forecasting errors*. This argument seems particularly strong when the model generates a 2-cycle, even in the presence of (small) exogenous shocks. When producers expect a high (low) price, they will supply a high (low) quantity and consequently the realized market price will be low (high). Along a ‘hog cycle’ of up and down price oscillations, expectations are thus *systematically wrong*, and forecasting errors are strongly correlated. Rational agents would learn from their systematic errors and revise expectations accordingly, so the argument goes. These considerations led Muth (1961) to introduce *rational expectations*, where producers’ subjective price expectations equal the objective conditional mathematical expectation of the market price, i.e.  $p_t^e = E_t[p_t]$ . Using market equilibrium (5) with the linear demand curve (4), taking conditional mathematical expectation on both sides, we can solve for the rational expectations forecast

$$p_t^e = E_t[p_t] = p^*, \quad (8)$$

where  $p^*$  is the unique price corresponding to the intersection point of demand and supply. Given producers' rational price forecast  $p_t^e = p^*$ , the actual law of motion (6) becomes

$$p_t = p^* + \frac{\epsilon_t}{d}. \quad (9)$$

The cobweb model therefore has a unique REE, given by an IID process with mean  $p^*$ . Along a REE expectations are *self fulfilling* and producers make no systematic forecasting errors, since forecasting errors are uncorrelated. It is important to note that, in order to form rational expectations, perfect knowledge of underlying market equilibrium equations is required and, in particular, agents must be able to compute the intersection point  $p^*$ .

**2.3 Adaptive expectations** It is worthwhile to reconsider the issue of 'systematic forecasting errors' in the light of the recent discovery of chaotic dynamics in simple nonlinear deterministic systems and under the more plausible assumption of *bounded rationality*, where agents do not know underlying market equilibrium equations, but only use time series observations to forecast<sup>2</sup>. As an example, consider the cobweb model with *adaptive expectations*, i.e.,

$$p_t^e = (1 - w)p_{t-1}^e + wp_{t-1}, \quad 0 \leq w \leq 1, \quad (10)$$

where  $w$  is the expectations weight factor. The expected price is a weighted average of yesterday's expected and realized prices, or equivalently, the expected price is adapted by a factor  $w$  in the direction of the most recent realization. Adaptive expectations may thus be seen as 'error learning' with a constant factor. Notice that for  $w = 1$ , adaptive expectations reduces to naive expectations. Under adaptive expectations and, given the linear demand curve (4), the dynamics of expected prices in the cobweb model becomes

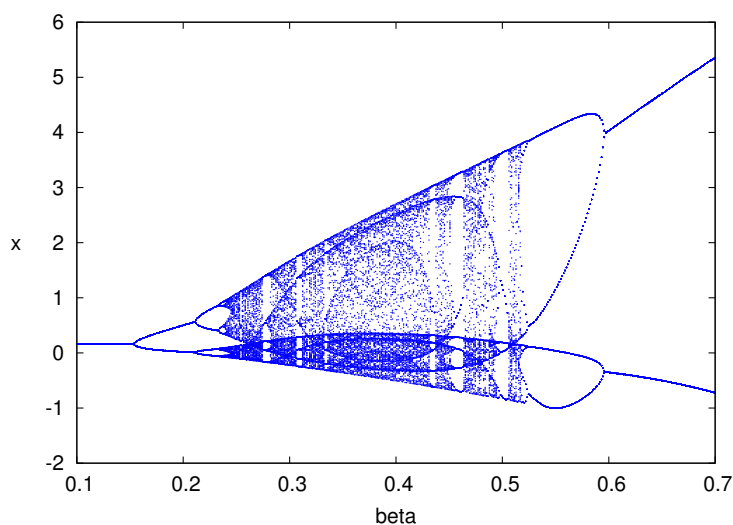
$$p_t^e = (1 - w)p_{t-1}^e + w\left(\frac{a + \epsilon_t - S(p_{t-1}^e)}{d}\right). \quad (11)$$

Chiarella (1988) and Hommes (1991,1994) have shown that, without any random shocks  $\epsilon_t$ , for *nonlinear*, but monotonic, demand and/or supply curves, this nonlinear deterministic difference

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<sup>2</sup>At the time of the introduction of rational expectations by Muth (1961) and its introduction into macroeconomics by Lucas (1972) and others, the phenomenon of deterministic chaos was still largely unknown to economists.

equation can easily generate chaotic fluctuations in expected prices, and therefore also in prices, quantities and forecasting errors. Figure 1 shows a bifurcation diagram with respect to the expectations weight factor  $w$ , with the nonlinear, S-shaped supply curve (3). For high values of  $w$ , sufficiently close to  $w = 1$  (i.e. close to naive expectations) prices converge to a stable 2-cycle, whereas for small values of  $w$ , sufficiently close to  $w = 0$ , prices converge to the RE steady state. For intermediate  $w$ -values however, chaotic price oscillations arise.



**Figure 1:** Bifurcation diagram with respect to the expectations weight factor  $w$ ,  $0.1 \leq w \leq 0.7$ , with the other parameters fixed at  $a = 0.7$ ,  $d = 0.25$  and  $\lambda = 4.8$  ( $x$  is the deviation from the inflection point of the nonlinear, S-shaped supply curve (3)). For large values of  $w$  prices converge to a regular 2-cycle with large amplitude. As  $w$  decreases the amplitude of price fluctuations decreases and a bifurcation route to chaos occurs. When  $w$  becomes very small, chaotic fluctuations are stabilized and prices converge to REE.

When prices fluctuate chaotically, the corresponding forecasting errors will be highly unpredictable and the question arises whether boundedly rational agents would be able to detect any structure in these chaotic forecasting errors and improve upon their simple adaptive forecasts. If patterns are indeed hard to discover, then adaptive expectations with chaotic price fluctuations might be a satisfactory (long run) boundedly rational equilibrium.



### 3 Laboratory Experiments

There is a lot of theoretical work on expectations formation and learning when agents are boundedly rational, but surprisingly few laboratory experiments with human subjects have been performed to study how individuals form expectations and learn from experience, and how the market aggregates individual forecasts.

Early experiments on expectations have been done in Schmalensee (1976), who uses historical data on wheat prices and asks subjects to predict the mean wheat price for the next 5 periods. In Dwyer et al. (1993) and Hey (1994) subjects have to predict a time series generated by a stochastic process such as a random walk or a simple linear first order autoregressive process. More recently, Kelley and Friedman (2002) consider learning in an Orange Juice Futures price forecasting experiment, where prices are driven by a linear stochastic process with two exogenous variables (weather and competing supply). A drawback of these papers is that subjects are forecasting an exogenous process, and there is no feedback from individual expectations to realizations.

Williams (1987) considers expectation formation in an experimental double auction market which varies from period to period by small shifts in the market clearing price. Participants predict the mean contract price for 4 or 5 consecutive periods. The participant with the lowest forecast error earns \$1.00. Peterson (1993) studies price predictions in repeated double auction experimental asset markets, as in the famous bubble experiments of Smith et al. (1988), and shows that forecasts tend to be biased and inconsistent with RE, but there is a tendency of forecasts to evolve in the direction of RE.

Marimon, Spear and Sunder (1993) and Marimon and Sunder (1993, 1994, 1995) have studied expectation formation in laboratory experiments in inflationary overlapping generations economies. Marimon, Spear and Sunder (1993) find experimental evidence for expectationally driven cycles and coordination of beliefs on a sunspot 2-cycle equilibrium, but only after agents have been exposed to exogenous shocks of a similar kind. Marimon and Sunder (1995) present experimental evidence that a “simple” rule, such as a constant growth of the money supply, can help coordinate agents’ beliefs and help stabilize the economy. More recently, Adam (2007) conducted laboratory

experiments on inflationary expectations.

Here we discuss some recent laboratory experiments of Hommes et al. (2007) on individual expectations and learning in the cobweb framework. See also Hommes et al. (2005) for similar experiments in an asset pricing framework. The participants in the experiments were asked to predict next periods price of a certain, unspecified, good. The realized price  $p_t$  in the experiment was determined by the (unknown) cobweb market equilibrium equation

$$D(p_t) = \frac{1}{K} \sum_{i=1}^K S(p_{i,t}^e), \quad (12)$$

where  $D(p_t)$  is the demand for the good at price  $p_t$ ,  $K$  is the size of the group,  $p_{i,t}^e$  is the price forecast by participant  $i$  and  $S(p_{i,t}^e)$  is the supply of producer  $i$  depending upon the forecast by participant  $i$ . Demand and supply curves  $D$  and  $S$  were fixed during all experiments (except for small random shocks to the demand curve) and unknown to the participants. We focus on the group experiments with  $K = 6$ , as in Hommes et al. (2007). Hommes et al. (2000) ran one-person experiments (i.e.  $K = 1$ ); Colucci and Valori (2006) use these one-person experiments to estimate various learning models. Solving (12) for the market equilibrium price, with a linear demand curve as in (4), yields

$$p_t = \frac{a - \frac{1}{K} \sum_{i=1}^K S(p_{i,t}^e)}{d} + \epsilon_t, \quad (13)$$

where  $\epsilon_t$  are IID demand shocks, which are drawn from a normal distribution  $N(0, 0.5)$ . In the experiments the parameters were fixed at  $a = 2.3$ ,  $d = 0.25$  and  $K = 6$ , and we used the nonlinear, S-shaped supply curve (geometrically similar to the S-shaped supply curve (3)):

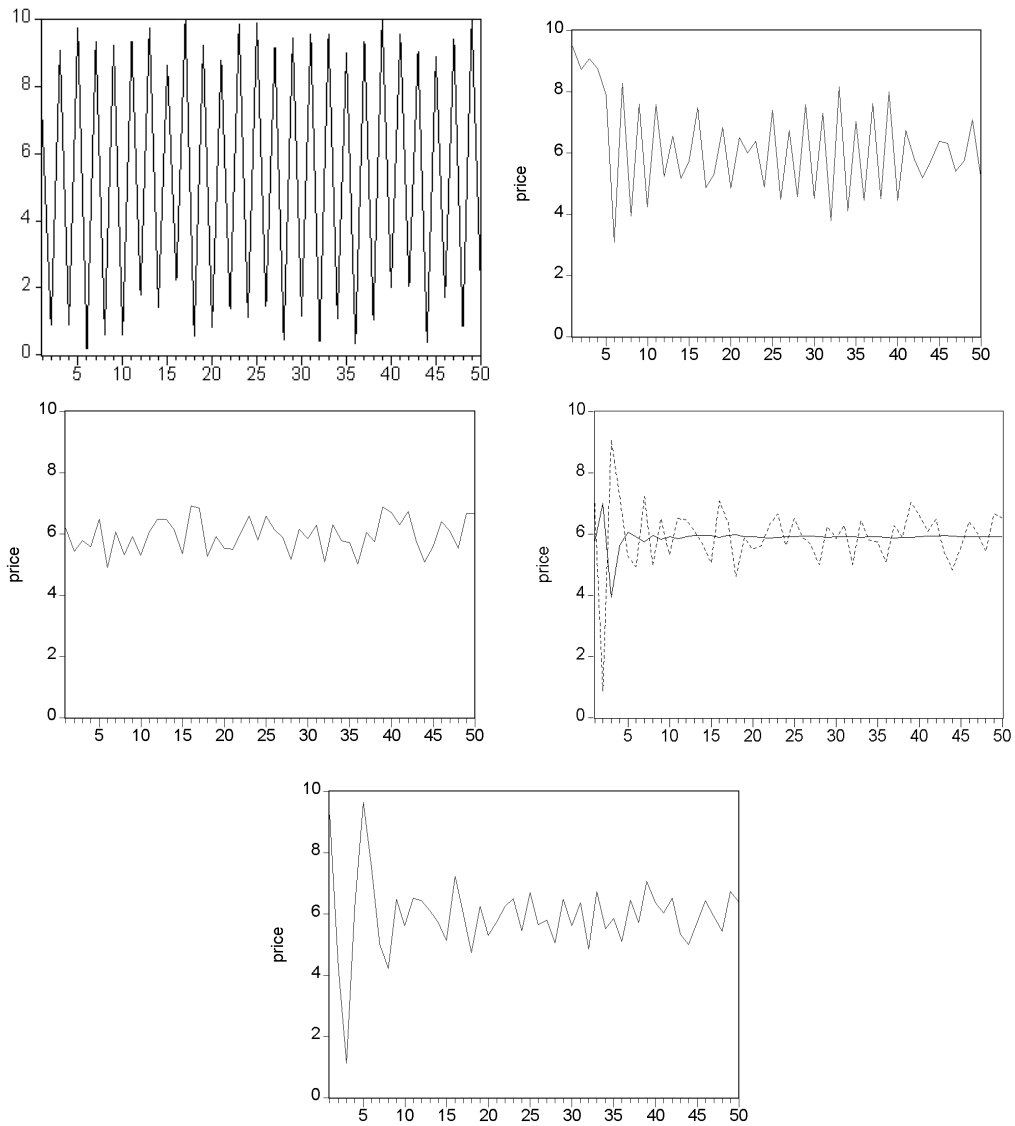
$$S(p_{i,t}^e) = \text{Tanh}(\lambda(p_{i,t}^e - 6)) + 1. \quad (14)$$

Expectation formation of the producers is the only part of the model that is affected by the participants in the experiments. Participants did not know underlying market equilibrium equations, nor were they informed about the distribution of any exogenous shocks to demand and/or supply. The participants were told that they were advisors to producers of an unspecified good and that the

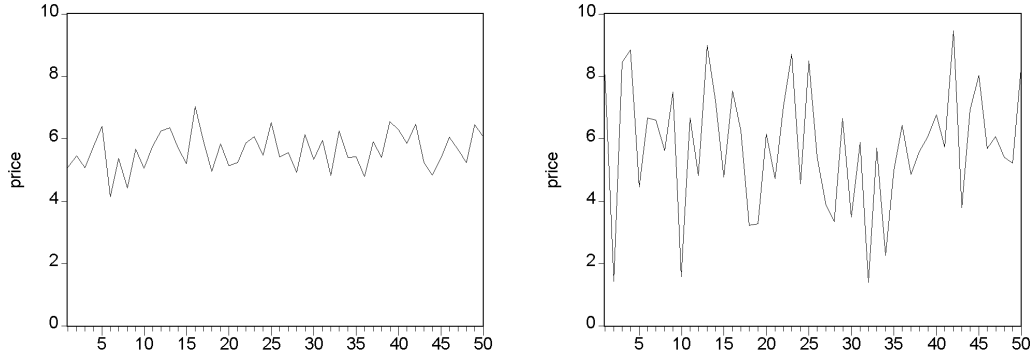
price was determined by market clearing. Based upon this information the participants were asked to predict next period's price. The predicted price had to be between 0 and 10 and the realized price was also always between 0 and 10. Participants' earnings in each period were a quadratic function of their squared forecasting error. The better their forecast, the higher their earnings. After every period the participants were informed about the realized price in the experiment. Also a time series of the participant's own prediction and a time series of the realized price in the experiment was shown on their computer screen.

Participants in the experiments therefore had little information about the price generating process and had to rely mainly upon time series observations of past prices and predictions. The information in the experiment was thus similar to the information assumption underlying much of the bounded rationality literature, where agents form expectations based upon time series observations. Our setup enables us to test the expectations hypothesis in a controlled dynamic environment. The main question was whether agents can learn and coordinate on the unique REE, in a world where consumers and producers act as if they were maximizing utility and profits, but where they do *not* know underlying market equilibrium equations and only observe time series of prices and expected prices. Our choice for a nonlinear, S-shaped supply curve enables us to investigate whether agents can avoid systematic forecasting errors, as would e.g. occur along a 2-cycle under naive expectations, or can even learn a REE steady state.

In their experiment, Hommes et al. (2007) considered a stable and an unstable treatment, which only differ in the parameter  $\lambda$  tuning the nonlinearity of the supply curve (14). In the stable treatment, if all subjects use naive expectations, prices converge to the RE steady state. In contrast, in the unstable treatment, if all subjects use naive expectations, prices diverge from the RE steady state and converge to the stable 2-cycle, with systematic forecasting errors, as illustrated in Figure 2 (top left panel). Figure 2 also illustrates what would happen in the unstable treatment of the experiment if all subjects would use one of the other well known benchmark expectations rules, namely adaptive expectations ( $w = 0.2$ ), rational expectations (i.e. use the RE price  $p^*$  as forecast),



**Figure 2:** Price fluctuations in the cobweb model under naive expectations (top left), adaptive expectations (top right), rational expectations (middle left), average price forecast (middle right) and SAC-learning (bottom).



**Figure 3:** *Realized market prices in two different cobweb group experiments. In the stable treatment (left panel;  $\lambda = 0.22$ ) the price quickly converges to the RE price, whereas in the unstable treatment (right panel;  $\lambda = 2$ ) prices do not converge but exhibit excess volatility, characterized by strongly fluctuating prices around the RE price.*

learning by average, that is, use the sample average

$$\bar{p}_t^e = \left( \sum_{j=0}^{t-1} p_j \right) / t, \quad (15)$$

as forecast as in Carlson (1969)), and sample autocorrelation (SAC) learning (i.e. updating sample average and first order sample autocorrelation coefficient, as discussed in detail in Section 4).

Figure 3 shows time series of the realized prices in two typical group experiments, one stable and one unstable treatment, and Table 1 summarizes the sample mean and sample variances, over the subsamples 1–25, 26–50 and the full sample 1–50, for both treatments and for the corresponding RE benchmarks. For both treatments, the sample mean of realized prices is very close to the (unknown) RE price. Moreover, in the stable treatment, the sample variance (0.44, 0.29 and 0.36 respectively, over the first half, the second half and the full sample) is close to the variance (0.25) of the RE benchmark. In contrast, in the unstable treatment the sample variance (4.75, 3.32 and 4.04 respectively, over the first half, the second half and the full sample) is significantly higher than the variance (0.25) of the RE benchmark, so that the unstable treatment exhibits excess volatility. Hommes et al. (2007) also look at autocorrelations in realized market prices, and find that there is no statistically significant autocorrelations in realized market prices, for both the stable and the

	1-25		26-50		1-50	
	sample average	sample variance	sample average	sample variance	sample average	sample variance
<b>Stable treatment (<math>\lambda = 0.22</math>)</b>						
RE	5.57	0.25	5.57	0.25	5.57	0.25
experiment	5.59	0.44	5.66	0.29	5.63	0.36
<b>unstable treatment (<math>\lambda = 2</math>)</b>						
RE	5.91	0.25	5.91	0.25	5.91	0.25
experiment	6.07	4.75	5.50	3.32	5.79	4.04

**Table 1:** Sample mean and sample variance of realized market prices in the laboratory experiments for the stable and the unstable treatment, over the full sample of 50 periods and over the subsamples of the first 25 and the last 25 periods, together with rational expectations benchmarks.

unstable treatments. Apparently, the heterogeneous interaction of individual forecasting rules has washed out all linear predictable structure in realized market prices. The *stylized facts* of realized market prices in the cobweb experiments may thus be summarized as follows:

1. the sample mean of realized market prices is very close to the RE price;
2. the sample variance of realized market prices depends on the treatment
  - (a) in the stable treatment the sample variance is very close to the RE benchmark;
  - (b) in the unstable treatment the sample variance is significantly higher than the RE benchmark;
3. there is no linear autocorrelation left in realized market prices.

One may say that the stable treatment converges to RE<sup>3</sup>, whereas the unstable treatment exhibits excess volatility, with prices fluctuating irregularly (no autocorrelations) and with high amplitude

<sup>3</sup>For different market settings, these results may of course change. The cobweb model has negative expectations feedback. Heemeijer et al. (2007) show in fact that the results are quite different in markets with positive feedback,

around the RE benchmark.

It is useful to compare these experimental results to the theoretical benchmarks illustrated in Figure 2<sup>4</sup>. These are *representative agent* benchmarks, where all agents use the same forecasting rule, and demand and supply are exactly the same as in the unstable treatment in the experiment. Naive expectations is clearly very different from the experiments, since it leads to high amplitude price fluctuation with regular, predictable up and down (noisy) period 2 oscillations. Adaptive expectations is also inconsistent with the experiments. Although the amplitude is smaller, the price fluctuations are too regular, with frequent up and down oscillations. In contrast to the experiments, the price series under adaptive expectations, for example, exhibits strong negative first order autocorrelation. The time series under rational expectations is very similar to the time series in the stable treatment (the exogenous shocks in the experiments are the same as for the RE benchmark simulation), but very different from the unstable treatment, which has a much larger amplitude. RE is therefore a good description in the stable treatment, but not in the unstable treatment. Finally, learning by average or by sample autocorrelation always leads to (quick) convergence to RE, which is inconsistent with the observed excess volatility in the unstable treatment of the experiments. None of these representative agent learning models thus can explain the cobweb experiments, suggesting that *heterogeneous expectations* play a key role in expectation formation of boundedly rational agents. Before turning to heterogeneous expectations models in Section 5, we discuss adaptive learning by an “average agent” in Section 4.

## 4 Adaptive Learning

Adaptive learning usually refers to the situation where agents use some parameterized rule, and update the parameters over time as additional observations become available. Agents thus try to learn the parameters of their rule, for example behaving as a time series econometrician using a such as demand driven speculative asset markets. Positive feedback may lead to persistent deviations from the fundamental benchmark, with the sample mean of realized prices e.g. much higher than the RE fundamental benchmark.

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<sup>4</sup>Arifovic (1994) investigates genetic algorithm learning in the cobweb model and compares the results to the cobweb experiments of Wellford (1989).

recursive ordinary least squares (OLS) updating rule. Marcet and Sargent (1989) contains early examples of such an approach; Evans and Honkapohja (2001) contains an extensive and excellent overview of adaptive learning in macroeconomics. Within the cobweb framework adaptive learning has been applied by Bray and Savin (1986).

Adaptive learning may provide a *learning story* how agents may learn a REE, without structural knowledge of market equilibrium equations but based on time series observations. In fact, we have seen an example already, since the average price forecast rule (15) can be obtained from OLS regression of prices on a constant. As we have seen, the average price forecast rule enforces convergence to the unique REE in the cobweb model. In cases when there are multiple REE, adaptive learning may be used as an equilibrium selection device, providing a justification of RE equilibria that are stable under learning.

However, adaptive learning need not always converge to REE. In particular, when the perceived law of motion (i.e. the law of motion agents believe in) is *misspecified* (i.e. different from the true law of motion), the learning process need not converge to a REE steady state, but may lead to some *boundedly rational learning equilibrium*, leading to expectations driven periodic or even chaotic fluctuations. Well known examples are Bullard (1994), Schönhofer (1999)<sup>5</sup>, and Bullard and Duffy (1998); Grandmont (1998) contains a discussion of (in-)stability conditions of adaptive learning rules. Evans and Honkapohja (2001) use the notion of *restricted perception equilibria* to describe a situation where agents' perceived law of motion is misspecified, and agents try to learn a rule which is optimal within a limited class of misspecified rules. Branch (2006) wrote a stimulating recent review on restricted perception equilibria, and their importance for macro.

The purpose of this section is to discuss a simple adaptive learning scheme, sample autocorrelation (SAC-)learning, as introduced by Hommes and Sorger (1998). In this setting agents are trying to learn the best *linear* rule (according to forecasting performance) in an unknown, *nonlinear*

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<sup>5</sup>Recently Tuinstra and Wagener (2007) have argued that cycles and chaos are the result of the estimation procedure in Bullard (1994) and Schönhofer (1999). If agents regress inflation rates on a constant, instead of regressing (nonstationary) prices on past prices, adaptive learning does converge to REE. Interestingly, Tuinstra and Wagener (2007) also show that in the case of heterogeneous agents, with agents switching between the two estimation procedures based upon past performance, complicated price fluctuations arise again.



economy. In equilibrium the linear rule has the same autocorrelation structure as the unknown nonlinear system, and Hommes and Sorger (1998) called this situation a consistent expectations equilibrium. Within the cobweb model, we will see that SAC-learning may sometimes converge to a REE steady state, but may also fail to converge and even lead to chaotic fluctuations and excess volatility.

**4.1 Consistent Expectations Equilibrium (CEE)** We start from a motivating example in chaos theory. Consider the dynamics

$$x_{t+1} = T_\beta(x_t), \tag{16}$$

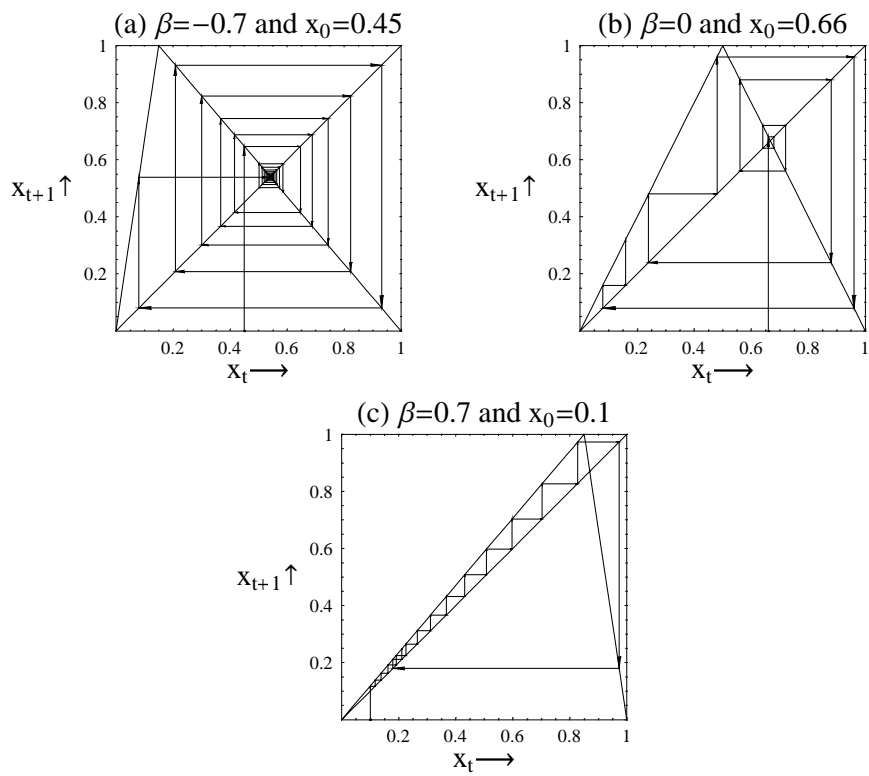
where  $-1 < \beta < +1$  and  $T_\beta(x) : [0, 1] \rightarrow [0, 1]$  is the 1-D piecewise linear *asymmetric tent map* (see the graphs in Figure 4)

$$T_\beta(x) = \begin{cases} \frac{2}{1+\beta}x, & 0 \leq x \leq \frac{(\beta+1)}{2} \\ \frac{2}{1-\beta}(1-x), & \frac{\beta+1}{2} < x \leq 1. \end{cases} \tag{17}$$

This piecewise linear map is expanding, that is,  $|T'_\beta(x)| > 1$ , and typical trajectories are chaotic. In particular, the following properties of the dynamics are well known (Bunow and Weiss (1979), Sakai and Tokumaru (1980)):

1. almost all time paths  $\{x_t\}_{t=0}^\infty$  are chaotic and dense in  $[0, 1]$ .
2. for almost all initial states  $x_0 \in [0, 1]$ , the *sample average* of the (chaotic) time path is  $\bar{x} = \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=0}^T x_t = \frac{1}{2}$ .
3. for almost all initial states  $x_0 \in [0, 1]$ , the *sample autocorrelation coefficient* at lag  $j$  is  $\rho_j = \beta^j$ .

These properties imply that the nonlinear dynamical system (16) has the same autocorrelation structure as a stochastic AR(1) process. Boundedly rational agents observing time series generated by the unknown nonlinear process (16) and using linear statistical techniques, might wrongly



**Figure 4:** Graphs of the asymmetric tentmap for different values of the parameter  $\beta$ . In each case, a typical trajectory is chaotic with sample mean  $1/2$ . The first order sample autocorrelation coefficient of a chaotic trajectory equals  $\beta$ .

believe that the time series is generated by a stochastic AR(1) process. This example motivated the concept of consistent expectations equilibrium, introduced in Hommes and Sorger (1998), building on earlier work in Hommes (1998) and Sorger (1998), and the concept of *self-fulfilling mistake* of Grandmont (1998).

We discuss the CEE concept within the cobweb model. Assume that agents believe that prices are generated by a stochastic AR(1) process. Given this perceived law of motion and prices known up to  $p_{t-1}$ , the predictor for  $p_t$  minimizing the mean squared prediction error is

$$p_t^e = \alpha + \beta(p_{t-1} - \alpha), \quad (18)$$

where the parameters  $\alpha$  and  $\beta$ ,  $\beta \in [-1, 1]$ , represent the long run average and the first order autocorrelation coefficient. Given that agents use the linear predictor (18), the *implied actual law of motion* for the cobweb model becomes

$$p_t = F_{\alpha, \beta}(p_{t-1}) := D^{-1}S(\alpha + \beta(p_{t-1} - \alpha)). \quad (19)$$

The *sample average* of a time series  $(p_t)_{t=0}^{\infty}$  is defined as

$$\bar{p} = \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=0}^T p_t \quad (20)$$

and the *sample autocorrelation coefficients* are given by

$$\rho_j = \lim_{T \rightarrow \infty} \frac{c_{j,T}}{c_{0,T}}, \quad j \geq 1, \quad (21)$$

where

$$c_{j,T} = \frac{1}{T+1} \sum_{t=0}^{T-j} (p_t - \bar{p})(p_{t+j} - \bar{p}), \quad j \geq 0. \quad (22)$$

A CEE is now defined as

DEFINITION. A triple  $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ , where  $(p_t)_{t=0}^{\infty}$  is a sequence of prices and  $\alpha$  and  $\beta$  are real numbers,  $\beta \in [-1, 1]$ , is called a *consistent expectations equilibrium* (CEE) if

1. the sequence  $(p_t)_{t=0}^{\infty}$  satisfies the implied actual law of motion (19),

2. the sample average  $\bar{p}$  in (20) exists and is equal to  $\alpha$ , and
3. the sample autocorrelation coefficients  $\rho_j, j \geq 1$ , in (21) exist and the following is true:
  - a. if  $(p_t)_{t=0}^\infty$  is a convergent sequence, then  $\text{sgn}(\rho_j) = \text{sgn}(\beta^j), j \geq 1$ ;
  - b. if  $(p_t)_{t=0}^\infty$  is not convergent, then  $\rho_j = \beta^j, j \geq 1$ .

Stated differently, a CEE is a price sequence together with an AR(1) belief such that expectations are self-fulfilling in terms of the *observable* sample average and sample autocorrelations. Along a CEE expectations are thus correct in a linear statistical sense.

**4.2 Sample Autocorrelation Learning** So far, the notion of CEE involves a given AR(1) belief, with fixed parameters  $\alpha$  and  $\beta$ . Now consider the more flexible situation of *adaptive learning* with agents updating their AR(1) belief parameters  $\alpha_t$  and  $\beta_t$  over time, as additional observations become available. A natural learning scheme fitting the framework of CEE is based upon sample average and sample autocorrelation coefficients.

For any finite set of past observations  $\{p_0, p_1, \dots, p_t\}$  the sample average is

$$\alpha_t = \frac{1}{t+1} \sum_{i=0}^t p_i, \quad t \geq 1 \quad (23)$$

and the first order sample autocorrelation coefficient is

$$\beta_t = \frac{\sum_{i=0}^{t-1} (p_i - \alpha_t)(p_{i+1} - \alpha_t)}{\sum_{i=0}^t (p_i - \alpha_t)^2}, \quad t \geq 1. \quad (24)$$

When, in each period, the belief parameters are updated according to (23) and (24) the (temporary) law of motion (19) becomes

$$p_{t+1} = F_{\alpha_t, \beta_t}(p_t) = D^{-1}S(\alpha_t + \beta_t(p_t - \alpha_t)), \quad t \geq 0. \quad (25)$$

We call the dynamical system (23) - (25) the actual dynamics with *sample autocorrelation learning* (SAC-learning)<sup>6</sup>. The initial state for the system (23– 25) can be any triple  $(p_0, \alpha_0, \beta_0)$  with  $\beta_0 \in [-1, 1]$ .

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<sup>6</sup>SAC-learning is closely related to OLS-learning. For both learning schemes  $\alpha_t$  is the same, but  $\beta_t$  has one extra term in the denominator under SAC-learning, ensuring that  $-1 \leq \beta_t \leq +1$ . Simulations for OLS-learning lead to similar results and would not change our general conclusions below.

Which type of CEE exist in the cobweb model, and to which of them will the SAC-learning dynamics converge? Hommes and Sorger (1998) show that in the most relevant case, when demand is decreasing and supply is increasing, the *only* CEE is the RE steady state price  $p^*$ . This means that, even when underlying market equilibrium equations are *not* known, agents should be able to learn and coordinate on the REE price simply by looking at sample averages and sample autocorrelations. Although other simple forecasting rules, such as adaptive expectations, might lead to chaotic price fluctuations, these forecasting rules are *inconsistent* in terms of sample autocorrelations. Hence, in a nonlinear cobweb economy with monotonic demand and supply, boundedly rational agents should, at least in theory, be able to learn the unique REE from time series observations<sup>7</sup>.

In general however, given an AR(1) belief, there are at least three possible types of CEE:

- a *steady state CEE* in which the price sequence  $(p_t)_{t=0}^{\infty}$  converges to a steady state  $p^*$ , with  $\alpha = p^*$  and  $\beta = 0$ ;
- a *2-cycle CEE* in which the price sequence  $(p_t)_{t=0}^{\infty}$  converges to a period two cycle  $\{p_1^*, p_2^*\}$ ,  $p_1^* \neq p_2^*$ , with  $\alpha = (p_1^* + p_2^*)/2$  and  $\beta = -1$ ;
- a *chaotic CEE* in which the price sequence  $(p_t)_{t=0}^{\infty}$  is chaotic, with sample average  $\alpha$  and autocorrelations  $\beta^j$ .

Which of these cases occurs in the cobweb model depends on the composite mapping  $D^{-1}S$  in (19), determined by demand and supply curves. We will discuss a chaotic example below.

**4.3 CEE in a fishery model with backward bending supply** As an illustration of chaotic CEE, we briefly discuss the fishery model in Hommes and Rosser (2001), with a backward bending supply curve derived from optimal management of the fish resources, by a sole owner maximizing

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<sup>7</sup>Bray and Savin (1986) show that OLS learning also converges to the REE steady state in the cobweb model. Arifovic (1994) shows that agents using genetic algorithms can learn the REE steady state.

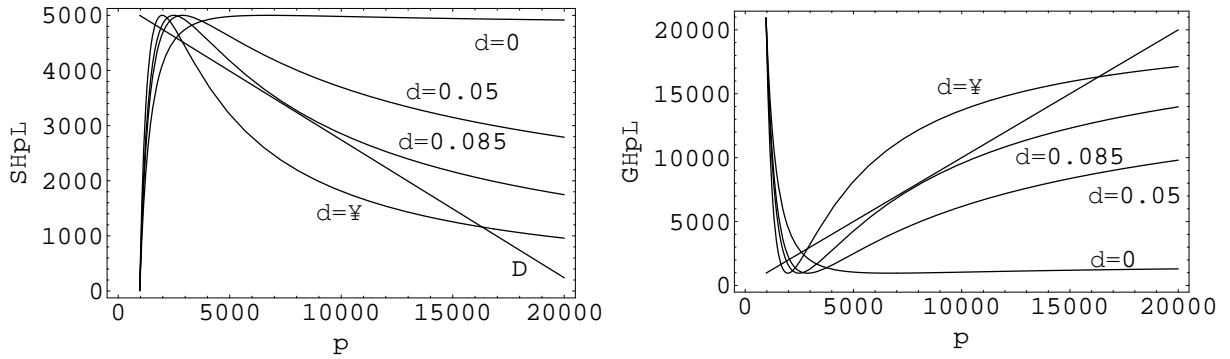
discounted revenues from harvesting. The backward bending supply curve is given by

$$S_\delta(p) = h = F(x_\delta^*(p)) = rx_\delta^*(p)\left(1 - \frac{x_\delta^*(p)}{k}\right). \quad (26)$$

Supply equals harvest  $h$ , which in sustained yield equilibrium has been set equal to the (logistic) growth of the fish population  $x$ . Moreover,  $x_\delta^*(p)$  is the optimal bioeconomic equilibrium fish population derived from maximization of discounted future net revenues, which depends on the discount rate  $\delta$  and the fish price  $p$ . In the case of logistic growth of the fish population and a standard specification of the harvesting cost function  $c(x) = c/(qx)$ , it is given by

$$x_\delta^*(p) = \frac{k}{4} \left\{ 1 + \frac{c}{pqk} - \frac{\delta}{r} + \sqrt{\left(1 + \frac{c}{pqk} - \frac{\delta}{r}\right)^2 + \frac{8c\delta}{pqkr}} \right\}. \quad (27)$$

For details, the interested reader is referred to Hommes and Rosser (2001), and to Clark (1985,1990) for an extensive treatment of fishery and other renewable resource models<sup>8</sup>.



**Figure 5:** (a) Graphs of the demand and the discounted equilibrium supply curves  $S_\delta$  in (26) (left panel) and (b) graphs of the implied law of motion  $G_\delta$  in (29) under naive expectations (right panel) for several discount factors  $\delta$ . As the discount factor  $\delta$  increases the supply curve becomes strongly backward bending and two additional steady states are created for  $\delta \approx 0.085$ .

We refer to  $S_\delta(p)$  in (26) as the *discounted equilibrium supply curve*. For consumer demand for fish, we will choose the same linear form (4) as before. Figure 5 shows plots of the equilibrium

<sup>8</sup>Parameters will be fixed as follows: the fish carrying capacity  $k = 400.000$ , catchability (per vessel per day)  $q = 0.000013$ , the marginal cost of effort  $c = 5000$  and the growth rate of fish  $r = 0.05$ .

demand and supply system, for different values of the discount rate  $\delta$ <sup>9</sup>. Figure 5 (left panel) shows that, as the discount rate  $\delta$  increases, the supply curve becomes more backward bending. The most backwardly bent supply curve corresponds with the totally myopic case of  $\delta = \infty$ , which corresponds to the open access bionomic equilibrium case studied by Gordon (1954) and which is associated with overfishing behavior. We note that the supply curve bends backwards quite quickly at values of the discount rate that are empirically and socially meaningful. Figure 5 illustrates the fact that a backward-bending supply curve together with a sufficiently inelastic demand curve may lead to multiple steady state equilibria even for the static case. In the extreme case  $\delta = 0$  there is a unique steady state equilibrium price, whereas at the other extreme  $\delta = +\infty$  there are three different steady state equilibrium prices.

The market equilibrium price at date  $t$  is determined by demand and supply, i.e.,

$$D(p_t) = S_\delta(p_t^e). \quad (28)$$

With linear consumer demand  $D$  as before in (4), the discounted supply curve  $S_\delta$  in (26), and price expectations given by SAC-learning, the implied actual law of motion becomes

$$p_{t+1} = G_\delta(\alpha_t + \beta_t(p_t - \alpha_t)) = D^{-1}S_\delta(\alpha_t + \beta_t(p_t - \alpha_t)) = \frac{a - S_\delta(\alpha_t + \beta_t(p_t - \alpha_t))}{d}, \quad (29)$$

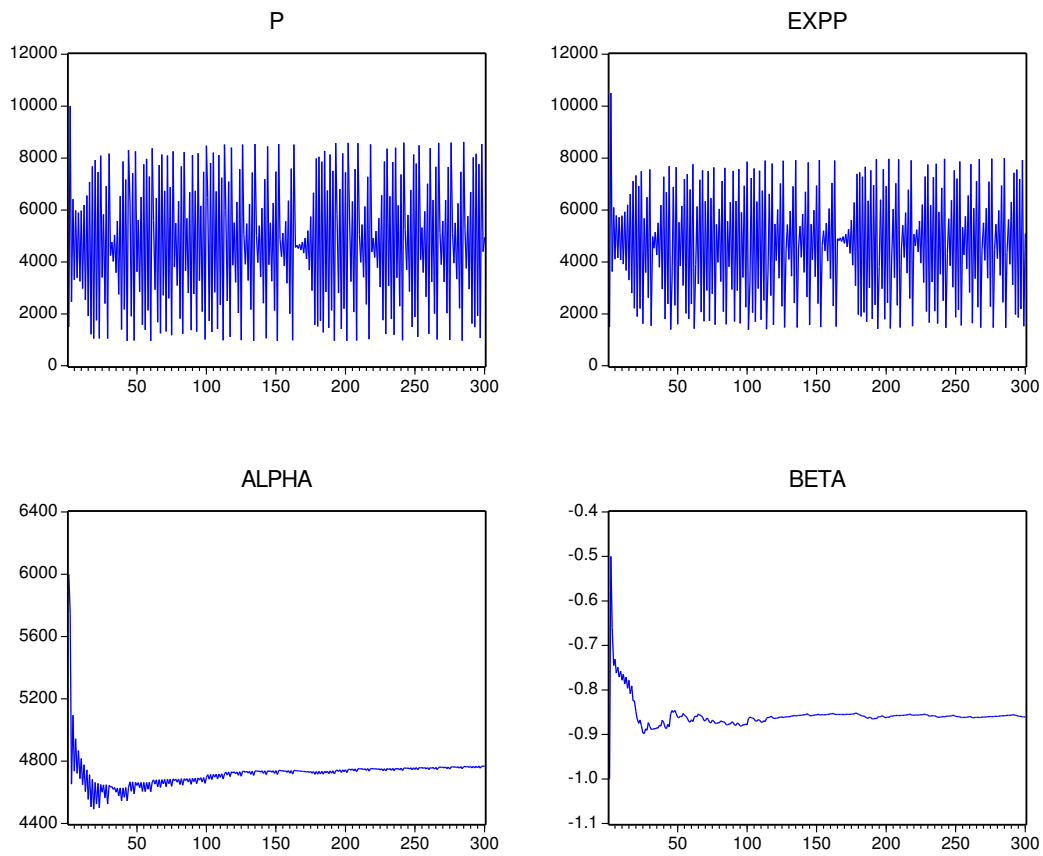
with  $\alpha_t$  given by (23) and  $\beta_t$  by (24). Figure 5 (right panel) shows graphs of the implied actual law of motion  $G_\delta$ , for different values of the discount rate. In our simulations of the adaptive SAC-learning process (23), (24) and (29), we have observed three typical outcomes:

- convergence to the “good” steady state equilibrium with a low price and a high fish stock
- convergence to the “bad” steady state equilibrium with a high price and a low fish stock
- convergence to a chaotic CEE, with prices and fish stock irregularly jumping between low and high values

Simulations of the SAC-learning dynamics suggest that for low values of the discount rate convergence to the “good” equilibrium steady state is the “most likely” outcome of the SAC learning

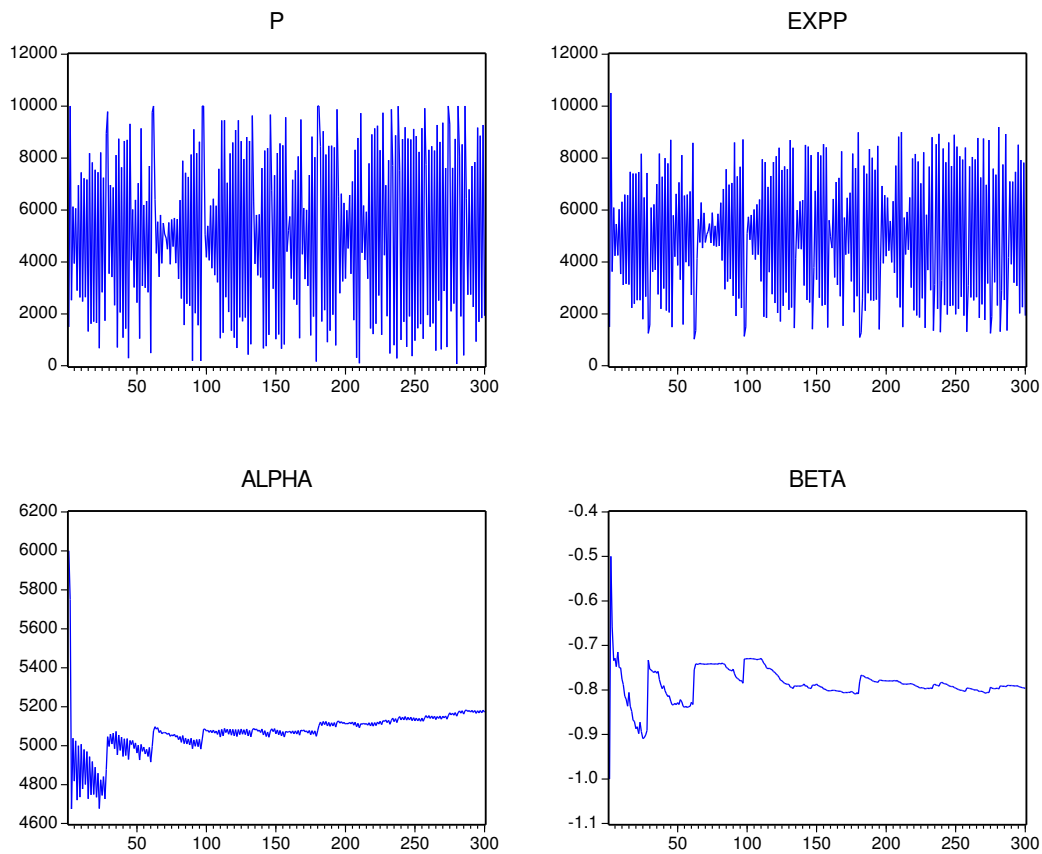
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<sup>9</sup>The parameters of the demand curve are fixed at  $d = 0.25$  and  $a = 5240.5$ .



**Figure 6:** Learning to believe in chaos, that is, belief parameters converge (bottom panels), while prices and forecasts (top panel) fluctuate chaotically. In this example, belief parameters  $(\alpha_t, \beta_t) \rightarrow (\alpha^*, \beta^*) \approx (4988, -0.87)$ .





**Figure 7:** Learning to believe in noisy chaos, that is, belief parameters converge (bottom panels), while dynamics of prices and forecasts (top panel) follow a noisy chaotic process. In this example, belief parameters  $(\alpha_t, \beta_t) \rightarrow (\alpha^*, \beta^*) \approx (5200, -0.8)$ .

process, whereas for high values of the discount rate convergence to the “bad” steady state is most likely. For intermediate discount rates the outcome of the learning process is uncertain and in general depends on the initial states, i.e. on the initial belief parameters  $\alpha_0, \beta_0$  and the initial fish stock  $x_0$ . The system may settle down to either the “good” or the “bad” steady state, possibly after a long (chaotic) transient. However, it may also happen that belief parameters  $\alpha_t$  and  $\beta_t$  converge to constants  $\alpha^*$  and  $\beta^*$ , while prices never converge to a steady state (or to a cycle), but keep fluctuating chaotically, as illustrated in Figure 6 for  $\delta = 0.1$ . This situation is referred to as *learning to believe in chaos* and it seems to occur with positive probability, that is, for an open set of initial states  $(x_0, \alpha_0, \beta_0)$ . Learning to believe in chaos means that the SAC-learning dynamics converges to a chaotic system, when  $\alpha_t$  and  $\beta_t$  have converged to constants  $\alpha^*$  and  $\beta^*$ , while prices keep fluctuating chaotically<sup>10</sup>.

Next we investigate the effect of noise upon the learning dynamics. SAC-learning with additive dynamic noise is given by (23), (24), as before, and adding a noise term to the implied actual law of motion, i.e.

$$p_{t+1} = G_{\delta, \alpha_t, \beta_t}(p_t) = G_{\delta}(\alpha_t + \beta_t(p_t - \alpha_t)) + \epsilon_t, \quad t \geq 0, \quad (30)$$

where  $\epsilon_t$  is an independently identically distributed (IID) random process and  $G_{\delta} = D^{-1}S_{\delta}$  in (29) as before. Notice that the noise is not merely observational noise, but dynamic noise, e.g. due to exogenous demand or supply shocks, affecting the dynamic law of motion in each period of time. Figure 7 illustrates a typical example, with  $\epsilon_t$  drawn from a uniform distribution over the interval  $[-1000, +1000]$ ; for this choice of the noise process, the signal to noise ratio, as measured by the ratio  $\sigma_p/\sigma_{\epsilon}$  of standard deviations of the noise free price series to the noise, is about 5. Surprisingly, even in the presence of dynamic noise, the SAC-learning dynamics still settles down to a chaotic CEE as illustrated in Figure 7. The noisy chaotic series has an autocorrelation pattern very similar to that of an AR(1) process with strongly negative first order autocorrelation. In fact, estimation of an AR(1) process of the noisy chaotic series yields estimated parameters  $\hat{\beta}$  and  $\hat{\alpha}$  which are close

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<sup>10</sup>Schönhofer (1999, 2001) presents a related case of learning to believe in chaos in an OLG-model. In Schönhofer’s examples belief parameters of the OLS-learning scheme do *not* converge but keep fluctuating chaotically, while at the same time, due to inflation, prices diverge to infinity.

to the coefficients of the underlying chaotic CEE  $\beta^* \approx -0.87$  and  $\alpha^* \approx 4988$ . Standard statistical tests such as the Q-statistic indicate that the null hypothesis that prices follow a stochastic AR(1) process can *not* be rejected, not even at the 10% level. Learning to believe in noisy chaos is thus a possibility which is not rejected by linear statistical theory. Agents are therefore satisfied with their linear forecasting rules, and have no reason to abandon their belief and will stick to their AR(1) belief in an unknown stochastic nonlinear economy.

The key feature of a (noisy) chaotic CEE is that learning parameters converge to constants, whereas prices do not converge but fluctuate chaotically on a (noisy) strange attractor, with the correct sample average and sample autocorrelations. A chaotic CEE may be seen as an example of what Sargent (1993) calls an *approximate rational expectations equilibrium*, with optimal misspecified forecasts. A chaotic CEE is not a REE, because expectations do not coincide with the conditional mathematical expectations, which could only be derived if underlying market equilibrium equations would be known. Agents are using a simple, but misspecified model to forecast an unknown, possibly complicated actual law of motion. In the presence of (small) dynamic noise, the misspecification is hard to detect and boundedly rational agents using linear statistical techniques can do no better than stick to their optimal, simple linear model of an unknown stochastic, nonlinear economy.

## 5 Heterogeneous Beliefs and Evolutionary Selection

So far we have focused on a representative agent cobweb model, where all producers have identical expectations. But why would all agents have the same expectations? Laboratory experiments have shown that, even when individuals face the same information, they may disagree and take different decisions. In a complex market it seems more appropriate to model agents as boundedly rational and heterogeneous, using different types of forecasting rules. But this raises an immediate problem: which rules will boundedly rational agents choose from the infinitely many possibilities?

Models with heterogeneous agents are becoming increasingly popular. In particular, in finance models with fundamentalists and chartists have received much attention. Examples include Zeeman (1974), Frankel and Froot (1986), DeLong et al. (1990), Kirman (1991), Lux (1995), Brock

and Hommes (1998) and DeGrauwe and Grimaldi (2006). Hommes (2006) and LeBaron (2006) have recently reviewed this rapidly expanding literature.

In this section we discuss a model with heterogeneous expectations, as proposed in Brock and Hommes (1997) (henceforth BH) based on three underlying assumptions: (i) agents choose from a class of rules varying from very simple to very sophisticated; (ii) more sophisticated rules require more effort and are therefore more costly than simple rules, and (iii) agents tend to switch to rules that have performed better in the recent past. Evolutionary selection thus disciplines the forecasting rules to be used. In the cobweb framework, producers can choose between different forecasting rules  $H_j$ . The fractions  $n_{j,t}$  of producers using predictor  $H_j$  at date  $t$ , will be updated over time based upon a publically available evolutionary fitness measure, given by realized net profits, associated to each predictor.

BH focus on a simple two type case with rational expectations, which can be obtained at costs  $C \geq 0$  per time period, versus naive expectations, which is freely available. This case may be viewed as an extreme case, with rational expectations representing the most sophisticated forecasting rule, and naive expectations representing the simplest forecasting rule. BH show the occurrence of a *rational route to randomness*, i.e. a bifurcation route to strange attractors and chaos as traders become more rational in the sense that they become more sensitive to differences in past performance and switch more quickly to a better predictor.

Rational agents have perfect knowledge about market equilibrium equations and are aware of the fact that the market equilibrium price is affected by the presence of naive traders. Hence, in a heterogeneous world rational agents have perfect knowledge about prices and quantities, but also about *beliefs* of *all* other traders. Although this case is theoretically appealing, it seems highly unrealistic in real markets that some agents have (perfect) information about beliefs of other agents. Therefore we will focus here on some perhaps more realistic cases, where agents only use information extracted from observable quantities, such as prices. As a starting point of the discussion, we consider the case of two simple linear predictors. Two special cases will be discussed, the case of *fundamentalists* versus naive expectations and the case of *contrarians* versus naive.

### 5.1 Linear forecasting rules

Consider the two linear AR(1) prediction rules

$$H_j(p_{t-1}) = \alpha_j + \beta_j p_{t-1}, \quad j = 1, 2, \quad (31)$$

with fixed parameters  $\alpha_j$  and  $\beta_j$ . Throughout this section we focus on the case where the supply curve is *linear* as in (2), with corresponding cost function  $c(q) = q^2/(2s)$ . The market clearing price in the cobweb model with linear demand and supply and two trader types, with linear predictors as in (31), is determined by<sup>11</sup>

$$a - dp_t = n_{1t}s(\alpha_1 + \beta_1 p_{t-1}) + n_{2t}s(\alpha_2 + \beta_2 p_{t-1}), \quad (32)$$

where  $n_{1t}$  and  $n_{2t}$  denote the fractions of agents using respectively  $H_1$  and  $H_2$ , at the beginning of period  $t$ . These fractions will be updated according to an evolutionary fitness measure based on past realized profits. Realized net profit in period  $t$  for traders using predictor  $H_j$  is given by

$$\pi_{j,t} = sp_t H_j(p_{t-1}) - \frac{s}{2}(H_j(p_{t-1}))^2 - C_j, \quad (33)$$

where  $C_j$  represents the average costs per time period for obtaining predictor  $H_j$ . For a simple habitual rule of thumb predictor, such as naive or adaptive expectations, these costs  $C_j$  will be zero, whereas for more sophisticated predictors such as fundamentalists beliefs based on fundamental analysis, information gathering costs  $C_j$  may be positive. The fitness measure underlying evolutionary selection is given by

$$U_{jt} = wU_{j,t-1} + (1 - w)\pi_{j,t}, \quad (34)$$

where  $0 \leq w \leq 1$  is a memory parameter. A smaller  $w$  puts more memory on recent observations and in the case  $w = 0$  fitness is given by most recent observed realized net profits.

Brock and Hommes (1997) considered this model with *synchronous* updating of strategies, that is, in each period *all* agents update their strategies. Here we consider the more general case of *asynchronous* updating. Per time unit only a fraction  $1 - \delta$  of agents, distributed randomly among

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<sup>11</sup>In our simulations we will work in deviations  $x_t = p_t - p^*$  from the fundamental RE steady state price  $p^*$ . This is equivalent to setting the parameter  $a = 0$ , so that the RE steady state  $p^* = a/(d + s) = 0$ .

agents of both types and independently across time, is assumed to reconsider their strategy on the basis of the most recent information available. The remaining fraction  $\delta$  sticks to their current strategy. The corresponding dynamics of the fractions is given by a modified version of the discrete choice, logit probabilities:

$$n_{jt} = (1 - \delta)e^{\gamma U_{j,t-1}}/Z_{t-1} + \delta n_{j,t-1}, \quad (35)$$

where  $Z_{t-1} = \sum_h e^{\gamma U_{h,t-1}}$  is a normalization factor so that fractions add up to 1. For  $\delta = 0$ , we are back in the case of synchronous updating. In evolutionary games there has been a discussion whether asynchronous updating may lead to more stability (cf. Nowak and May (1992), Huberman and Glance (1993) and Nowak et al. (1994)). Financial market models with asynchronous updating have been considered by Diks and van der Weide (2005) and Hommes, Huang and Wang (2005). A key feature of this evolutionary predictor selection is that agents are *boundedly rational*, in the sense that predictors with higher evolutionary fitness attract more followers. The parameter  $\gamma$  is called the *intensity of choice*, measuring how fast producers switch between different prediction strategies. For  $\gamma = 0$  the fractions always converge to equal shares  $1/H$ , whereas for the other extreme  $\gamma = \infty$ , in each period *all* producers who update in that period (i.e., a fraction  $1 - \delta$ ) switch to the optimal predictor. Hence, the higher the intensity of choice, the more rational agents are in the sense that they switch more quickly to the best strategy in terms of past performance.

The timing of the coupling between the market equilibrium equation (32) and the evolutionary selection of strategies (35) is important. The market equilibrium price  $p_t$  in (32) depends upon the fractions  $n_{ht}$ . These fractions  $n_{ht}$  depend upon *past* fitness  $U_{h,t-1}$ , which in turn depends upon past prices  $p_{t-1}$  in periods  $t - 1$  and further in the past. After the equilibrium price  $p_t$  has been revealed by the market, it is used in evolutionary updating of beliefs and determining the new fractions  $n_{h,t+1}$ . These new fractions  $n_{h,t+1}$  will then determine a new equilibrium price  $p_{t+1}$ . Market equilibrium prices and fractions of different trading strategies thus co-evolve over time.

**5.2 Fundamentalists versus naive expectations** The linear predictors (39) specialize to the case with fundamentalists versus naive expectations when  $\alpha_1 = p^* = a/(d + s)$  (the steady state

price),  $\beta_1 = 0$ ,  $\alpha_2 = 0$  and  $\beta_2 = 1$ :

$$H_1(p_{t-1}) = p^* = \frac{a}{d+s} \quad (36)$$

$$H_2(p_{t-1}) = p_{t-1}. \quad (37)$$

Figure 8 shows attractors for different values of the intensity of choice  $\gamma$  and some corresponding time series. The Bifurcation diagram and largest LE-plot in Figure 9 illustrate a *rational route to randomness*, i.e. a bifurcation route from simple to complicated, chaotic dynamics as the intensity of choice increases. The market switches between periods of low volatility, with prices close to the fundamental price, and high volatility, with irregularly switching prices. Prices diverge slowly from the fundamental steady state price, as long as most agents use the simple, freely available naive forecast. When forecasting errors increase, it becomes worthwhile to buy the sophisticated fundamental forecast, and more agents start switching to the fundamental forecast, thus stabilizing price fluctuations, etc. Due to the *asynchronous* updating of strategies, agents switch more gradually between strategies, and the time series of fractions of fundamentalists shows much more persistence than in the case with synchronous updating. Figure 8 (bottom panel) also illustrates the sample average and first order sample autocorrelation (SAC) of the price series. Sample average quickly settles down to a value close to  $0^{12}$ , whereas the first order SAC is clearly negative, converging to approx.  $-0.85$ .

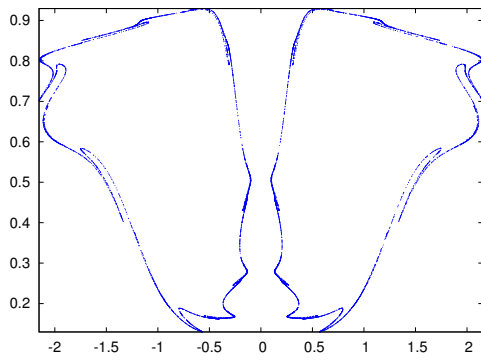
It is remarkable that both the attractor and the price time series are similar to the case of rational versus naive expectations studied in BH, and in particular for  $\gamma$  large, the system is close to a homoclinic tangency<sup>13</sup>.

**5.3 Contrarians versus naive expectations** In the case of fundamentalists versus naive, price series exhibit strong first order negative autocorrelations, even when the dynamics is chaotic. This

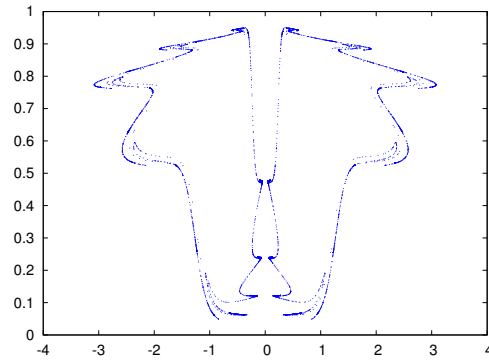
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<sup>12</sup>Recall that the simulations are in deviations  $x_t = p_t - p^*$  from the fundamental, so that the sample average of prices converges to fundamental value.

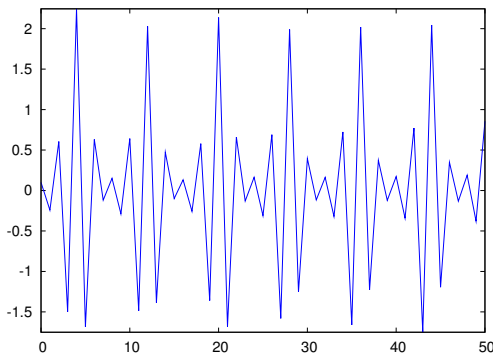
<sup>13</sup>See the original working paper Brock and Hommes (1995) for more details concerning the case of fundamentalists versus naive expectations.



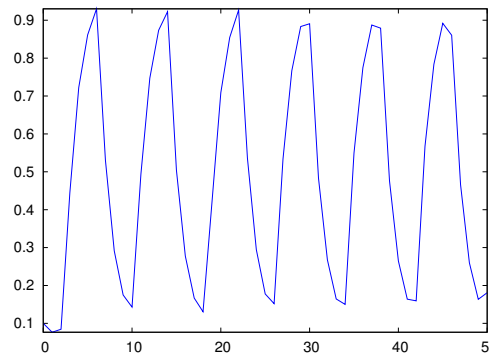
(a)  $\gamma = 3$ : strange attractor



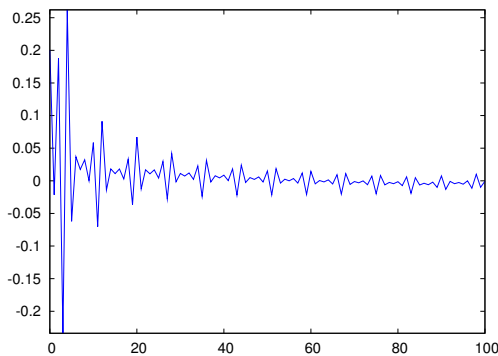
(b)  $\gamma = 6$ : strange attractor



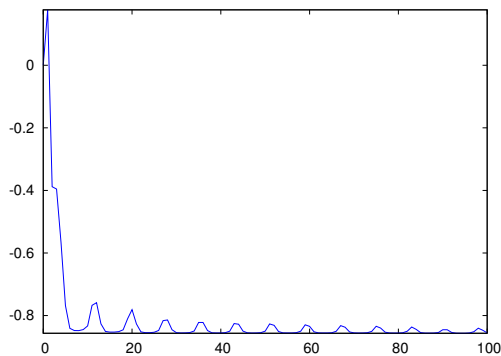
(c)  $\gamma = 3$ : price deviations from fundamental



(d)  $\gamma = 3$ : fraction of fundamentalists



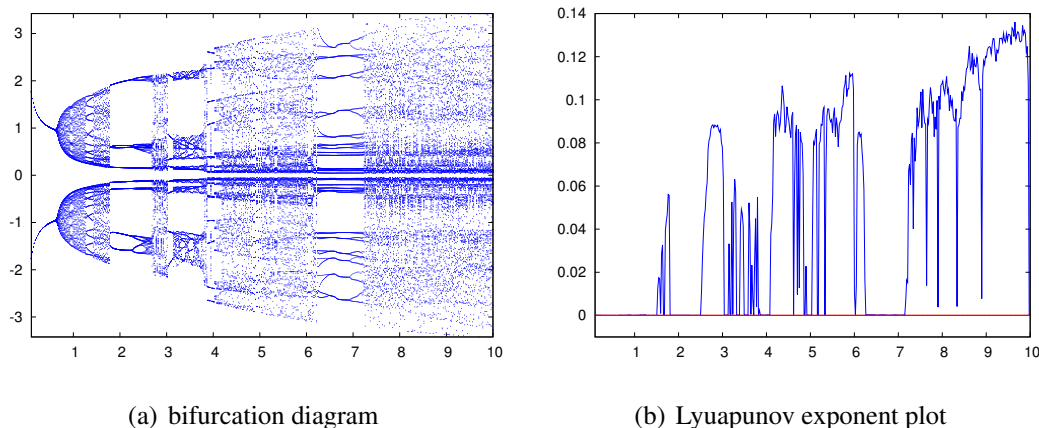
(e)  $\gamma = 3$ : sample average of prices



(f)  $\gamma = 3$ : sample autocorrelation of prices

**Figure 8:** *Fundamentalists versus naive. Strange attractors and time series for different  $\gamma$ -values, with other parameters fixed at  $a = 0$ ,  $d = 0.5$ ,  $s = 1.35$ ,  $\delta = 0.5$ ,  $\alpha_1 = 0$ ,  $\beta_1 = 0$ ,  $C_1 = 1$ ,  $\alpha_2 = 0$ ,  $\beta_2 = 1$  and  $C_2 = 0$ . Although price dynamics is chaotic, there is still clear linear autocorrelation structure. Sample average of prices converges (close) to fundamental value, while sample autocorrelations converge (close) to  $-0.85$ , indicating significantly negative first order autocorrelation.*





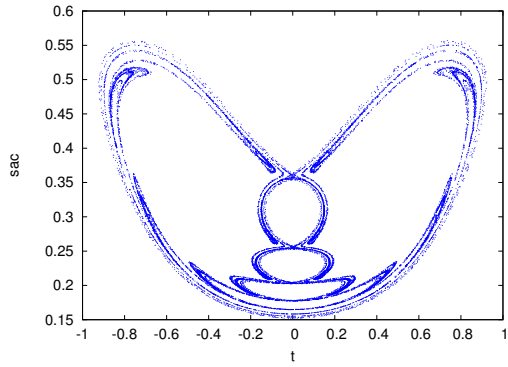
**Figure 9:** *Fundamentalists versus naive: a rational route to randomness as the intensity of choice increases. Parameters:  $a = 0$ ,  $d = 0.5$ ,  $s = 1.35$ ,  $\delta = 0.5$ ,  $\alpha_1 = 0$ ,  $\beta_1 = 0$ ,  $C_1 = 1$ ,  $\alpha_2 = 0$ ,  $\beta_2 = 1$  and  $C_2 = 0$ .*

has been illustrated in Figure 8 showing that, for  $\gamma = 3$ , the sample autocorrelations of prices converges to a negative value around  $-0.85$ . An agent who behaves as a time series econometrician would easily detect this strong negative autocorrelation and adapt her forecasts. Even without the use of any statistical software, a smart agent might detect negative autocorrelation, simply by observing that positive (negative) deviations from the average price are always followed by negative (positive) deviations. What would happen if agents recognize this structure from observing realized market prices?

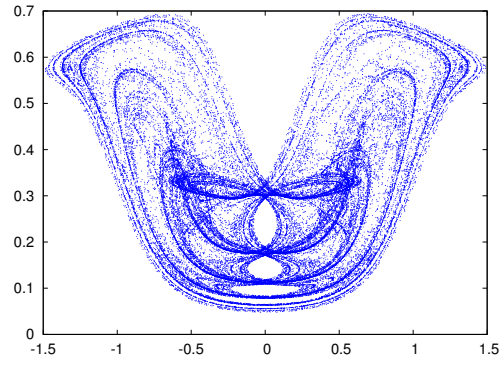
Consider a group of *contrarians*, who take the negative first order autocorrelation in prices into account, and predict that next period's deviation from the fundamental price will be on the opposite side, that is, we replace the fundamental forecast by a contrarian rule

$$H_1(p_{t-1}) = p^* + \beta_1(p_{t-1} - p^*). \quad (38)$$

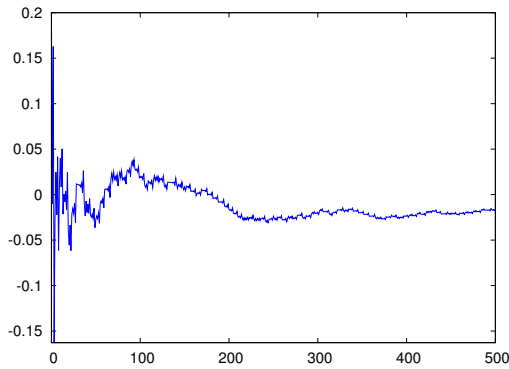
Figure 10 illustrates an example with  $\beta_1 = -0.85$  (with other parameters as in Figure 8), that is, contrarians recognize the autocorrelation structure present in the previous example. The structure of the strange attractors in Figure 10 seems more complicated than in the case of fundamentalists versus naive in Figure 8. In particular, the negative autocorrelation in prices becomes weaker, since



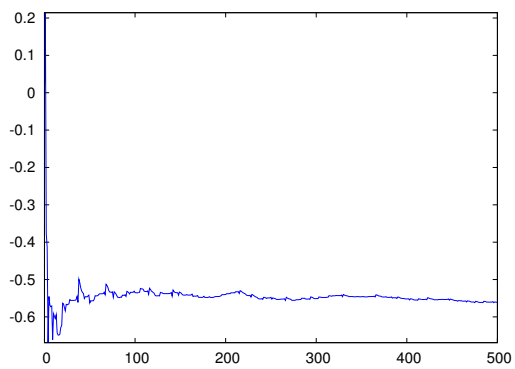
(a) strange attractor for  $\gamma = 1.72$



(b) strange attractor for  $\gamma = 3$



(c)  $\gamma = 3$ : sample average of prices



(d)  $\gamma = 3$ : sample autocorrelation of prices

**Figure 10:** *Contrarians versus naive. Strange attractors (top panel) for different  $\gamma$ -values, with other parameters fixed at:  $a = 0$ ,  $d = 0.5$ ,  $s = 1.35$ ,  $\delta = 0.5$ ,  $\alpha_1 = 0$ ,  $\beta_1 = -0.85$ ,  $C_1 = 1$ ,  $\alpha_2 = 0$ ,  $\beta_2 = 1$  and  $C_2 = 0$ . Time series of sample average and (first order) sample autocorrelation converge. Compared to fundamentalists, contrarians weaken the negative first order autocorrelations in prices, and in the long run sample autocorrelations converge to  $-0.57$ .*

the sample autocorrelation  $\rho_t \rightarrow -0.57$  (instead of  $-0.85$ ). Due to the presence of contrarians in the market, the strongly negative first order autocorrelation has become weaker<sup>14</sup>. In a sense, contrarians have “arbitraged away, at least partly, predictable linear structure in the price time series<sup>15</sup>.

**5.4 Adaptive learning versus naive expectations** In this subsection, we combine evolutionary strategy selection and adaptive learning. In the previous example we have seen that the presence of contrarians in the market weakens the first order autocorrelations in realized prices. A time series econometrician might note however, that contrarian behaviour is still *inconsistent* with realized prices since contrarians expect strong negative autocorrelation  $\beta_2 = -0.85$ , while realized prices exhibit weaker SAC, with first order SAC  $\beta_t \rightarrow -0.57$ . It is natural to go one step further and introduce a type of agent with adaptive learning, trying to optimize the parameters of her linear forecasting rule:

$$H_1(p_{t-1}) = \alpha_{t-1} + \beta_{t-1}(p_{t-1} - \alpha_{t-1}), \quad (39)$$

where  $\alpha_t$  and  $\beta_t$  are determined through SAC-learning as in (23) and (24) respectively. This approach widens the range of forecasting rules to all linear AR(1) rules. The sophisticated agent type tries to learn the optimal linear rule through adaptive learning, within a heterogeneous agent environment. Recently, Branch and Evans (2006) have studied a related cobweb model with two types of agents, both using OLS-learning of a misspecified model, and allowing for endogenous, evolutionary switching between the two predictors. DeGrauwe and Markiewicz (2006) compare evolutionary learning and adaptive (or statistical) learning in an asset pricing framework, and investigate how these different learning schemes replicate the stylized facts (disconnect puzzles, excess volatility) in exchange rates. Diks and Dindo (2006) consider an asset pricing model with heterogeneous information combining adaptive learning of the growth rate of dividends and evolutionary switching between free riding and costly information gathering.

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<sup>14</sup>The attractors in Figure 10 suggest that there are transversal intersections between the stable and the unstable manifolds of the steady state. Apparently homoclinic bifurcations are weakening the autocorrelation structure.

<sup>15</sup>See Dindo (2006) and Dindo and Tuinstra (2006) for similar results in the context of the well known El-Farol bar problem and related models.

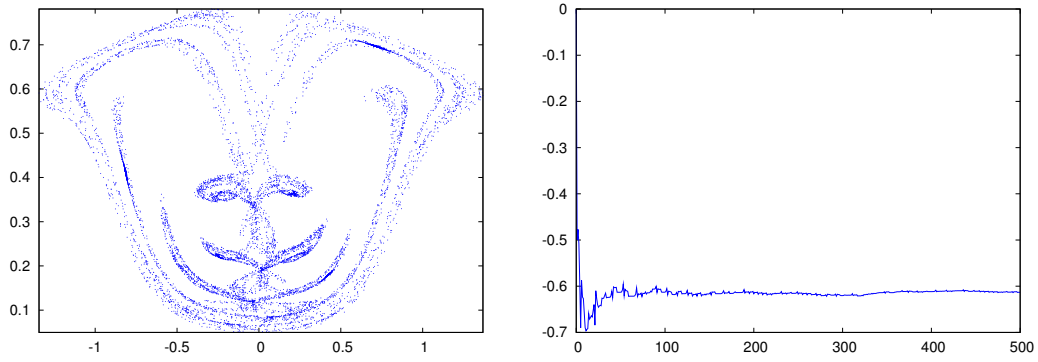
Figure 11 illustrates the dynamics in the case of SAC-learning versus naive expectations. Agents *learn to be contrarians*, as  $\beta_t \rightarrow -0.62$ , consistent with the SAC in realized prices. In this example there is still fairly strong negative first order autocorrelation in prices, but it is consistent with the behavior of the sophisticated type, who have learned the first order autocorrelation coefficient consistent with realized market prices. Figure 12 illustrates an example with memory in the fitness measure, where the (first order) autocorrelation in prices becomes even weaker ( $\beta_t \rightarrow -0.48$ ).

## 6 Concluding Remarks

We have reviewed bounded rationality and learning in the familiar cobweb, hog-cycle framework. Two stories of bounded rationality have been emphasized. The story of adaptive learning assumes a representative “average” agent trying to optimize a simple, (linear) misspecified rule in an unknown complex (nonlinear) economy. The other story assumes heterogeneous forecasting strategies and endogenous, evolutionary selection based upon past performance. We have also presented an example where both stories are integrated, with evolutionary selection between an adaptive learning rule and a simple, fixed rule.

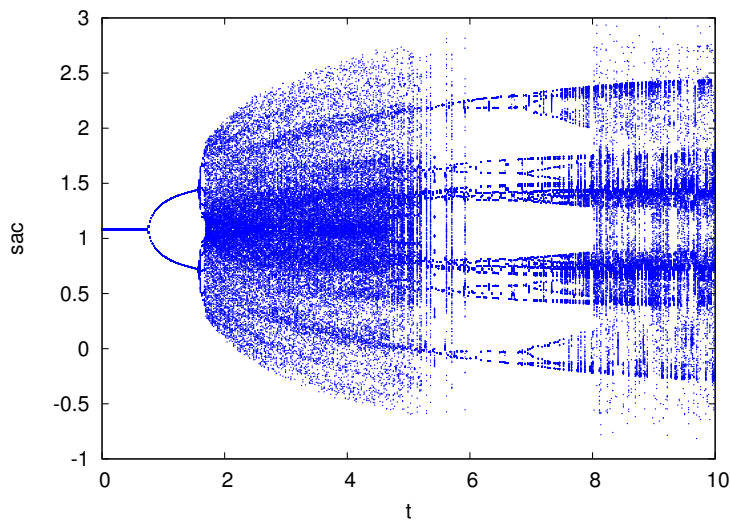
In a cobweb economy with nonlinear, monotonic demand and supply curves, many adaptive learning processes enforce convergence to the unique REE steady state price. The steady state price forecast is e.g. the only (linear) forecast, where sample averages and sample autocorrelations of realized market prices are consistent with beliefs. Simply by looking at sample averages and sample autocorrelations, in particular trying to learn the negative first order autocorrelation so typical for the ‘hog cycle’, boundedly rational agents should be able to learn the unique REE.

Laboratory experiments with human subjects show however that this is not as easy as theory suggests. Only in the stable treatment of the experiment (i.e. when the market is stable under naive expectations) do market prices converge to REE. In the unstable treatment of the experiments, realized market prices are characterized by three stylized facts: (i) the sample mean is close to the RE price; (ii) there is excess volatility, i.e., the sample variance is much higher than the RE variance, and (iii) there is no linear predictability (no autocorrelations) in realized prices. The observed ex-



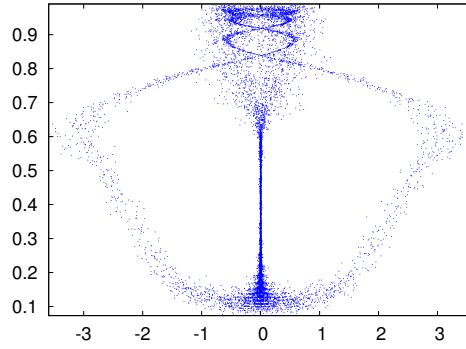
(a) strange attractor for  $\gamma = 3$

(b) sample autocorrelation  $\rho_t$  for  $\gamma = 3$

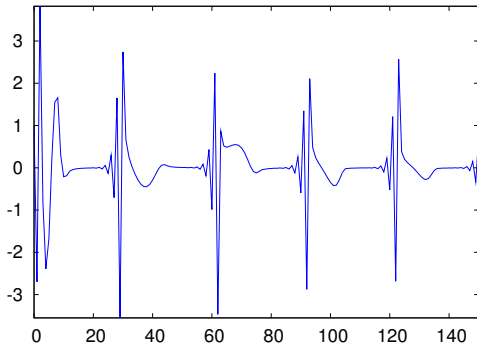


(c) bifurcation diagram

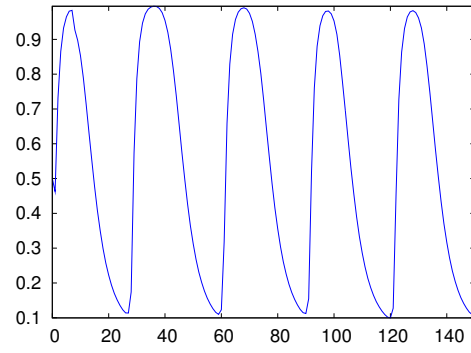
**Figure 11:** SAC-learning versus naive. Agents learn to be contrarians, as the first order autocorrelation coefficient converges,  $\beta_t \rightarrow -0.62$ . The bifurcation diagram shows a rational route to randomness, as the intensity of choice  $\gamma$  increases. Parameters:  $a = 0$ ,  $d = 0.5$ ,  $s = 1.35$ ,  $\delta = 0.5$ ,  $w = 0$ ,  $C_1 = 1$ ,  $\alpha_2 = 0$ ,  $\beta_2 = 1$  and  $C_2 = 0$ .



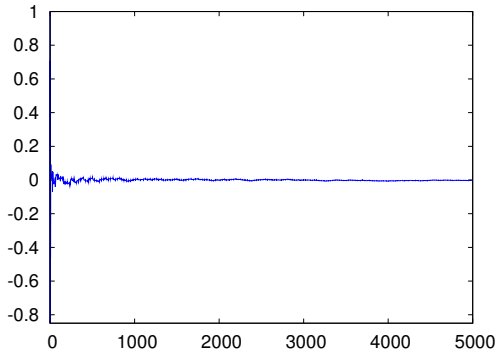
(a) strange attractor



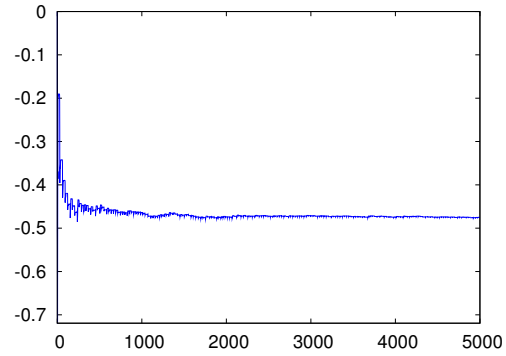
(b) price deviation from fundamental



(c) fraction of SAC-learners



(d) sample average



(e) sample autocorrelation coefficient

**Figure 12:** SAC-learning versus naive expectations with memory in the fitness measure. Attractor (a) and time series of prices  $p_t$  (deviations from fundamental), fraction  $n_{1t}$  of SAC-learners, sample average  $\alpha_t$  and sample autocorrelation coefficient  $\beta_t$ . With more memory in the fitness measure, the remaining autocorrelation in prices is weaker ( $\beta_t \rightarrow -0.48$ ). Parameters:  $\gamma = 3$ ,  $a = 0$ ,  $d = 0.5$ ,  $b = 1.35$ ,  $\delta = 0.5$ ,  $w = 0.9$ ,  $C_1 = 1$ ,  $\alpha_2 = 0$ ,  $\beta_2 = 1$  and  $C_2 = 0$ .

cess volatility is inconsistent with convergence of adaptive learning of a representative agent. For other simple expectations rules, such as adaptive expectations, irregular price fluctuations around the RE benchmark arise, but these fluctuations, even when chaotic, typically still exhibit negative first order autocorrelations, inconsistent with stylized fact (iii) in the experiments. Some form of heterogeneity is therefore needed to explain the laboratory experiments.

We have also reviewed some results on heterogeneous agent models with endogenous, evolutionary strategy selection, including several two-type cases with a costly sophisticated forecasting rule (fundamentalists, contrarians or SAC-learning) versus a free, simple forecasting rule (naive expectations). These two type models will converge to the RE price in the stable treatment of the experiment, and at the same time may generate instability and excess volatility in the unstable treatment, when agents switch fast enough between strategies, similar to the stylized facts in the experiments. However, it is not clear whether a two type model can simultaneously explain stylized fact (iii), i.e. linear unpredictability. A two type model with fundamentalists versus naive expectations generates strongly negative first order autocorrelation in prices, even when the system is chaotic. The typical up and down ‘hog cycle’ oscillations are still present, and would be observable to a careful agent. When fundamentalists are replaced by contrarians, who try to exploit the negative first order autocorrelation in prices, the first order autocorrelation gets weaker, but does not disappear completely. When contrarians are replaced by SAC-learning the results are similar, first order autocorrelation becomes weaker but again does not completely vanish. In the cobweb framework, adaptive agents *learn to become contrarians* and “arbitrage away” part of the linear predictability, but do not completely wash out the autocorrelations in market prices. These results suggest that, in order to match all stylized facts of the experiments, either the simple strategy (naive expectations) in these 2-type models needs to be replaced by a somewhat more complicated strategy (perhaps adaptive expectations or a 2-period average forecast), or more heterogeneity, i.e. more types of forecasting rules, are needed to fully explain the laboratory experiments. Matching the stylized facts of laboratory experiments on expectations formation remains an important challenge for theories of bounded rationality and learning, in the simple cobweb framework as well as for other, more realistic expectations feedback settings.

## References

- [1] Adam, K., (2007), Experimental Evidence on the Persistence of Output and Inflation, *Economic Journal* 117, 603-636.
- [2] Anderson, P.W, Arrow, K.J. and Pines, D. (eds.), (1988), *The Economy as an Evolving Complex System II* Addison-Wesley, Reading, MA.
- [3] Arifovic, J., (1994) Genetic algorithm learning and the cobweb model, *Journal of Economic Dynamics and Control* 18, 3-28.
- [4] Arthur, W.B., (2006), Out-of-equilibrium economics and agent-based modeling, In: *Hand-book of Computational Economics, Volume 2: Agent-Based Computational Economics*, Edited by L. Tesfatsion and K.L. Judd, Elsevier Science.
- [5] Arthur, W.B., Durlauf, S.N and Lane, D.A. (eds.), (1997), *The economy as an evolving complex system II* Addison-Wesley, Reading, MA.
- [6] Böhm, V., and Wenzelburger, J., (1999) Expectations, forecasting and perfect foresight. A dynamical systems approach, *Macroeconomic Dynamics* 3, 167-186.
- [7] Branch, W.A. (2006), Restricted perception equilibria and learning in macroeconomics, In: Colander, D. (ed.), *Post Walrasian Macroeconomics: Beyond the Stochastic General equilibrium Model*, Cambridge University Press.
- [8] Branch, W.A. and Evans, G.W. (2006), Intrinsic heterogeneity in expectation formation, *Journal of Economic Theory* 127, 264-295.
- [9] Bray, M.M., and Savin, N.E., (1986) Rational expectations equilibria, learning and model specification, *Econometrica* 54, 1129-1160.
- [10] Brock, W.A., and Hommes, C.H., (1995) Rational routes to randomness, SSRI working paper 9506, University of Wisconsin.
- [11] Brock, W.A., and Hommes, C.H., (1997) A rational route to randomness, *Econometrica*, 65, 1059-1095.
- [12] Bullard, J., (1994) Learning equilibria, *Journal of Economic Theory* 64, 468-485.
- [13] Bullard, J. and Duffy, J., (1998), Learning and Excess Volatility, Federal Reserve Bank of St. Louis, October 1998.
- [14] Bunow, B. and Weiss, G.H., (1979), How chaotic is chaos? Chaotic and other 'noisy' dynamics in the frequency domain. *Mathematical Biosciences* 47, 221-237.
- [15] Carlson, J. (1969), An invariably stable cobweb, *Review of Economic Studies* 69, 360-362.



- [16] Chiarella, C., (1988) The Cobweb Model: Its Instability and the Onset of Chaos, *Economic Modeling*, 5, 377-384.
- [17] Clark, C. W. (1985), *Bioeconomic Modelling and Fisheries Management*, New York: Wiley-Interscience, 1985.
- [18] Clark, C. W. (1990), *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*, 2nd ed. New York: Wiley-Interscience, 1990.
- [19] Colucci, D. and Valori, V. (1986), Ways of learning in a simple economic setting: a comparison, *Chaos, Solitons and Fractals* 29, 653-670.
- [20] Conlisk, J. (1996), Why Bounded Rationality?, *Journal of Economic Literature* 34, 669 - 700.
- [21] DeGrauwe, P. and Grimaldi, M. (2006), Exchange rate puzzles: A tale of switching attractors, *European Economic Review* 50, 1-33.
- [22] DeGrauwe, P. and Markiewicz, A. (2006), Learning to Forecast the Exchange Rate: Two Competing Approaches, *CESifo Working Paper* no. 1717, May 2006
- [23] DeLong, J.B., Shleifer, A., Summers, L.H. and Waldmann, R.J., (1990) Noise trader risk in financial markets, *Journal of Political Economy* 98, 703-738.
- [24] Diks, C.G.H. and Weide, R. van der (2005), Herding, a-synchronous updating and heterogeneity in memory in a CBS, *Journal of Economic Dynamics and Control* 29, 741-763.
- [25] Dindo, P.D.E., (2007), Bounded Rationality and Heterogeneity in Economic Dynamic Models, Ph-D thesis Tinbergen Institute Research Series 396.
- [26] Diks, C.G.H. and Dindo, P.D.E., (2006), Informational differences and learning in an asset market with boundedly rational agents, CeNDEF Working paper 06-11 University of Amsterdam.
- [27] Dindo, P.D.E. and Tuinstra, J., (2006), A behavioral model for participation games with negative feedback, CeNDEF Working paper 06-10, University of Amsterdam.
- [28] Dwyer, G.P., A.W. Williams, R.C. Battalio and T.I. Mason, (1993), Tests of Rational Expectations in a Stark Setting, *Economic Journal*, 103, 586-601.
- [29] Evans, G.W. and Honkapohja, S. (2001), *Learning and expectations in macroeconomics*, Princeton University Press, Princeton.
- [30] Ezekiel, M., (1938) The cobweb theorem, *Quarterly Journal of Economics* 52, 255-280.
- [31] Frankel, J.A. and Froot, K.A., (1986) Understanding the US Dollar in the Eighties: The Expectations of Chartists and Fundamentalists, *Economic Record*, special issue, pp. 24–38; also published as NBER working paper No. 0957, December 1987.

- [32] Friedman, M., (1953) The case for flexible exchange rates, In: Essays in positive economics, Univ. Chicago Press.
- [33] Goeree, J.K. and Hommes, C.H., (2000), Heterogeneous beliefs and the nonlinear cobweb model, *Journal of Economic Dynamics and Control* 24, 761-798.
- [34] Gordon, H. S. (1954), The economic theory of a common-property resource: the fishery, *Journal of Political Economy* 62, 124-142.
- [35] Grandmont, J.-M., (1998) Expectations Formation and Stability in Large Socio-Economic Systems, *Econometrica*, 66, 741-781.
- [36] Heemeijer, P., Hommes, C.H., Sonnemans, J. and Tuinstra, J. (2007), Price Stability and Volatility in Markets with Positive and Negative Expectations Feedback: An Experimental Investigation, *CeNDEF Working Paper* 07-01, University of Amsterdam.
- [37] Hey, J.D. (1994), Expectations formation: rational or adaptive or ...?, *Journal of Economic Behaviour and Organization*, 25, 329-349.
- [38] Hommes, C.H., (1991) Adaptive learning and roads to chaos. The case of the cobweb, *Economics Letters* 36, 127-132.
- [39] Hommes, C.H., (1994) Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand, *Journal of Economic Behavior & Organization* 24, 315-335.
- [40] Hommes, C. H., (1998) On the Consistency of Backward-Looking Expectations: The Case of the Cobweb, *Journal of Economic Behavior and Organization*, 33, 333-362.
- [41] Hommes, C.H. (2006), Heterogeneous Agent Models In Economics and Finance, In: *Hand-book of Computational Economics, Volume 2: Agent-Based Computational Economics*, Edited by L. Tesfatsion and K.L. Judd, Elsevier Science, 1109-1186.
- [42] Hommes, C.H., Huang, H. and Wang, D. (2005), A robust rational route to randomness in a simple asset pricing model, *Journal of Economic Dynamics and Control* 29, 1043-1072.
- [43] Hommes, C.H., and Rosser, J. Barkley Jr., (2001) Consistent Expectations Equilibria and Complex Dynamics in Renewable Resource Markets, *Macroeconomic Dynamics*.
- [44] Hommes, C.H., Sonnemans, J., Tuinstra, J., and van de Velden, H., (2005), Coordination of expectations in asset pricing experiments, *Review of Financial Studies* 18, 955-980.
- [45] Hommes, C.H., Sonnemans, J., Tuinstra, J. and van de Velden, H., (2007), Learning in cobweb experiments, *Macroeconomic Dynamics* 11 (S1), 8-33.

- [46] Hommes, C.H., Sonnemans, J., and van de Velden, H., (2000) Expectation formation in a cobweb economy: some one person experiments, In: Delli Gatti, D., Gallegati, M. and Kirman, A. (eds.), *Interaction and Market Structure. Essays on Heterogeneity in Economics, Lecture Notes in Economics and Mathematical Systems* 484, Springer Verlag, Berlin/Heidelberg, pp.253–266.
- [47] Hommes, C. H., and Sorger, G., (1998) Consistent Expectations Equilibria, *Macroeconomic Dynamics* 2, 287-321.
- [48] Huberman, B.A. and Glance, N.S., (1993) Evolutionary games and computer simulations, *Proceedings of the National Academy of Sciences (USA)*, 90, 7716–7718.
- [49] Kelley, H. and D. Friedman, (2002), Learning to Forecast Price, *Economic Inquiry* 40, 556–573.
- [50] Kirman, A., (1991) Epidemics of opinion and speculative bubbles in financial markets, In M. Taylor (ed.), *Money and financial markets*, Macmillan.
- [51] LeBaron, B., (2006), Agent-based Computational Finance, In: *Hand-book of Computational Economics, Volume 2: Agent-Based Computational Economics*, Edited by L. Tesfatsion and K.L. Judd, Elsevier Science.
- [52] Lucas, R.E., (1972) Econometric testing of the natural rate hypothesis, In: O. Eckstein (ed.) *The econometrics of price determination Conference*. Board of Governors of the Federal Reserve System and Social Science Research Council.
- [53] Lux, T., (1995) Herd Behavior, Bubbles and Crashes, *The Economic Journal* 105, 881–896.
- [54] Marcet, A. and Sargent, T.J., (1989), Convergence of Least Squares Learning Mechanisms in Self Referential Linear Stochastic Models, *Journal of Economic Theory* 48, 337-368.
- [55] Marimon, R., S.E. Spear and S. Sunder, (1993), Expectationally driven market volatility: an experimental study, *Journal of Economic Theory* 61, 74–103.
- [56] Marimon, R. and S. Sunder, (1993), Indeterminacy of equilibria in a hyperinflationary world: experimental evidence, *Econometrica* 61, 1073-1107.
- [57] Marimon, R. and S. Sunder, (1994), Expectations and learning under alternative monetary regimes: an experimental approach, *Economic Theory* 4, 131-162.
- [58] Marimon, R. and S. Sunder, (1995), Does a constant money growth rule help stabilize inflation? *Carnegie-Rochester Conference Series on Public Policy* 43, 111-156.
- [59] Muth, J.F., (1961) Rational expectations and the theory of price movements, *Econometrica* 29, 315-335.
- [60] Nicholson, W. (1995), *Microeconomic theory. Basic principles and extensions*, Sixth Ed., Dryden Press, Fort Worth.

- [61] Nowak, M. and May, R.M., (1992) Evolutionary games and spatial chaos, *Nature* 359, 826–929.
- [62] Nowak, M., Bonhoeffer, S. and May, R.M., (1992) Spatial games and the maintenance of cooperation, *Proceedings of the National Academy of Sciences (USA)* 91, 4877–4881.
- [63] Peterson, S.P. (1993) Forecasting dynamics and convergence to market fundamentals. Evidence from experimental asset markets, *Journal of Economic Behaviour and Organization* 269-284.
- [64] Rosser, J.B., (2004), *Complexity in Economics: The International Library of Critical Writings in Economics* 174, (3 volumes), Aldergate: Edward Elgar, 2004.
- [65] Sakai, H., and Tokumaru, H., (1980), Autocorrelations of a certain chaos, *IEEE Transactions on Acoustics, Speech and Signal Processing* 28, 588-90.
- [66] Sargent, T.J., (1993) Bounded rationality in macroeconomics, Clarendon Press, Oxford.
- [67] Schmalensee, R., (1976), An Experimental Study of Expectation Formation, *Econometrica* 44, 17–41.
- [68] Schönhofer, M. (1999), Chaotic Learning Equilibria, *Journal of Economic Theory* 89, 1-20.
- [69] Schönhofer, M., (2001) Can agents learn their way out of chaos?, *Journal of Economic Behavior & Organization* 44, 71-83.
- [70] Simon, H.A. (1957), *Models of man*, Wiley, New York, NY.
- [71] Smith, V., Suchanek, G.L. and Williams, A.W., (1988) Bubbles, crashes and endogenous expectations in experimental spot asset markets, *Econometrica* 56, 1119-51.
- [72] Sonnemans, J., Hommes, C.H., Tuinstra, J. and van de Velden, H., (2004), The instability of a heterogeneous cobweb economy: a strategy experiment on expectation formation, *Journal of Economic Behavior and Organization* 54, 453–481.
- [73] Sorger, G. (1994) Imperfect foresight and chaos: an example of a self-fulfilling mistake, *Journal of Economic Behavior & Organization* 33, 363-383.
- [74] Tuinstra, J. and Wagener, F.O.O. (2007), On learning equilibria, *Economic Theory* 30, 493-513.
- [75] Tversky, A. and Kahneman, D. (1974), Judgment under uncertainty: heuristics and biases, *Science* 185, 1124-1131.
- [76] Wellford, C.P. (1989), A laboratory analysis of price dynamics and expectations in the cobweb model, *Discussion Paper* No. 89-15, Department of Economics, University of Arizona.
- [77] Williams, A.W., (1987), The formation of price forecasts in experimental markets, *Journal of Money, Credit and Banking* 19, 118.
- [78] Zeeman, E.C., (1974) On the unstable behaviour of stock exchange, *Journal of Mathematical Economics* 1, 39-49.