On the Micro-Dynamics of a Cash-in-Advance Economy

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Abstract

The purpose of this paper is to develop a general equilibrium model with money and trade taking place at disequilibrium prices. There are multiple markets being visited sequentially and transactions occur along the adjustment path. This implies quantity rationing to clear the market and we assume that there are cash-in-advance constraints on the transactions. The updating of the prices and cash balances along the way makes it necessary for agents to reconsider their trading plans subject to new information due to substitution and spill-over effects. The dynamics of this disequilibrium re-optimization process are shown to depend crucially on the exchange mechanisms that are imposed. One of the results is that the introduction of a cash-in-advance constraint does not help in stabilizing the fluctuations of cash balances, even though it does prevent debts from occurring outside of equilibrium.

1 Introduction

An important but still largely unresolved problem in economic theory is the question of stability and the convergence towards equilibrium. It is often assumed that equilibrium prices are generated by a market mechanism that is driven by the law of supply and demand, a model described by the Walrasian tâtonnement process, in which a central price setting mechanism, or market maker, adjusts prices on all markets simultaneously based on the demand and supply of all the market participants. It is essentially a nonlinear dynamical system that is driven by a central processing unit, which coordinates the decentralized decisions of self-interested, autonomously acting agents.

It is well-known in the literature on (price) adjustment processes that the Walrasian tâtonnement process need not converge to a Walrasian equilibrium. Some examples in which cycles or chaotic behavior are shown to occur are in Scarf (1960), Saari (1985), Goeree, Hommes and Weddepohl (1997) and Tuinstra (2000). In dynamic price adjustment processes it is not uncommon to use as the dynamic process a sequence of temporary equilibria. An equilibrium is simply obtained in every period of the process by solving a market-clearing equation. But the question of how these temporary equilibria can be reached as the result of some dynamic disequilibrium process is then usually left unspecified. The simplest model in which such a process can be studied is in an exchange economy without a production sector or a financial sector. This model should be seen as a pedagogical device: it is a first step towards a more comprehensive theory that explains how equilibria can be reached. The question is also strongly related to the stability of Keynesian or underemployment equilibria under dynamic adjustment processes that take place outside of equilibrium. This refers back to the Austrian tradition of disequilibrium modelling (e.g. Hayek (1928), Lindahl (1939b) and Hicks (1939)); for a general survey see Zappia (1999).

In this paper we develop a dynamic model with trade outside of equilibrium in a monetary exchange economy with cash-in-advance constraints. The markets are visited sequentially and the transactions are taking place at disequilibrium prices. A direct motivation for such a sequential model is that not all markets may be open simultaneously, and that the information flow between the markets is sequential.

The model with sequential market visits is related to the Dual-Decisions Hypothesis

of Clower (1965), which states that the decisions for multiple markets have to be taken sequentially by taking into account the trades that have already occurred during previous market visits. This forces the agents to reconsider their trading plans subject to new information due to spill-overs. An interpretation of the Dual Decisions Hypothesis (due to Benassy, 1975) is that the markets can be seen as 'trading posts', with a single commodity being traded at each post. The trading posts are visited sequentially according to a predetermined market order.[1]

The trading period is divided into subperiods, or 'market days'. On each market day only one trading post (market) is visited so only one good can be traded. The markets are visited in sequence and prices adjust sequentially. There is trade at disequilibrium prices, markets clear by quantity rationing and monetary and spill-over effects play a role outside of equilibrium. During the market visits the quantities are traded against prevailing market prices, which have been predetermined during the previous round of trading. In this sense the model belongs to the class of fix-price models (Hicks, 1939). After every round of trades a new trading round begins and the sequence of market visits is indefinitely repeated, see Figure 4.

Prices are temporarily fixed and taken as given, but can adjust after a market has been visited. Only the price for the commodity that is traded on the current market is updated, since the other markets are temporarily closed for trading. After every round of market visits other economic variables than the prices can be updated such as the budget constraints and the expectations of the agents.

According to Magill and Quinzii (1996, Ch.7) a satisfactory modelling of money requires an open-ended future. Trade should take place in a sequence economy in which the imperfections in the trading opportunities of the agents play a role (Ibid. p. 488). The trading possibilities of the agents can be restricted due to multiple transaction constraints, including income, financial or quantity constraints. Such restrictions have consequences in monetary and in real terms. Not only are there spill-over and substitution effects due to trade at disequilibrium prices, but there are also wealth effects due to the monetary restrictions, i.e. there are cash-in-advance or liquidity effects.

The current model with sequential trade is a generalization of the model in Weddepohl

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[1]The market order is the same for all agents. A preferred model would be one in which the market order is heterogeneous and subject to choice by the agents themselves.

(1996). For disequilibrium trade to be possible an endogenous quantity rationing mechanism is needed to clear the market, otherwise no transactions can occur. To determine transactions we use a proportional rationing rule: agents on the *long* side of the market are rationed in proportion to their own demand or supply orders. If the agents know that the rationing is proportional then the rule can be manipulated by over-asking or over-supplying the market. For simplicity we will assume that agents ignore this type of strategic behavior.

The rest of the paper is organized as follows. In section 2 the formal structure of the model is introduced. Section 3 contains a numerical example and provides simulation results and section 4 concludes.

[INSERT FIGURE 1 ABOUT HERE]

2 The structure of the model

This section describes the physical characteristics of the economy, the monetary institutions, the timing of the model, the cash-in-advance equilibrium, the agents' decision-making problem, the budget accounting mechanism, the proportional rationing mechanism and the price dynamics. All these model ingredients provide the model with a modular structure in which the outputs of one module are the inputs for the next. This facilitates operationalizing the formal model into a computational model.

2.1 The exchange economy

We consider an exchange economy that remains the same over time, i.e. it is a stationary economy without growth. We further assume that agents are infinitely lived and that endowments are the same in every period. There is no storage facility and the commodities are perishable, that is they have to be traded or consumed during the current period. It is further assumed that there are N households and L markets. Households are indexed by $h \in \mathbf{N} = \{1, ..., N\}$ and commodities are indexed by $k \in \mathbf{M} = \{1, ..., m\}$. A household $h \in \mathbf{N}$ has a consumption set $X^h = \mathbb{R}_+^L$, a utility function U^h defined on X^h, and a vector of initial endowments $\mathbf{w}^h = (w_1^h, ..., w_m^h) \in \mathbb{R}_{++}^L$. Finally, there is a positive money stock

$\bar{M} \geq 0$, which is distributed across the households according to the cash distribution $\{M^1, ..., M^N\}$, with $\sum_h M^h = \bar{M}$.

2.2 Monetary institutions

The trade deficits and trade surpluses that accumulate during a trading round have to be corrected on the balance of account of the agents. This will affect the agents' budget constraints. Agents must not only repay the old debts that have already accumulated but must also prevent new debts from occurring. If these effects were ignored then it would be optimal for agents to run into debt indefinitely. For this we require the following assumptions:

Assumption 1 *All agents have a bank account and a credit card (or debit card) to perform transactions. Bank accounts are a temporary store of purchasing power between the sale and purchase of commodities.*

Assumption 2 *There is a positive money stock and all transactions are paid by cash transfers between one agents' bank account (the debtor) to another agents' bank account (the creditor). The bank accounts thus serve as the medium of exchange.*

In addition to the budget constraint there is a cash-in-advance constraint which only becomes binding on the market when the agent tries to spent more then the bank account allows. Trade deficits are therefore excluded, all bank saldi are non-negative. If an agent has a trade surplus at the end of a period, then this surplus can be carried over to the next period and can be spent on consumption. The cash-in-advance constraint causes agents to become rationed in their demand, implying that convergence towards equilibrium (if it occurs) takes place without the occurrence of any debts.

2.3 Timing

Time is discrete and every period is divided into subperiods. The time-index for a subperiod is (t, k), where t is the current period and k is the current subperiod in period t. Commodity k can be traded in subperiod (t, k) and will also be traded in all subperiods $(t + n, k)$, for all $n \geq 0$. The number of subperiods in every trading round is equal to the number of markets m. In every subperiod only one commodity can be

traded in exchange for money. To be sure, the sequence of time periods is as follows: $(..., (t, 1), ..., (t, m), (t + 1, 1), ..., (t + 1, m), ...)$. The subperiods should be seen as the unit-period, which is the smallest unit of time on which decisions are made, and can be interpreted as 'market days' (as in Lindahl 1939) since only one market is visited in every subperiod. A trading round consists of a sequence of such 'market days' and can be called a 'trading week' (as in Hicks 1939). But this interpretation should not be taken literally, since the period can have any arbitrary calendar length.

2.4 The state variables

The state vector $\mathbf{s} \in \mathcal{S}$ is a multidimensional vector that represents the state of the economy. It pertains to the state of the markets and to the internal states of the individual agents. For each agent we record a history of past revenues and past money balances, with a time lag of one period given the transactions of the previous round. All the state variables are given in Table 1. A state $\mathbf{s} \in \mathcal{S}$ is a tuple given by

$$\left(\{p_k, D_k, S_k\}_{k \in \mathbf{M}}, \{z_k^h, \hat{z}_k^h, \bar{z}_k^h, R_k^h, M_k^h, \mu^h\}_{h \in \mathbf{N}, k \in \mathbf{M}} \right) \in \mathcal{S}, \text{ where}$$
$$\mathcal{S} = (\mathbb{R}_{++}^L \times \mathbb{R}_+^L \times \mathbb{R}_+^L) \times (\mathbb{R}^{N \times L} \times \mathbb{R}^{N \times L} \times \mathbb{R}^{N \times L} \times \mathbb{R}^{N \times L} \times \mathbb{R}_+^{N \times L} \times \mathbb{R}_+^N).$$

It should be understood that the planned trade vectors (z_k^h) and the realized transactions $(\bar{z}_k^h)$ of each individual agent are unobservable to any of the other agents; these are private information. The total demand and supply on the market cannot be publicly observed, but there is a centralized mechanism per market which is assumed to aggregate all the buy and sell orders. The price vector is known to all agents; this is public information. We assume that new prices are announced at the moment that they are updated, which is after the transactions on a market have occurred, but before the next market is opened for trade. So the updated price for market k is known prior to visiting market $k + 1$. At every moment, the full price-vector is known one full round ahead (there is a public price list with fixed prevailing market prices). The only uncertainty in the model is therefore the quantities that will be traded, since agents cannot observe the planned transactions (the notional demands) of the other market participants.

Now that we have fixed the notation for all the state variables, and described the physical characteristics of the exchange economy, we will formulate the decision making process in the next section.

Table 1: State variables.

Variable	
$\mathbf{p} = (p_1, ..., p_m)$	prices
$\mathbf{w}^h = (w_1^h, ..., w_m^h)$	initial endowments of agent h
$\mathbf{z}^h = (z_1^h, ..., z_m^h)$	planned trades of agent h (notional excess demand)
$\hat{z}_k^h$	buy or sell signal after applying a cash-in-advance constraint
$\bar{z}_k^h$	realized trade of agent h (rationed demands, or transactions)
$s_k^h = \min\{0, z_k^h\} \leq 0$	planned supply of agent h (sell order)
$d_k^h = \max\{0, z_k^h\} \geq 0$	planned demand of agent h (buy order)
$S_k = -\sum_h s_k^h \geq 0$	market supply on market k
$D_k = \sum_h d_k^h \geq 0$	market demand on market k
$\bar{s}_k^h = \min\{0, \bar{z}_k^h\} \leq 0$	realized supply of agent h
$\bar{d}_k^h = \max\{0, \bar{z}_k^h\} \geq 0$	realized demand of agent h
$R_k^h = -p_k z_k^h$	net revenues of agent h on market k
$M_k^h \geq 0$	realized cash balance of agent h
$\mu_k^h \geq 0$	minimum cash position of agent h over previous trading round
$\mathbf{p}\mathbf{w}^h + \mu_k^h$	corrected budget constraint of agent h

Notional demand

The notional demand $\mathbf{x}^h \in \mathbb{R}_+^L$ is the best choice from the budget set $\mathbb{B}^h$, which depends on prices and the real wealth $\mathbf{p}\mathbf{w}^h + \tilde{M}^h$, consisting of the value of possessions and real balances:[2]

$$\mathbb{B}^h(\mathbf{p}, \mathbf{p}\mathbf{w}^h + \tilde{M}^h) = \{\mathbf{x} \in \mathbb{R}^L | \mathbf{p}\mathbf{x}^h \leq \mathbf{p}\mathbf{w}^h + \tilde{M}^h, x_k^h \geq 0, \forall k \in \mathbf{M}\}. \qquad (1)$$

The notional demand function $\mathbf{D}^h : \mathbb{R}_{++}^L \times \mathbb{R} \to \mathbb{R}_+^L$ is defined by:

$$\mathbf{D}^h(\mathbf{p}, \mathbf{p}\mathbf{w}^h + \tilde{M}^h) = \arg \max_{\mathbf{x}} \{U^h(\mathbf{x}^h) | \mathbf{x}^h \in \mathbb{B}^h(\mathbf{p}, \mathbf{p}\mathbf{w}^h + \tilde{M}^h)\}. \qquad (2)$$

The notional excess demand function $\mathbf{z}^h : \mathbb{R}_{++}^L \times \mathbb{R} \to \mathbb{R}^L$ is:

$$\mathbf{z}^h(\mathbf{p}, \tilde{M}^h) = \mathbf{D}^h(\mathbf{p}, \mathbf{p}\mathbf{w}^h + \tilde{M}^h) - \mathbf{w}^h. \qquad (3)$$

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[2]We include unanticipated debts and claims $\tilde{M}^h$ in addition to the income from resources: $\mathbf{p}\mathbf{w}^h + \tilde{M}^h$, such that $\sum_h \tilde{M}^h = 0$. For now we assume that $\tilde{M}^h = 0$ for all $h = 1, ..., N$, but in later sections the value of $\tilde{M}^h$ will fluctuate and can take on positive values.

The aggregate excess demand is a function of the price system $\mathbf{p}$ and the income distribution $(\mathbf{p}\mathbf{w}^1 + \tilde{M}^1, ..., \mathbf{p}\mathbf{w}^N + \tilde{M}^N)$, where $\sum_h \tilde{M}^h = 0$:

$$\mathbf{Z}(\mathbf{p}, \tilde{M}^1, ..., \tilde{M}^N) = \sum_{h \in \mathbf{N}} \mathbf{D}^h(\mathbf{p}, \mathbf{p}\mathbf{w}^h + \tilde{M}^h) - \sum_{h \in \mathbf{N}} \mathbf{w}^h. \tag{4}$$

In an equilibrium there are no unanticipated debts or claims: $\tilde{M}^h = 0$, $\mathbf{D}^h(\mathbf{p}, \mathbf{p}\mathbf{w}^h + \tilde{M}^h) = \mathbf{D}^h(\mathbf{p}, \mathbf{p}\mathbf{w}^h)$, $\mathbf{z}^h(\mathbf{p}, \tilde{M}^h) = \mathbf{z}^h(\mathbf{p}, 0)$, and $\mathbf{Z}(\mathbf{p}, \tilde{M}^1, ..., \tilde{M}^N) = \mathbf{Z}(\mathbf{p}, 0, ..., 0)$.

Definition 1 *(Walrasian equilibrium) A Walrasian equilibrium (WE) consists of a price system $\mathbf{p}^\star \in \mathbb{R}^L_{++}$, a feasible allocation $\mathbf{x}^{h\star} \in X^h$ for all $h \in \mathbf{N}$ and an income distribution $(\mathbf{p}\mathbf{w}^1, ..., \mathbf{p}\mathbf{w}^N)$ such that:*

1. $\mathbf{x}^{h\star} = \mathbf{D}^h(\mathbf{p}, \mathbf{p}\mathbf{w}^h)$ *(optimality)*

2. $\mathbf{Z}(\mathbf{p}^\star, 0, ..., 0) = \mathbf{0}$ *(market clearing)*

2.5 The cash-in-advance constraint

Suppose that agents are not permitted (or do not want) to have a debt during the trade sequence and do not want to hold more money than is strictly necessary to perform transactions. An equilibrium must then satisfy the additional condition that agents' money holdings do not become negative along the sequence. In other words, their trade balance should always remain positive (or non-negative). This requires a positive money stock, which can now be called 'cash', and it defines a 'cash-in-advance equilibrium' (cf. Clower, 1967). The minimum amount of cash an agent needs in order to prevent a trade deficit is exactly equal to the largest trade deficit that would occur along the sequence of trades if all transactions would take place in terms of credit transfers between buyers and sellers.

In addition to the budget constraint we now introduce a cash-in-advance constraint, which becomes binding only if the planned consumption at subperiod (t, k) violates the cash position:

$$p_k(t, k) z_k^h(t, k) \leq M^h(t, k). \tag{5}$$

This constraint is equivalent to setting the planned trade equal to the minimum

$$\hat{z}_k^h(t, k) = \min\{z_k^h(t, k), M^h(t, k)/p_k(t, k)\}. \tag{6}$$

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The cash-in-advance constraint implies that if the money holdings available to agent h at the beginning of subperiod (t, k) suffices only to buy the quantity $M^h(t, k)/p_k(t, k)$, instead of the optimal amount $z_k^h(t, k)$, then the minimum amount will be ordered. If however the cash balance is sufficient to buy the optimal quantity – the planned trade is not too expensive – then that amount will be ordered. Note that a cash-in-advance constraint strictly prohibits the occurrence of debts because the cash balance cannot become negative.

In fact the cash-in-advance constraint in combination with the budget constraint ensures that the agents are not running a deficit nor are accumulating a surplus. This gives the monetary dynamics the tendency to converge towards an equilibrium. If prices are too low (high) with respect to the equilibrium levels, then there is an excess demand (supply) for all commodities, because the purchasing power of the money balances is too high (low). The general price level will increase (decrease) as all prices adjust to the total money stock $\bar{M}$ in the economy.

Cash-in-advance equilibrium

In general, the sequence of equilibrium transactions produces a sequence of net revenues for each agent $h = 1, ..., N$:

$$R_k^{h\star} = -p_k^\star(t, k) z_k^{h\star}(t, k), \qquad k = 1, ..., m \qquad (7)$$

where $R_k^{h\star} > 0$ means the proceeds from sales on market k and $R_k^{h\star} < 0$ is an expenditure on market k. A trade deficit or surplus at subperiod (t, k) is then the sum of net revenues $\sum_{j=1}^k R_j^{h\star}$. The largest trade deficit that would occur if a negative trade balance would be allowed is: $\min_{1 \le k \le m}\{\sum_{j=1}^k R_j^{h\star}\}$. The amount of cash an agent needs in order to prevent this trade deficit from occurring is therefore

$$M^{h\star} = - \min_{1 \le k \le m} \{\sum_{j=1}^k R_j^{h\star}\}. \qquad (8)$$

The term $M^{h\star}$ can be interpreted as the equilibrium demand for real balances, which is simply the demand for money. This gives us the necessary condition for defining a cash-in-advance equilibrium.

Definition 2 *(Cash-in-advance equilibrium) A cash-in-advance equilibrium consists of a positive money stock $\bar{M} > 0$, a distribution of money holdings $\{M^{1\star}, ..., M^{N\star}\}$, an allocation $\mathbf{x}^\star$ and a price system $\mathbf{p}^\star$, such that:*

1. *$(\mathbf{x}^\star, \mathbf{p}^\star)$ is a Walrasian equilibrium;*

2. *$M^{h\star} = -\min_{1 \leq k \leq m}\{\sum_{j=1}^{k} R_j^{h\star}\}$;*

3. *$\sum_h M^{h\star} = \bar{M}$.*

The cash-in-advance equilibrium (henceforth CIA equilibrium) has two properties that are important to mention. The first property is that the sequence of equilibrium transactions can be carried out without any agent having a debt along the way, since the equilibrium money holdings $M^{h\star}$ are precisely sufficient to prevent any debts from occurring. A second property of the CIA equilibrium is that the cash-in-advance constraint is never strictly binding. It can be 'just binding', which means that relaxing it would not lead to a change in the optimum. In a Walrasian equilibrium, which is a special case of a CIA eqm., the CIA constraint is non-binding.

2.6 Decision-making

We assume that agents are *myopic* moving-horizon decision-makers. By a moving-horizon optimization we will understand the following. Although the agents in our model are infinitely lived they do not plan further ahead than one 'trading week', i.e. one period. This means that the agents have to re-optimize and revise their trading plans during the trading period. A similar procedure was introduced in Weddepohl (1996), where a *'gliding'* optimization procedure allows the agents to optimize along a *'gliding time horizon'*.[3]

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[3]The concept of moving-horizon optimization that will be used here is in order to describe the behavior of the agents. In control theory this optimization technique is known as 'closed-loop feedback control'. It has a long history in the literature on dynamic programming, dating back at least to the late 1950s and early 1960s, when it was explicitly used to solve infinite horizon optimal control problems. For example in the state-space optimal control methods used by Bellman and others, Bellman's principle of optimality involves the sequential updating of the optimal control as a new state is realized. Furthermore, the method of closed-loop feedback control often uses a 'moving window' or 'moving horizon' of fixed length instead of an infinite horizon to solve the infinite horizon problem. But note that the concept of moving

At the beginning of a market day an L-dimensional problem is solved based on the current state and the forecasted data for the next market days. Only the transactions for the current market are made. On the next market day, the state is again observed and the forecasts for the future market days are updated. A new L-dimensional problem is solved at the beginning of every market day, and the transactions for the current market are made. This procedure is repeated on every market day, hence the term *moving horizon*. The state of the economy is updated after every market visit and the agents' planning horizon is one 'trading week'. The essential aspect of the moving-horizon approach is that the planning horizon has a fixed length and is 'rolled over' after every subperiod. Such a procedure can also be described as *'solving an online optimization at every step'*, since the problem state in the optimization problem changes over time and the problem has to be re-solved 'online' during the process.

2.6.1 Moving horizon optimization

We assume that consumers maximize utility over a single period (one round of market visits) and have no intertemporal preferences. At every subperiod (t, k) the agents' utility ranges over a single commodity bundle and money is not included in the utility function. The extension to multiple periods with intertemporal preferences over multiple commodity bundles (multiple rounds of trade) is straightforward, but only possible when the intertemporal utility function is time-additive: $U^h(u_1^h, ..., u_T^h) = \sum_t u_t^h$. This strong requirement is necessary in order to ensure that the marginal rate of substitution between any two commodities in two consecutive periods is time-invariant: $MRS(x_{i,t}, x_{j,t+1}) = MRS(x_{i,t+1}, x_{j,t+2})$, where $MRS(x, y) = (\partial U/\partial x)/(\partial U/\partial y)$. This rules out utility functions that are based on time-discounting, which would introduce a

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horizon optimization is being used here in a different context. Here we use it as a behavioral postulate which states that, although agents are infinitely lived in relation to their planning horizon, they are myopic moving-horizon optimizers who only look one period ahead. Therefore they have to re-optimize after every sequential step of the process. This is somewhat similar to the closed-loop feedback control method, but different in the sense that in our model the agents do not try to solve an infinite horizon problem by means of optimal control theory. Furthermore there are multiple agents, which makes the problem an N-agent optimal control problem. Whether or not the aggregate behavior of all the agents together leads to the solution of some infinite horizon problem is a different matter altogether which does not concern us here.

time-inconsistency in the moving-horizon decision-making procedure.

The optimization problem for subperiod (t, k) is:

$$\max_{\mathbf{z}}\{U^h(\mathbf{z}^h + \mathbf{w}^h)|\mathbf{p}(t,k) \cdot \mathbf{z}^h(t,k) \leq \tilde{M}^h(t,k), z_k^h(t,k) + w_k^h \geq 0\}. \quad (9)$$

The consumption plan $\mathbf{z}^h(t, k)$ made at subperiod (t, k) ranges over the future sequence of subperiods: $T_k^+ = \{(t,k), ..., (t,m)\} \cup \{(t+1,1), ..., (t+1,k-1)\}$. The new plan $\mathbf{z}^h(t, k+1)$, which is made at the start of market $k + 1$, ranges over the shifted horizon T_{k+1}^+. Table 2 depicts the structure of the re-optimization process with the moving planning horizon.

Table 2: 1-period moving-horizon planning.

Planning date	$(t,1)$	$(t,2)$	$(t,3)$	$(t+1,1)$	$(t+1,2)$	$(t+1,3)$	$(t+2,1)$	$(t+2,2)$	$(t+2,3)$
Market	1	2	3	1	2	3	1	2	3
$(t,1)$	X	O	O						
$(t,2)$		X	O	O					
$(t,3)$			X	O	O				
$(t+1,1)$				X	O	O			
$(t+1,2)$					X	O	O		
$(t+1,3)$						X	O	O	

X=realized trades, O=planned trades. Moving-horizon planning scheme with a re-optimization after every subperiod. The planning horizon is one period, so there are no intertemporal transfers of purchasing power.

2.6.2 Utility specification

We assume that all agents have a CES utility function of the following form:

$$\begin{aligned} U^h(\mathbf{x}) &= \left(\sum_{k=1}^m \alpha_k^h (x_k^h)^\nu\right)^{1/\nu}, & \nu \in (-\infty, 1] \\ &= \left(\sum_{k=1}^m \alpha_k^h (x_k^h)^{(\varepsilon-1)/\varepsilon}\right)^{\varepsilon/(\varepsilon-1)}, & \varepsilon \in [0, +\infty). \end{aligned} \quad (10)$$

The parameter ν is the substitution parameter, which is considered to be the same for all agents. The (constant) elasticity of substitution is derived from the substitution parameter by $\varepsilon = 1/(1-\nu)$. The notional demand function resulting from utility maximization is such that agent h plans to spent a fraction of the real wealth $(\mathbf{p} \cdot \mathbf{w}^h + \tilde{M}^h)/p_k$ on commodity k. This fraction is a function of the prices, the preference parameters $\alpha^h = (\alpha_1^h, ..., \alpha_m^h)$, $\alpha_k^h \geq 0$ and the elasticity parameter ε:

$$x_k^h = \mathbf{D}_k^h(\mathbf{p}; \tilde{M}^h, \alpha^h, \varepsilon) = \left(\frac{p_k(\alpha_k^h/p_k)^\varepsilon}{\sum_{k=1}^m p_k(\alpha_k^h/p_k)^\varepsilon}\right) \cdot \left(\frac{\mathbf{p} \cdot \mathbf{w}^h + \tilde{M}^h}{p_k}\right). \quad (11)$$

The parameter ν is used as a bifurcation parameter in the stability analysis. It lies in the interval $\nu \in (-\infty, 1]$, where $\nu \to -\infty$ corresponds to Leontief utility (no substitutes,

$\varepsilon = 0$), $\nu = 0$ corresponds to Cobb-Douglas utility (unitary substitutes, $\varepsilon = 1$), and $\nu = 1$ corresponds to a linear utility function (perfect substitutes, $\varepsilon = +\infty$). The default values for the parameters $(\alpha, \mathbf{w}^h, \nu)$ will be specified when they become relevant for simulations.

2.7 Budget updating in the cash model

In this section we describe how the budget constraint is updated by the term $\tilde{M}^h$. In a disequilibrium model with infinitely lived agents wealth effects play an important role since trade takes place outside equilibrium and therefore debt and claims accumulate over time. The repayment of debts then becomes relevant, so an updating procedure for the budget constraints is needed. The frequency at which the re-enforcement of debt repayment is performed can have a large impact on the dynamics, since there are nominal and real wealth effects for the individual traders. In this section we will show how agents sequentially update their budget constraints to take into account the debts and claims.

The balance of account of agent h at the beginning of subperiod $(t, 1)$ is denoted by $M^h(t, 1)$. It can be positive or negative, depending on whether the agent has a debt or a claim on other agents. The balance of account is the sum of net revenues up to and including the last transaction during the previous market visit. After every transaction the realized balance of account is updated:

$$M^h(t + n, k) \quad = M^h(t + n, k - 1) + R^h(t + n, k - 1) \qquad \forall k \neq 1 \qquad (12)$$

$$M^h(t + n + 1, 1) \quad = M^h(t + n, m) + R^h(t + n, m) \qquad\qquad k = 1. \qquad (13)$$

Here $R^h(t, k) = -p_k(t, k)\bar{z}_k^h(t, k)$ is the money value of the transaction $\bar{z}_k^h$. Subperiod $(t + n, m)$ is the last subperiod of round n, just before subperiod $(t + n + 1, 1)$. Note that the balance of account $M^h(t, k)$, for $k \neq 1$, is determined at the beginning of subperiod (t, k) and can therefore only include the receipts and expenditures up to and including the last transaction of subperiod $(t, k - 1)$.

The sequential market structure makes it necessary to distinguish between the realized balance of account M^h and the unanticipated debts and claims $\tilde{M}^h$. This last term is used by the agents as a 'rule of thumb' in the decision-making process, in correction on the budget constraint. The actual balance of account M^h will fluctuate due to the time-lag between income and consumption (this is due to the sequential structure of the economy). Because of these fluctuations we have to find a more sophisticated way to

update the planned balance of account, by taking into account that the transactions are occurring sequentially and outside of equilibrium. This procedure has to be consistent with a steady state, since in a CIA equilibrium the sequence of equilibrium transactions causes fluctuations in the actual balance of account M^h. Therefore we can not simply put the money balance on the right-hand side of the budget constraint because then it would not remain constant in a CIA equilibrium. What does remain constant is the *change* in the cash positions along the trading sequence. At every point along the equilibrium sequence the same cash positions are repeated. Also the *minimum* (and maximum) cash position computed over the entire sequence is the same in every trading round. Therefore the actual cash balance M^h can be replaced by the *minimum* cash position computed over a full trading round. The model now contains the following additional concepts.

The minimum cash position over the previous round, computed at the start of a new trading round $(t, 1)$, is given by:

$$\mu^h(t, 1) = \min\{M^h(t-1, 1), ..., M^h(t-1, m)\}. \tag{14}$$

The minimum cash position over the previous round, computed at the start of every subperiod (t, k), is given by:

$$\mu^h(t, k) = \min\{M^h(t-1, k), ..., M^h(t-1, m), M^h(t, 1), ..., M^h(t, k-1)\} \tag{15}$$

In general, we write $\mathbf{pw}^h + \mu^h$ for the corrected budget constraint and for the individual excess demands we write

$$\mathbf{z}^h(\mathbf{p}, \mu^h) = \mathbf{D}^h(\mathbf{p}, \mathbf{pw}^h + \mu^h) - \mathbf{w}^h. \tag{16}$$

The planned trade vectors are $\mathbf{z}^h(\mathbf{p}(t, k), \mu^h(t, k))$ and at subperiod (t, k) they automatically satisfy the budget constraint, since all commodities are included in the optimization problem. By including μ^h on the RHS the budget constraint is corrected for any positive cash balance that arises due to a trade surplus. Such a positive cash balance arises if an agent first sells some of its endowments and then turns out to be demand-rationed later on. The resulting trade surplus can then be spend immediately on consumption during the next market visit.

In a CIA equilibrium $\mu^{h\star} = 0$ and $\mathbf{z}^h(\mathbf{p}, 0) = \mathbf{z}^h(\mathbf{p})$. This trading plan is consistent in the sense that agents do not plan to violate their budget constraint, so agents are not planning to make a surplus but the positive cash balances are unanticipated (and unintentional).

2.8 Proportional quantity rationing mechanism

This section describes the market rationing mechanism. Rationing of demand or supply is required if the trading plans $\mathbf{z}^h$ are incompatible, in the sense that for some markets demand is not equal to supply, i.e. $\sum_h z_k^h \neq 0$. Consequently, a system of quantity rationing assignments is needed in order for transactions to occur outside of equilibrium. The mechanism by which agents are rationed depends on the market structure, and can be heterogeneous for different markets. For the setup of the disequilibrium framework we follow Bénassy (1974). An agent is rationed if and only if the individual excess demand is on the 'long side' of the market, that is if it has the same sign as the aggregate excess market demand: $z_k^h \cdot \sum_h z_k^h > 0$. For proportional rationing this mechanism is described by a map $\mathbf{F}_k^h$ that maps individual excess demands (z_k^h) to assignments of rationed demands $(\bar{z}_k^h)$ as follows:

$$\bar{z}_k^h = \mathbf{F}_k^h(z_k^1, ..., z_k^N), \text{ such that } \sum_h \bar{z}_k^h = 0. \tag{17}$$

We assume that $\mathbf{F}_k^h = \mathbf{F}$ for all h, k, where:

1. if $D_k \geq S_k$ then $(D_k/S_k) \geq 1$ and

$$\bar{z}_k^h = \begin{cases} (S_k/D_k)z_k^h & \text{for } z_k^h \geq 0 \\ z_k^h & \text{for } z_k^h \leq 0 \end{cases} \tag{18}$$

2. if $D_k \leq S_k$ then $(D_k/S_k) \leq 1$ and

$$\bar{z}_k^h = \begin{cases} z_k^h & \text{for } z_k^h \geq 0 \\ (D_k/S_k)z_k^h & \text{for } z_k^h \leq 0. \end{cases} \tag{19}$$

The transactions that are realized on market k are by definition equal to the rationed demand and supplies $\bar{z}_k^h$ on market k. Note that the rationing function is only a function of the excess demands on market k. The function $\mathbf{F}_k^h$ can be market specific, and it can also discriminate among the agents if it is made heterogeneous across the agents and/or the markets.

2.9   Price dynamics

2.9.1   Sequential price dynamics

In the sequential trading process the price adjustments are taking place *sequentially*, instead of simultaneously for all markets at once. Let f_k denote some price adjustment function for the price on market k, which must be necessarily a function of the entire price vector since the agents' trading plans are a function of all prices. Recall that the price-vector $\mathbf{p}(t,1)$ is given at the start of subperiod $(t,1)$, and the price for market 1 is updated after market 1 has been visited, but before market 2 opens. Therefore, $p_1(t,2) = f_1(\mathbf{p}(t,1))$. The price dynamics in the sequential model therefore evolves according to:

$$
\begin{aligned}
\mathbf{p}(t,1) &= (p_1(t,1), p_2(t,1), ..., p_m(t,1)) \\
\mathbf{p}(t,2) &= (f_1(\mathbf{p}(t,1)), p_2(t,1), ..., p_m(t,1)) \\
\mathbf{p}(t,3) &= (f_1(\mathbf{p}(t,1)), f_2(\mathbf{p}(t,2)), p_3(t,1), ..., p_m(t,1)) \\
&\ \ \vdots \\
\mathbf{p}(t,k+1) &= (f_1(\mathbf{p}(t,1)), f_2(\mathbf{p}(t,2)), ..., f_k(\mathbf{p}(t,k)), p_{k+1}(t,1), ..., p_m(t,1)) \\
&\ \ \vdots \\
\mathbf{p}(t+1,1) &= (f_1(\mathbf{p}(t,1)), f_2(\mathbf{p}(t,2)), ..., f_m(\mathbf{p}(t,m-1))).
\end{aligned}
\tag{20}
$$

It is clear that this system is not yet written in closed-form, but it can be turned into a simultaneous system of difference equations by substituting the first equation into the second, the first and second into the third, etc. This does not alter the dimension of the system, which remains m-dimensional.

2.9.2   Proportional price adjustments

We will use a proportional price adjustment rule that is based on the ratio of aggregate market demand and aggregate market supply. We assume that the price mechanism is the same on all markets: $f_k = f$:[4]

$$
p_k(t+1,k) = f_k(\mathbf{p}(t,k)) \equiv p_k(t,k) \left(\frac{D_k(\mathbf{p}(t,k))}{S_k(\mathbf{p}(t,k))} \right)^{\lambda_k}.
\tag{21}
$$

This price rule has the property that: $p_k(t+1,k) = p_k(t,k)$ if and only if $D_k = S_k$, for all λ_k. Hence in an equilibrium the prices do not change. The parameter λ is the 'speed

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[4]This proportional price rule can also be derived from the ratio of excess demand over total market supply: $p_{k,t+1} = p_{k,t}(1 + z_k/S_k) = p_k(1 + (D_k - S_k)/S_k) = p_k(D_k/S_k)$.

of adjustment' parameter, which plays a crucial role for the stability of the price process. It can be interpreted as the *price flexibility*. High values of λ indicate that prices are very flexible; low values of λ indicate that prices are rigid; $\lambda = 0$ means that prices are completely fixed. We assume that the price flexibility is the same on all markets: $\lambda_k = \lambda$. The price flexibility can be assumed to be homogeneous across markets without loss of generality, by a rescaling of the units in which the quantities of commodities are measured. Since *a priori* there are no theoretical arguments for the value of the parameter λ, we will use it as a bifurcation parameter to investigate the stability of the dynamics with respect to changes in the price flexibility.

Another nice property of the DS-ratio rule is that it can be transformed into a log-transformation of the classical discrete-time version of the tatonnement process that is often used in the literature: $[p_k(t+1) - p_k(t)]/p_k(t) = \lambda z_k(t)$. Taking the logarithm on both sides of the DS-rule, we obtain

$$log(p_k(t+1)) - log(p_k(t)) = \lambda \cdot (log(D_k) - log(S_k)), \tag{22}$$

which is linear in both log-demand and log-supply.

2.9.3 Price normalization

Since nominal price levels are indeterminate, we normalize prices using a simplex-normalization rule. The real-balance effect (the effect that old debts denominated in the old currency become cheaper in terms of purchasing power when the price level increases) should not affect the agents' demand for commodities, since the demand functions are homogeneous. The purchasing power of the old balance of account should remain constant in real terms if all prices are normalized by the normalization rule. Therefore we will also have to re-normalize all the cash balances by the same normalization rule.

The budget constraint after normalization, i.e., the real wealth consisting of (i) the value of possessions $\mathbf{p}\mathbf{w}^h$ and (ii) the initial real balances M^h, is given by:

$$\frac{\mathbf{p} \cdot \mathbf{x}^h}{N(\mathbf{p})/c} \leq \frac{\mathbf{p} \cdot \mathbf{w}^h}{N(\mathbf{p})/c} + \frac{M^h}{N(\mathbf{p})/c}, \text{ where } N(\mathbf{p}) = \sum_{i=1}^{m} p_i. \tag{23}$$

The last term is the normalization of the real balance of account. In simulations we have used the normalization rule $\sum p_i = 3$, since all the examples we consider have three markets, hence $c \equiv m = 3$. The normalization is performed after every change in the price-vector, i.e. after every sequential market visit.

2.9.4 Price growth restrictions

We introduce price rigidities in the form of 'ceilings' and 'floors' on the *growth rates* of prices. This is to prevent the nominal prices from imploding or exploding, and in order to keep the fluctuations of nominal prices bounded within economically meaningful regions. Let the price growth factor be given by $\pi_k(t+1) \equiv p_k(t+1)/p_k(t)$. We define limits to growth by $\pi_k^+ = (1+r^+)$ and $\pi_k^- = (1-r^-)$, where r^+ and r^- are the maximum rates of upward and downward growth, respectively. The upward and downward price rigidities are given by the following restrictions:

$$\pi_k^- \leq \pi_k(t_k+1) \leq \pi_k^+, \qquad (1-r^-) \leq \frac{p_k(t_k+1)}{p_k(t_k)} \leq (1+r^+). \qquad (24)$$

In terms of the growth rates of the log-prices, the price rigidities are linear restrictions and the following are equivalent:

$$\begin{aligned} \log \pi_k^- &\leq \log \pi_k(t+1) & &\leq \log \pi_k^+ \\ \log(1-r^-) &\leq (\log p_k(t+1) - \log p_k(t)) & &\leq \log(1+r^+) \\ -r^- &\leq \log p_k(t+1) - \log p_k(t) & &\leq r^+. \end{aligned} \qquad (25)$$

The last line follows since $\log(1+x) = x$ in first approximation. For the proportional price rule in (21) the price changes are restricted by the growth rates π_k^+ and π_k^- as follows:

$$\pi_k = \min\{\pi_k^+, \max\{\pi_k^-, (D/S)^\lambda\}\}. \qquad (26)$$

Redefining the restrictions to take the parameter λ into account we obtain new bounds: $\eta^- = (-r^-/\lambda)$ and $\eta^+ = (r^+/\lambda)$. The log-price changes are now restricted by the log-growth rates η^- and η^+ (note that λ no longer appears in the restrictions due to the re-parametrization):

$$\log \frac{p_k(t_k+1)}{p_k(t_k)} = \begin{cases} \lambda \log(D/S), & if \quad \eta^- \leq \log(D/S) \leq \eta^+ \\ -r^-, & if \quad \log(D/S) \leq \eta^- \\ r^+, & if \quad \log(D/S) \geq \eta^+. \end{cases} \qquad (27)$$

The terms η^- and η^+ simultaneously take into account the effect of the price adjustment speed λ and the restrictions r^+ and r^- respectively, and can be interpreted as the minimum and maximum growth rate in log-prices.

Model parameters

All parameters and their simulation specifications are given in Table 3.

Table 3: Model parameters.

Parameter	label	range
preferences	α_k^h	$\{0.1, 0.2, 0.3, 0.4, 0.5\}$
substitution parameter	ν	$(-\infty, 1)$
elasticity of substitution	ε	$(0, +\infty)$, $\varepsilon = 1/(1 - \nu)$
price flexibility	λ	$[0, 2]$
maximum price growth rates	r^-, r^+	$r^- = 0.09$, $r^+ = 0.10$
downward price rigidity	π^-	$\pi^- = (1 - r^-) = 0.91$
upward price rigidity	π^+	$\pi^+ = (1 + r^+) = 1.10$

3   Simulation

Having described the details of all the model elements we can now formulate the dynamical system. Figure 2 provides a flowchart for the sequence of events that takes place during a single market visit. Figure 3 gives a pseudo-code or algorithm which encodes these events into a computational model.

[INSERT FIGURE 2 ABOUT HERE]

[INSERT FIGURE 3 ABOUT HERE]

Given are the initial conditions: $\mathbf{p}(0)$ is random, $\mu_1^h(0) = 0$, $M_1^h(0) = 100$. The following steps are performed for every agent $h = 1, .., N$ during market visit k in subperiod (t, k), for $k = 1, ..., m$ and $t = 0, ..., T$ (see Figure 3):

1. Determine the desired trades $z_k^h(t, k)$ for market k given the notional excess demand function $\mathbf{z}^h(\mathbf{p}(t, k), \mu^h(t, k))$ (eqn. 3, specified in eqn. 11).

2. Apply the cash-in-advance constraint to the buy and sell orders (eqn. 6).

3. Determine the transactions $\bar{z}_k^h(t, k)$ on market k (the rationed demand and supply), by the function $\mathbf{F}$ (eqn. 17).

4. Determine the net revenues $R_k^h(t, k)$ from trading on market k and update the cash balance $M^h(t, k)$ (eqn. 13).

5. Update the minimum cash position over the current round $\mu_k^h(t, k)$ (eqn. 15).

6. Update the price $p_k(t + 1, k)$ for market k (eqn. 21).

In step 4 we have to collect a history of previous cash balances for each individual agent: $\{M_{k+1}^h, ..., M_m^h, M_1^h, ..., M_k^h\}$. This is necessary in order to calculate the agents' minimum cash position over one complete past round of trades. The sequence $\{M_{k+1}^h, ..., M_m^h\}$ comes from the previous round, whereas the sequence $\{M_1^h, ..., M_k^h\}$ comes from the current round.[5]

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[5]It is possible to simulate a weak form of the cash-in-advance model by leaving out step 2 from the simulations. This removes the 'hard' cash-in-advance constraint, but there is still the correction on the

3.1 Example: CES utility

We consider an example with $N = 3$ agents and $m = 3$ commodities (markets). On every market there are two sellers and one buyer (a case of monopsony). The agents all have CES utility functions with preferences and endowments specified as follows:

$$U^a(x^a) = [0.2(x_1^a)^\nu + 0.4(x_2^a)^\nu + 0.4(x_3^a)^\nu]^{1/\nu}, \quad w^a = (50, 0, 50), M^a(0) = 100,$$
$$U^b(x^b) = [0.4(x_1^b)^\nu + 0.2(x_2^b)^\nu + 0.4(x_3^b)^\nu]^{1/\nu}, \quad w^b = (50, 50, 0), M^b(0) = 100, \quad (28)$$
$$U^c(x^c) = [0.4(x_1^c)^\nu + 0.4(x_2^c)^\nu + 0.2(x_3^c)^\nu]^{1/\nu}, \quad w^c = (0, 50, 50), M^c(0) = 100.$$

The preferences and endowments are such that every agent owns two commodities and strictly prefers one of these two. The agent is indifferent between the commodity that he does *not* own and the strictly preferred commodity that he does own. This implies that the agent is willing to part with some quantity of the least preferred commodity, in exchange for the strictly preferred commodities. With Cobb-Douglas preferences ($\nu = 0, \varepsilon = 1$) the planned consumption and the notional excess demands are given by (equilibrium income is $\mathbf{p} \cdot \mathbf{w}^h = 100$):

$$\mathbf{x}^\star = \begin{pmatrix} 20 & 40 & 40 \\ 40 & 20 & 40 \\ 40 & 40 & 20 \end{pmatrix}, \mathbf{z}^\star = \begin{pmatrix} -30 & 40 & -10 \\ -10 & -30 & 40 \\ 40 & -10 & -30 \end{pmatrix}.$$

3.2 Fluctuations in the balance of account in equilibrium

For the example given above, the equilibrium price system is $\mathbf{p} = (\theta, \theta, \theta), \theta \in \mathbb{R}_+$. To every arbitrary positive money stock $\bar{M} > 0$ there corresponds a price level θ which can be determined by the quantity equation $\bar{M} = 50\theta$. The number 50 is coincidental for this choice of preference parameters. In addition, the presence of money turns the indeterminate Walrasian equilibrium into a determinate CIA equilibrium (see def. 2).

The money stock acts as a numeraire and the price levels adjust to the CIA equilibrium that is associated to the money stock $\bar{M}$. The initial money holdings of every agent are

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budget constraint by the minimum cash position μ^h. This would imply that we allow for debts to occur again (a negative balance of account), but the largest trade deficit is still used as a correction mechanism on the budget constraint. This correction mechanism can be said to act as a "soft" cash-in-advance constraint, since it is driving the dynamics towards a CIA equilibrium without strictly constraining the consumption pattern. The difference with the current model in which we do put restrictions on the consumption pattern is subtle, and we have not investigated this matter any further.

set to $M^h(0) = 100$, as an initial condition. This fixes the money stock at $\bar{M} = 300$. For a substitution parameter $\nu = 0$ (the Cobb-Douglas case) the equilibrium cash distribution is then given by: $M^{1\star} = 60, M^{2\star} = 0, M^{3\star} = 240$. The equilibrium price system associated to the CIA equilibrium is $\mathbf{p}^\star = (6, 6, 6)$. This is a result of the fact that if prices were $\mathbf{p}^\star = (1, 1, 1)$ then agents' cash requirements are $\sum_h M^h = 50$. Since we have imposed $\bar{M} = 300$, the price level is now $\theta = 6$. In the CIA equilibrium the sequence of transactions in monetary terms is shown in Table 4.

Table 4: Equilibrium demand for money balances.

Agent	1	2	3	
$M^h(t, 1)$	60	0	240	Start of market round: initial money holdings.
$M^h(t, 2)$	240	60	0	Agent 1 receives 180, agent 2 receives 60 (both from agent 3).
$M^h(t, 3)$	0	240	60	Agent 2 receives 180, agent 3 receives 60 (both from agent 1).
$M^h(t, 1)$	60	0	240	Agent 3 receives 180, agent 1 receives 60 (both from agent 2).

The total amount of cash that is required for the transactions to occur at any time is the row sum 300. The money stock that supports a CIA equilibrium is therefore given by $\bar{M} = 300$. But given any other value of $\bar{M}$, the money stock will be automatically redistributed among the agent population. The cash distribution $\{M^{h\star}\}_h$ in a CIA equilibrium with a different money stock $\bar{M}$ depends on the proportional change in the money stock: if $\bar{M} \to \beta \bar{M}$ then $(p_1, ..., p_m) \to (\beta p_1, ..., \beta p_m)$.

At the start all cash is concentrated with agents 1 and 3. Agent 2 does not need any cash at the start of market 1 since he is a seller on markets 1 and 2, and a buyer on market 3. Agent 1 needs some cash to be able to buy on market 2 and agent 3 needs cash to buy on market 1. Agent 3 requires the most cash, since he will be buying commodity 1 from both agents 1 and 2 before he can sell commodity 2 to agent 1 and commodity 3 to agent 2. This reasoning holds from the perspective of starting on market 1. If we start the equilibrium sequence on market 2 then the cash requirements are rotated cyclically, as shown by the sequence of payments in Table 4.

3.3 Fluctuations in the balance of account out of equilibrium

We start by analyzing the model from the time series perspective. Below we show three examples of the dynamics:

1. The transition path towards a stable CIA equilibrium ($\nu = -0.1$).

2. Fluctuations around an unstable CIA equilibrium for low elasticity of substitution ($\nu = 0$).

3. Fluctuations around an unstable CIA equilibrium for high elasticity of substitution ($\nu = 0.4$).

The transition path towards a stable CIA equilibrium

In Figure 4 we show time series of the individual cash balances for each agent. The plot shows the entire transition path, starting from an initial cash distribution $\{M^1(0), M^2(0), M^3(0)\} = \{100, 100, 100\}$. The variables $M^1(t, 1)$, $M^2(t, 1)$, $M^3(t, 1)$ represent the cash balance of each agent at the start of every round (beginning of market 1), for 50 consecutive trading rounds ($0 < n < 50$). For the substitution parameter $\nu = -0.1$ the CIA equilibrium is asymptotically stable under the specified proportional price dynamics, and each agents' time series converges to its CIA-equilibrium value as shown in Table 4. The equilibrium values corresponding to the substitution parameter $\nu = -0.1$ are slightly shifted from the equilibrium values at $\nu = 0$ in the table. For agent 1 the equilibrium cash balance is therefore slightly above 60 and for agent 3 it is slightly below 240: $M^{1\star} = 62$, $M^{2\star} = 0$ and $M^{3\star} = 238$.

[INSERT FIGURE 4 ABOUT HERE]

Fluctuations around an unstable CIA equilibrium (low elasticity case)

In Figure 5 we show the typical dynamical behavior for the case of a CIA equilibrium that is unstable. The figure shows time series plots of the cash balances at the start of every market visit, for 450 consecutive trading rounds ($50 < n < 500$). For Cobb-Douglas utility ($\nu = 0$) the CIA equilibrium is locally unstable. The behavior can be characterized as quasi-periodic, since it never repeats exactly in the same pattern.

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A more detailed description of the dynamics outside of equilibrium is as follows.

- Every agent has two trading posts: there are two markets on which an agent wants to sell his endowments.

- Agent 1 is sometimes restricted by his cash constraint on market 2, since he is a buyer on market 2 and a seller on markets 1 and 3. If agent 1 was supply-constrained on market 1 then he will find himself short of cash and unable to consume on market 2. The cash constraint therefore becomes binding during his visit to market 2, just before entering market 3 on which he plans to sell again. At the start of market 3 the cash balance of agent 1 thus reaches the zero level in equilibrium (see plot (1,3) in fig. 5). At the start of market 1, the cash balance fluctuates around 60 (see plot (1,1)), and at the start of market 2 it fluctuates around 240 (see plot (1,2)). The equilibrium sequence for agent 1 is: $(60, 240, 0)$.

- Agent 2 wants to sell on markets 1 and 2 and buy on market 3. Therefore a cash constraint becomes binding during his visit to market 3, just before visiting market 1 on which he plans to sell. The cash balance will reach the zero level (in equilibrium) just after having visited market 3, and before market 1 opens for trade (see plot (2,1) in fig. 5). At the start of market 2 the cash balance fluctuates around 60 (plot (2,2)), and at the start of market 3 around 240 (plot (2,3)). The equilibrium sequence for agent 2 is: $(0, 60, 240)$.

- Agent 3 is a seller on markets 2 and 3, and a buyer on market 1. He is sometimes rationed by his cash constraint on market 1. Therefore we see that his cash balance reaches the zero level (in equilibrium) at the beginning of market 2 (see plot (3,2)). At the start of market 3 the cash balance fluctuates around 60 (see plot (3,3)), and at the start of market 1 around 240 (see plot (3,1)). The equilibrium sequence for agent 3 is: $(240, 0, 60)$.

A general observation is that for each agent the cash-in-advance constraint becomes binding on the market that is entered just before one of the own trading posts is visited. This means that in equilibrium each agent spents its entire cash holdings just before the agent earns an income by selling some of his endowments again.

[INSERT FIGURE 5 ABOUT HERE]

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Fluctuations around an unstable CIA equilibrium (high elasticity case)

In Figure 6 we show what happens for higher values of the substitution parameter $\nu = 0.4$ (this corresponds to an elasticity of substitution $\varepsilon = 1/(1 - 0.4) = 1.66$). The corresponding equilibrium cash distribution has shifted from $\{M^{h\star}(t, 1)\}_h = (60, 0, 240)$ to $\{M^{h\star}(t, 1)\}_h = (40.8, 0, 259.2)$. The phase plots in Figure 6 show that the cash-in-advance constraint becomes binding more often in case of a higher elasticity of substitution (higher parameters ν and ε). This is due to the higher sensitivity of the demand to price fluctuations.

Note that if the cash constraint becomes binding, then the fluctuations in the cash balance remain inside a bounding box in the phase space. This bounded movement seems to be a result of the fact that there is only one buyer per market: the maximum cash position of a seller on a particular market is restricted by the cash constraint of the single buyer on that market. The only way that a seller can obtain a higher income (i.e. reach points that lie outside of the bounding box) is when the buyer is not cash-constrained.

A secondary effect is that the prices bounce back and forth between the price boundaries. This is what we observe in the phase plots in terms of prices in Figure 7. There are upward and downward restrictions on the price growth rates, but not on the absolute levels. These price rigidities are restricting the price-feasibility region: there is a maximum positive growth rate of 10% and a maximum negative growth rate of 9%. The effect of these price rigidities is that the minimum cash position of the agents now fluctuates between 0 and 20, as shown in the time series of the cash balances in Figure 6, subplot (1,1).

In conclusion, two rather different types of restrictions play a role if the elasticity of substitution is increased: there is the cash-in-advance constraint which becomes binding more often and there are the price rigidities that become more restrictive. These restrictions are strongly related: if the price fluctuations become more erratic (by an increased price-elasticity of the demand for commodities) then also the income of the agents will fluctuate more violently and so will the cash balances. The fluctuations in the cash balances in turn affect the agents demand and supply decisions, which causes fluctuations in the excess demands. These excess demand fluctuations then feed back into the dynamics of the price fluctuations.

[INSERT FIGURE 6 ABOUT HERE]

[INSERT FIGURE 7 ABOUT HERE]

3.4 Stability analysis

Figure 8 shows a 2-parameter bifurcation diagram for the elasticity of substitution parameter $0 < \nu < 0.9$ and the price flexibility parameter $0 < \lambda < 2$. It illustrates some high-order periodic cycles that can occur in the model for different parameter combinations. Cycles of periodicity $\{1, 2, 3, 5, 6, 9, 10, 13, 15, 17, 19\}$ are shown to occur. Figure 9 shows a particular bifurcation scenario for the fixed parameter value $\lambda = 0.5$ and varying ν between $\nu = 0.67$ and $\nu = 0.69$.

As far as it can be determined by numerical methods, the bifurcation scenario that occurs in Figure 9a-f is a subcritical Hopf bifurcation of a steady state, followed by a saddle-node bifurcation, followed by a secondary Hopf bifurcation.

First the stable steady state becomes unstable through the subcritical Hopf bifurcation producing a sudden transition to a large invariant circle (Figure 9a). At the moment of bifurcation also an unstable invariant circle is created, but because the bifurcation is subcritical there is a jump from the steady state to the stable branch of the invariant circle. This invariant circle then undergoes a saddle-node bifurcation producing a periodic sink and a periodic saddle of period-11 (Figure 9b). The period-11 cycle loses stability through a secondary Hopf bifurcation which generates an attractor like the one shown in Figure 9c. This attractor consists of 11 quasi-periodic cycles around each of the 11 saddle-sink pairs. An 11-piece attractor emerges (Figure 9d) in which the 11 pieces are being visited cyclically. The dynamics along this 11-cyclic attractor undergoes phase-locking which turns the attractor into a stable period-11 cycle (Figure 9e). The unstable invariant circle in Figure 9b undergoes a period-doubling bifurcation, transforming it into a figure-eight shape. Figure 9f shows this figure-eight shape just after the period-doubling of the (repelling) invariant circle has taken place.

[INSERT FIGURE 8 ABOUT HERE]

[INSERT FIGURE 9 ABOUT HERE]

4 Conclusions

We have described in relative detail a model with disequilibrium trade and cash-in-advance constraints. There is quantity-rationing to clear the markets and transactions take place sequentially. Agents' cash balances are tracked over time by keeping account of the debts and claims they make outside of equilibrium.

We have used the cash-in-advance (Clower) constraint which leads to an alternative equilibrium concept, namely that the sequence of payments along the trading round remains the same and money has to be used for all transactions. The CIA equilibrium retains all the properties of a general equilibrium (it is a Walrasian equilibrium) except that it is no longer indeterminate since there is a positive money stock.

According to Magill and Quinzii (1996, Ch.7) a satisfactory modelling of money requires an open-ended future. Trade should take place in a sequence economy in which the imperfections in the trading opportunities of the agents play a role. Although the model presented here has a finite planning horizon instead of an infinite horizon, nonetheless the future is open-ended because agents are myopic: they only plan ahead one trading round which consists of multiple market visits. Due to the trade at disequilibrium prices and the spill-over effects from quantity rationing agents have to revise their trading plans subject to new information that arrives during the trading round. Such a moving-horizon time-structure results in a genuinely dynamic monetary model of an exchange economy with trade outside equilibrium.

The cash-in-advance constraint acts as a *"hard" correction mechanism* on fluctuations in the balance of account because the agents are not allowed to enter into credit arrangements. In combination with the corrections on the budget constraint the cash-in-advance constraint drives the economy back towards an equilibrium if the disequilibrium becomes too large. Due to the fact that agents are not allowed to make any debts along the adjustment path the cash-in-advance constraint causes the agents to become demand-rationed. The constraint *does not help to dampen the amplitude of fluctuations* or to prevent cycles from occurring because the correction mechanism can overshoot. When this occurs there are large swings in the prices which will directly influence the cash balances as well, since the transactions have to be paid at the fluctuating price levels. These fluctuations are then exacerbated by the cash-in-advance constraint rather than dampened.

The fluctuations in the cash balances increase with the elasticity of substitution due to

the higher sensitivity of the demand to price fluctuations. If the cash-in-advance process converges to a neighborhood of the cash-in-advance equilibrium then the process shows small fluctuations around the steady state before it converges and the convergence takes place without the occurrence of debts and claims along the adjustment path. However, for large values of the substitution parameter ν, the process may not converge to a steady state but to periodic or quasi-periodic orbits.

For the dynamics to be stable it seems necessary that the correction mechanism is not updating too fast, since this may exacerbate the fluctuations in the balance of account. This hints at a more general conjecture that the market mechanisms and the market structure are perhaps of greater importance than whether or not the agents in the model are in fact using optimal trading strategies that take into account all of the relevant market information. The main message of our simulation exercise is that even if agents take the relevant market signals into account, and update their budget constraints sequentially, the occurrence of erratic dynamics cannot be dismissed by introducing more sophisticated correction mechanisms.

References

[1] Amendola, M. and Gaffard, J.L., 1998. *Out of Equilibrium.* Oxford: Oxford University Press.

[2] Bénassy, J.-P., 1982. *The Economics of Market Disequilibrium.* New York: Academic Press.

[3] Bénassy, J.-P., 1986. *Macroeconomics: an introduction to the non-Walrasian approach..* San Diego: Academic Press.

[4] Citanna, A., Cres, H., Herings, P.J.-J., Drèze, J.H. and Villanacci, A., 2001. Underemployment of Resources and Self-fulfilling Beliefs: Non-Walrasian Allocations at Walrasian Prices. *Journal of Mathematical Economics* 36(1), 169-200.

[5] Clower, R.W., 1963. Classical Monetary Theory Revisited. *Economica* 30(118), 165-170.

[6] Clower, R.W., 1965. The Keynesian Counter-revolution: A Theoretical Appraisal. In F.H. Hahn and F.P.R. Brechling (eds.), *The Theory of Interest.* New York: Macmillan, 103-125.

[7] Clower, R.W., 1967. A Reconsideration of the Microfoundations of Monetary Theory. *Western Economic Journal* 6, 1-9.

[8] Clower, R.W. and Leijonhufvud, A., 1975. The Coordination of Economic Activities: A Keynesian Perspective. *American Economic Review* 65(2), 182-188.

[9] Clower, R.W., 1977. The Anatomy of Monetary Theory. *American Economic Review* 67(1), 206-212.

[10] Drèze, J.H., 1975. Existence of Exchange Equilibrium Under Price Rigidities. *International Economic Review* 16, 301-320.

[11] Drèze, J.H., 1997. Walras-Keynes Equilibria, Co-ordination and Macroeconomics, *European Economic Review* 41, 1735-1762.

[12] Drèze, J.H., 1999. On the Macroeconomics of Uncertainty and Incomplete Markets. Presidential address for the Twelfth World Congress of the International Economic Association. URL: http://ideas.repec.org/p/fth/louvco/9964.html.

[13] Fisher, F.M., 1983. *Disequilibrium Foundations of Equilibrium Economics.* Cambridge: Cambridge University Press.

[14] Gode, D.K. and Sunder, S., 1993. Allocative Efficiency of Markets with Zero-Intelligence Traders: Market as a Partial Substitute for Individual Rationality. *Journal of Political Economy* 101(1), 119-137.

[15] Gode, D.K. and Sunder, S., 1997. What Makes Markets Allocationally Efficient? *Quarterly Journal of Economics* 112(2), 603-630.

[16] Goeree, J.K., Hommes, C.H. and Weddepohl, H.N., 1997. Stability and Complex Dynamics in a Discrete Tâtonnement Model. *Journal of Economic Behavior and Organization* 33, 395-410.

[17] Hicks, J.R., 1935. A suggestion for simplifying the theory of money. In *Money, Interest and Wages: Collected Essays in Economic Theory, Vol. II.* Oxford: Basil Blackwell, 1982.

[18] Hicks, J.R., 1939. *Value and capital: An inquiry into some fundamental principles of economic theory.* Oxford: Clarendon Press, 1974.

[19] Lindahl, E., 1939. The dynamic approach to economic theory. In *Studies in the Theory of Money and Capital.* London: Allen and Unwin.

[20] Magill, M. and Quinzii, M., 1996, *Theory of Incomplete Markets: Volume I.* Cambridge: M.I.T. Press.

[21] Movshovich, S.M., 1994. A Price Adjustment Process in a Rationed Economy. *Journal of Mathematical Economics* 23(4), 305-321.

[22] Nusse, E.N. and Yorke, J.A., 1998. *Dynamics: Numerical explorations*, New York: Springer-Verlag.

[23] Schinkel, M.P., 2002. *Disequilibrium Theory: Reflections towards a rivival of learning.* Maastricht: Universitaire Pers Maastricht.

[24] Shubik, M., 1999. *The Theory of Money and Financial Institutions* Vol. 1 and 2. Massachusetts: MIT Press.

[25] Stiglitz, J., 1999. Toward a general theory of wage and price rigidities and economic fluctuations. *American Economic Review* 89(2), 75-80.

[26] Stiglitz, J., 2003. *The Roaring Nineties.* England: Allen Lane.

[27] Svensson, L.E.O., 1981. Effective Demand in a Sequence of Markets. *Scandinavian Journal of Economics*, 1-21.

[28] Tuinstra, J., 2000. *Price Dynamics in Equilibrium Models.* Amsterdam: Thela Thesis.

[29] Weddepohl, H.N., 1996. A Simulation Study of Disequilibrium Price Dynamics. In W.A. Barnett, G. Gandolfo and C. Hillinger (eds.), *Dynamic Disequilibrium Modelling.* Cambridge: Cambridge University Press, 421-443.

[30] Wicksell, K., 1898. *Value, Capital and Rent.* London: Allen and Unwin, 1954.

[31] Younès, Y., 1975. On the Role of Money in the Process of Exchange and the Existence of a Non-Walrasian Equilibrium. *Review of Economic Studies* 42(4), 489-501.

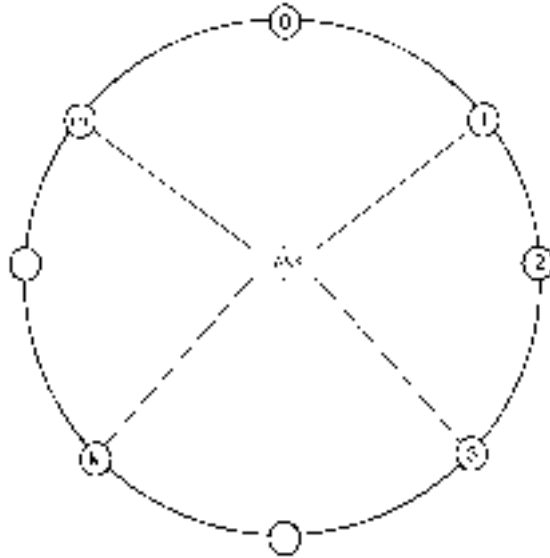[32] Zappia, C., 1999. Equilibrium and Disequilibrium Dynamics in the 1930s. Working Paper no. 243. University of Sienna. URL: http://www.econ-pol.unisi.it/quaderni/243.pdf.

Figure 1: Sequential market visits. Markets are considered as 'trading posts' that are being visited sequentially. The bank is central to all exchanges, since commodities can only be exchanged against money (barter is excluded).
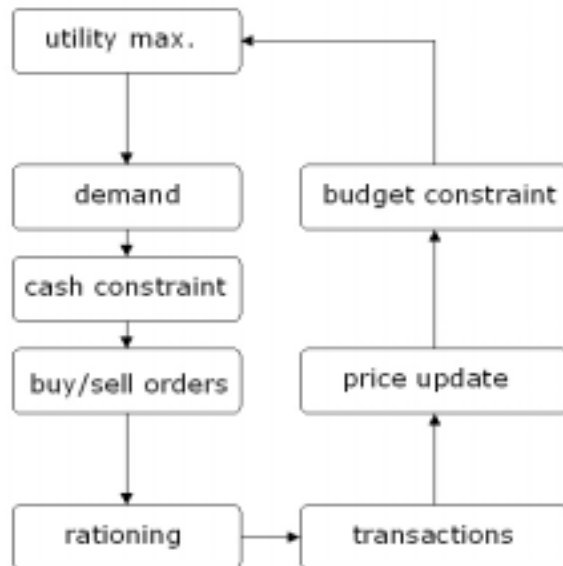
Figure 2: Flow diagram of market events. The diagram shows the sequence of events that occurs during a single market visit. The sequence is repeated for all markets $k = 1, ..., m$, after which the next trading round begins.

Cash-in-advance process.

Initial conditions: $\mathbf{p}(0), \mu_1^h(0) = 0, M_1^h(0) = 100$.

For Round $t = 0, ..., T$ **do**

    For Market $k = 1, ..., m$ **do**

1. $z_k^h := \mathbf{z}_k^h(\mathbf{p}, \mu_k^h)$

   $d_k^h := \max\{0, z_k^h\}, D_k := \sum_h d_k^h,$

   $s_k^h := \min\{0, z_k^h\}, S_k := -\sum_h s_k^h$

2. $\hat{z}_k^h := \min\{z_k^h, M^h/p_k\}$

3. $\bar{z}_k^h := \mathbf{F}_k^h(\hat{z}_k^1, ..., \hat{z}_k^N)$

4. $R^h(t, k) := -p_k(t, k)\bar{z}_k^h(t, k)$

   $M^h(t, k+1) := M^h(t, k) + R^h(t, k),$

5. $\mu^h(t, k+1) := \min\{M^h(t, 1), ..., M^h(t, k+1), M^h(t-1, k+2), ..., M^h(t-1, m)\}$

6. $p_k(t+1, k) := \max\{(1 - r^-) \cdot p_k(t, k), \min\{p_k(t, k) \left(\frac{D_k(t,k)}{S_k(t,k)}\right)^\lambda, (1 + r^+) \cdot p_k(t, k)\}\}$

    end

end.

Figure 3: Algorithm for the sequential cash-in-advance model.
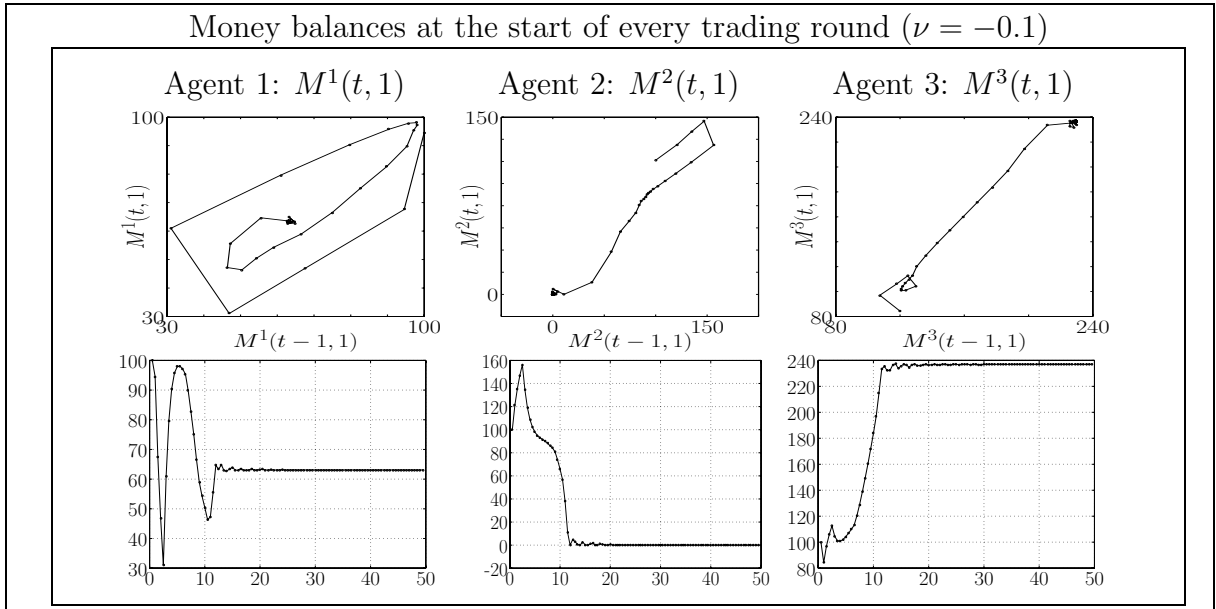
Figure 4: Timeseries of individual money balances at the start of every trading round, for 50 consecutive rounds ($0 < n < 50$). Parameters: $\nu = -0.1$, $\lambda = 1$. The equilibrium corresponding to the substitution parameter $\nu = -0.1$ is slightly shifted from the equilibrium values at $\nu = 0$. For agent 1 the equilibrium cash balance is therefore slightly above 60 and for agent 3 it is slightly below 240: $M^{1\star} = 62$, $M^{2\star} = 0$ and $M^{3\star} = 238$.
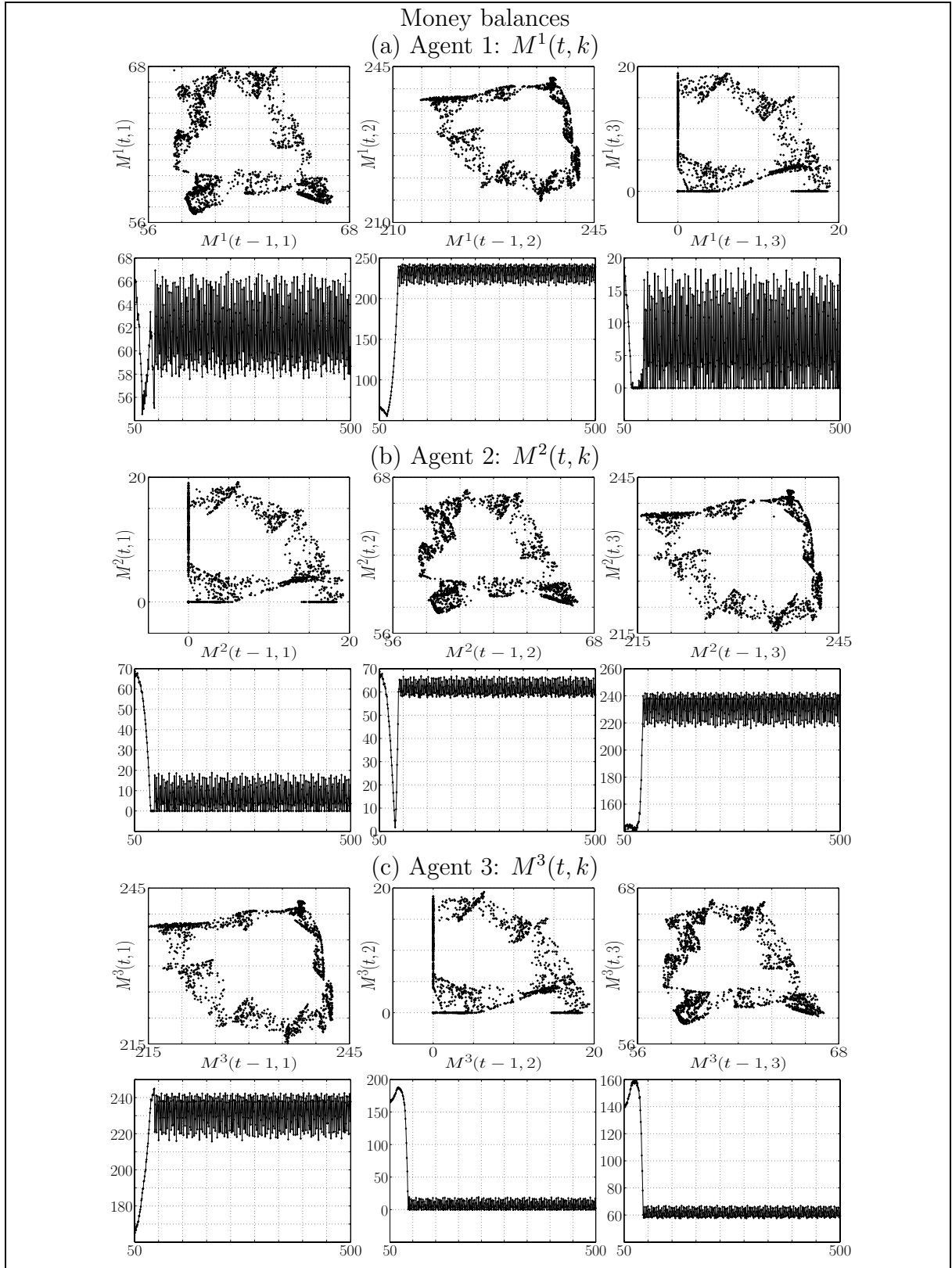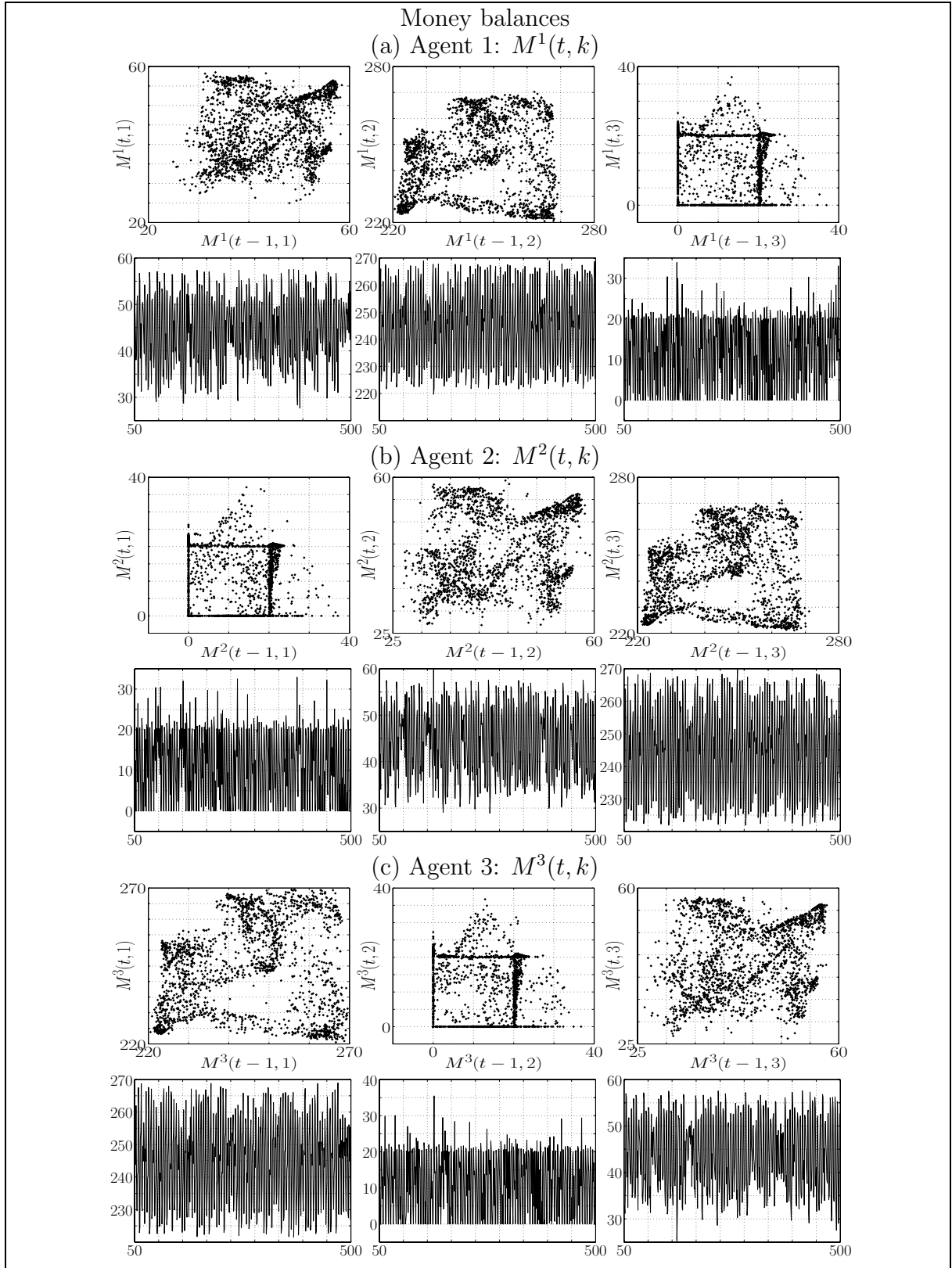
Figure 5: Parameters: $\nu = 0$, $\lambda = 1$. Money balances at the start of every market visit, for 450 consecutive rounds ($50 < n < 500$). Timeseries and phase plots of individual money balances: $M^1(t,k)$, $M^2(t,k)$ and $M^3(t,k)$, for $k = 1, 2, 3$. Eqm: $\mathbf{p} = (6,6,6)$, $M^{1\star} = (60, 240, 0)$, $M^{2\star} = (0, 60, 240)$, $M^{3\star} = (240, 0, 60)$.

Figure 6: Parameters: $\nu = 0.40$, $\lambda = 1$. Money balances at the start of every market visit, for 450 consecutive rounds ($50 < n < 500$). Timeseries and phase plots of individual money balances: $M^1(t,k)$, $M^2(t,k)$ and $M^3(t,k)$, for $k = 1, 2, 3$. Eqm: $\mathbf{p} = (6, 6, 6)$, $M^{1\star} = (40.8, 259.2, 0)$, $M^{2\star} = (0, 40.8, 259.2)$, $M^{3\star} = (259.2, 0, 40.8)$.
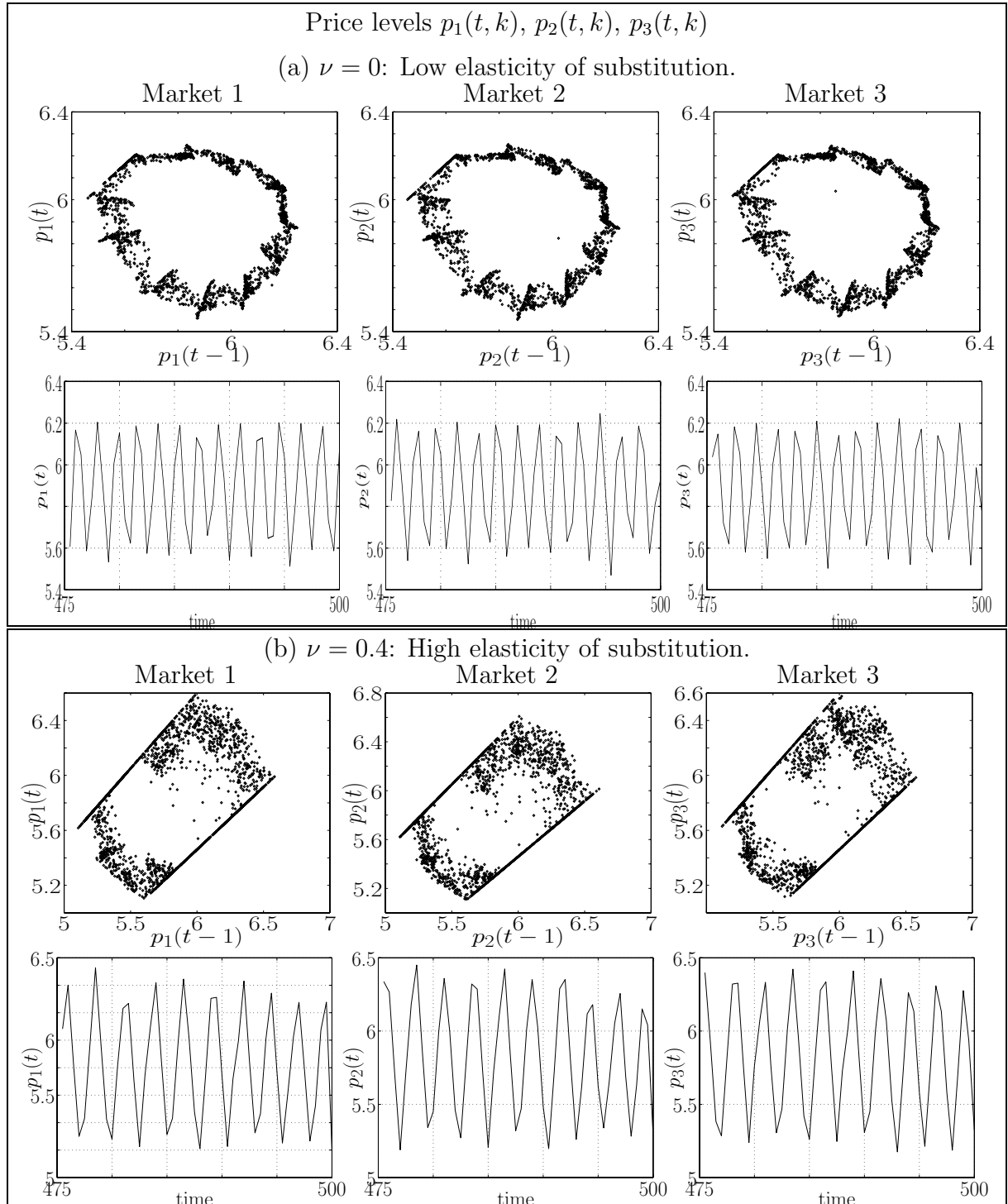
Figure 7: Parameters: (a) $\nu = 0$, $\lambda = 1$ (b) $\nu = 0.4$, $\lambda = 1$. Timeseries and phase plots of price levels: $p_1(t, 1)$, $p_2(t, 2)$ and $p_3(t, 3)$. Prices at the start of every market visit, for 25 consecutive rounds ($475 < n < 500$).
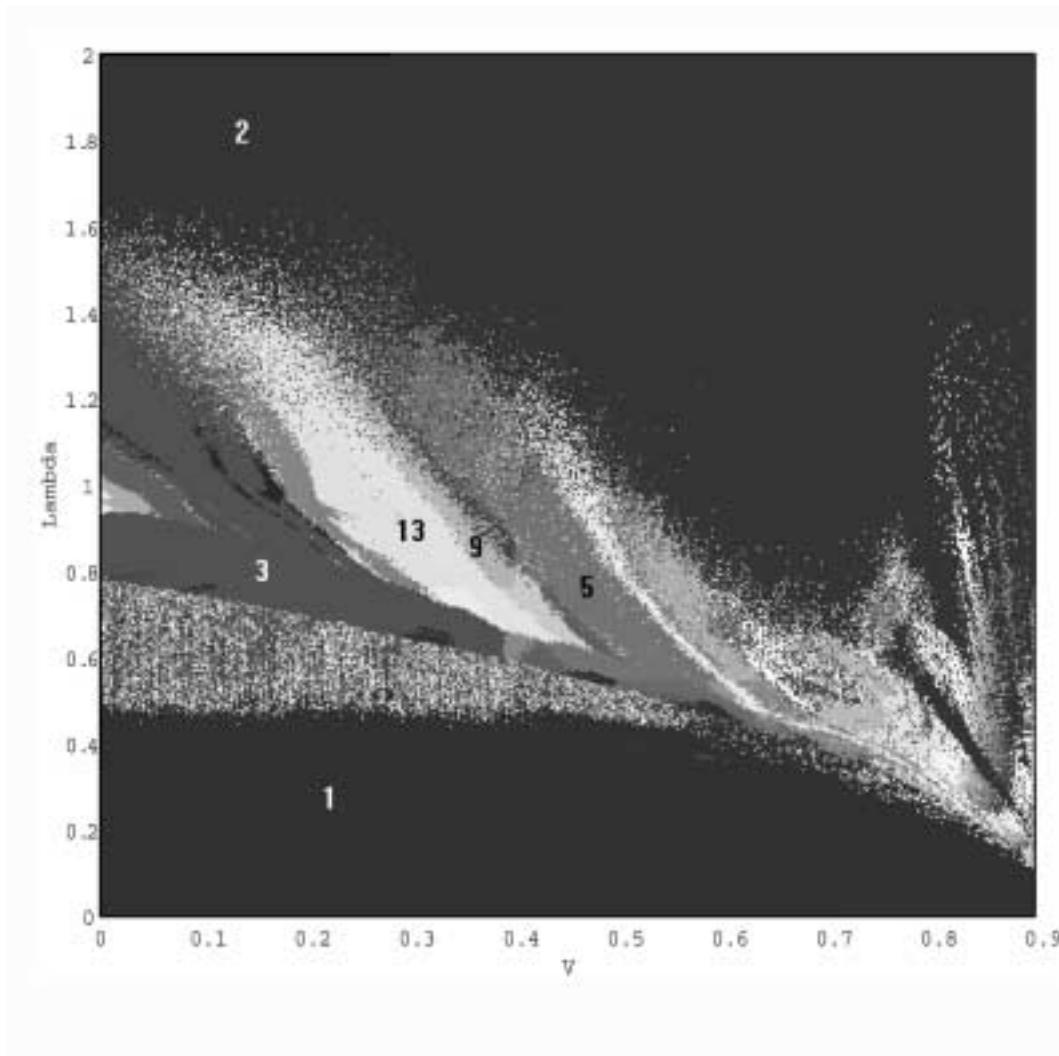
Bifurcation diagram

Figure 8: A 2-parameter bifurcation diagram in the (λ, ν)-plane, for $0 < \lambda < 2$, $0 < \nu < 0.9$. λ is the price flexibility, ν is the substitution parameter.
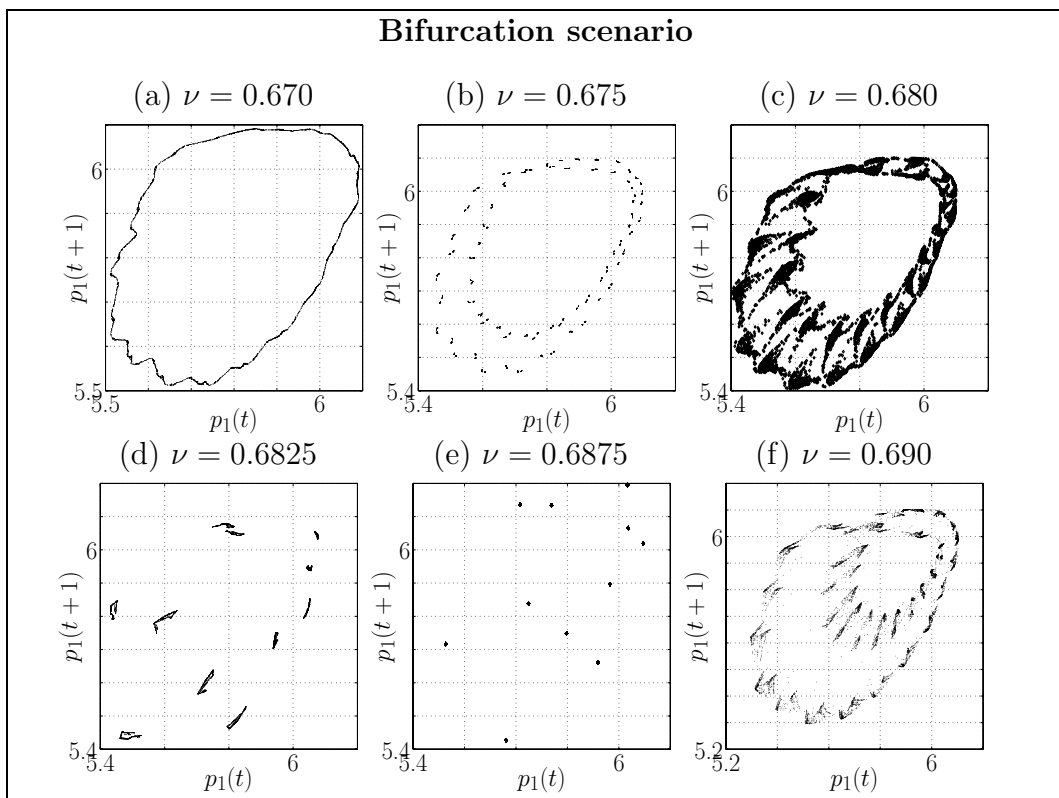
Figure 9: Phase plots in the price phase-space $(p_1(t), p_1(t+1))$ for $\nu \in [0.67, 0.69]$, $\lambda = 0.5$. $T = 10.000$, transient$= 1.000$. Initial condition: $M^h(0) = 100$. (a) An invariant circle becomes unstable and (b) undergoes a period-doubling. (c) An invariant torus is created. (d) The torus breaks up into an 11-piece attractor. (e) A stable 11-cycle is created due to resonance. (f) The attractor gets a figure eight-shape.