Structural analysis of optimal investment for firms with non-concave production

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Abstract

Qualitative properties of optimal investment strategies for a firm with quadratic costs and non-concave production are analysed. Organising the qualitative information in a bifurcation diagram, it is found that the structure of the bifurcation manifolds is determined by a so-called swallow-tail singularity. This implies the existence of threshold (Skiba) points for positive discount factors. The parameter region for which threshold points exist is determined numerically, and for small discount factors some of its properties are derived by an approximation method.

1 Introduction

If at a given point in time, a firm's capital stock is not optimal, that is, if the marginal product of capital does not equal the marginal cost of capital, a manager has to take a decision about investment or disinvestment. Under the assumption that there adjustment costs to be paid on changing the capital stock, it is not optimal to try to jump to the optimal level all at once; rather, the manager has to make an investment plan. That firms do not change their capital stock instantaneously has already been claimed by Keynes (1937); the work of Eisner and Strotz (1963), Lucas (1967), Gould (1968) and Treadway (1969) has shown how "Keynesian" investment schedules can be derived from optimality considerations by assuming adjustment costs of capital.

Increasing returns to scale over some regions of the firm size may lead to the existence of several optimal long term steady states, the initial size of the firm determining to which of these states an optimal investment plan will lead. This divides the possible initial sizes into 'optimal basins of attraction' of the respective long term steady states; these basins are separated by threshold points, which are often called Skiba or Dechert-Nishimura-Skiba points, after the seminal articles of Skiba (1978) and Dechert and Nishimura (1983). For a competent overview on the literature on threshold points, see Deissenberg *et al.* (2001).

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Whether such thresholds exist for a given situation is a delicate nonlinear problem, which depends on the convexity of the cost function and the non-concavity of the production function. The problem is investigated in the following pages by parametrising the strength of the non-concavity and the shape of the cost function, to obtain a detailed picture of the relative effects of these properties. This picture takes the mathematical form of a multi-dimensional bifurcation diagram.

Previously (Wagener (2003)) it has been shown that knowledge of the bifurcations of the state–co-state system associated to the optimality problem contains enough information to determine whether or not, for given values of the parameters, there are threshold points in the system, provided the state variable of the problem is one-dimensional: the set of parameters for which threshold points exist was shown to be bounded by manifolds of saddle-node and heteroclinic bifurcations. It is conjectured that this relation is general and holds for all dimensions of the state space.

A major problem in the investigation of nonlinear systems is that there are usually no global results to be had, unless there is some additional structure present in the model. In the present case, for small discounting, the state–co-state equations are close to a so–called Hamiltonian system; this Hamiltonian structure places strong restrictions on the possibilities of the phase flow, and there are several methods of perturbation theory that can be applied to the problem.

The investigation will try to elucidate the consequences of changes in the various parameters, but some statements will be of a somewhat tentative nature, as they have only numerical simulations to back them up. Though the main thrust of the following will be the analysis of a well-chosen model system, inspired on the model investigated by Haunschmied *et al.* (2003), it should be noted that the results obtained are robust, structurally stable in the mathematical sense.

Main ideas. To introduce the ideas of the present article, let the capital stock of a firm be denoted by the variable $x > 0$, and let the associated revenue from production be $R(x) > 0$; revenue is always assumed to increase with the capital stock, that is, it is assumed that $R'(x) > 0$ for all x. Investment in capital is denoted by $u \geq 0$; the dynamics of the capital stock are

$$\dot{x} = u - \sigma x, \qquad x(0) = x_0,$$

where $\sigma > 0$ is the depreciation rate. By choosing units of time and investment suitably, the parameter σ can be scaled to 1, so that the capital dynamics read simply as

$$\dot{x} = u - x, \qquad x(0) = x_0. \tag{1}$$

Investment at a rate u costs $C(u)$ per unit time; costs are increasing and convex in u.

Consider the (not optimal) stabilising investment strategy $u = u_s(x) = x$; note that for this investment level $\dot{x} = 0$, that is, the strategy stabilises the capital stock at $x(t) = x_0$. The profit of the firm per unit time is then given by $\Pi_s(x_0) = R(x_0) - C(x_0)$. For a revenue function of concave-convex-concave type, figure 1 shows two possible shapes of Π_s.

Imagine the case of a firm at an initial capital level x_I. In both cases of figure 1, the short-sighted strategy would be to decrease the level of investment steadily, until the capital stock is at level x_A. While in the left hand case, this is clearly the right strategy, in the right hand case a long-sighted manager has a more difficult choice to make: by increasing investment rather than decreasing it, the firm might reach a capital level close to x_B, yielding a higher profit

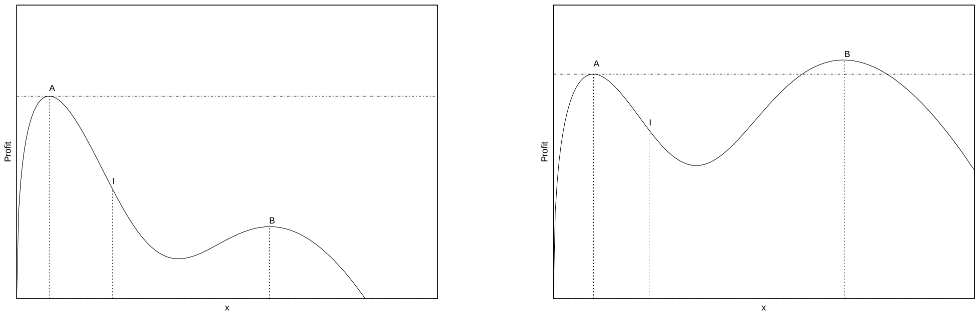


Figure 1: *Profit functions for stabilising investment strategies.*

stream. Whether this is the right strategy depends on how these future gains are weighted, and on the precise level of the firms initial capital stock. It might be the case that it is always best to let the firm expand in order to reach x_B, for instance if future gains are allowed to weigh heavily; conversely, if the future is discounted strongly, it might depend on the initial position of the firm whether it is best to reach x_B, or whether it should rather shrink towards x_A, and there will be a threshold point where both possibilities are equally attractive.

2 Setup

2.1 Control problem. The manager of the firm faces the following optimality problem: to maximise the discounted profit functional

$$J[x,u] = \int_0^\infty (R(x) - C(u))\,\mathrm{e}^{-\rho t}\,\mathrm{d}t, \tag{2}$$

$\rho > 0$ denoting the discount rate, subject to the condition that equation (1) holds for all t.

Introduce y as co-state variable; the Pontryagin function of the problem reads as

$$P(x,y,u) = -(R(x) - C(u)) + y(u - x);$$

minimising P over u yields the relation

$$C'(u) + y = 0.$$

Let c be the marginal cost function, given as $c(u) = C'(u)$; analogously, let the marginal revenue function r be given by $r(x) = R'(x)$. The relation between co-state and control can then be written as $y + c(u) = 0$, or $u = c^{-1}(-y)$, where c^{-1} denotes the inverse of c. The Hamilton function of the problem takes the form

$$H = -R(x) + C(c^{-1}(-y)) + y(c^{-1}(-y) - x);$$

the reduced Hamiltonian equations $\dot{x} = H_y$, $\dot{y} = \rho y - H_x$ read as

$$\dot{x} = c^{-1}(-y) - x, \qquad \dot{y} = \rho y + r(x) + y. \tag{3}$$

2.2 Equilibria. To determine the equilibria of the system (3), note that the equations $\dot{x} = 0$ and $\dot{y} = 0$ are equivalent to

$$y + c(x) = 0, \qquad (1+\rho)y + r(x) = 0,$$

which after elimination of y leads to

$$\delta(x;\rho) = r(x) - (1+\rho)c(x) = 0. \tag{4}$$

Also introduce $\Delta(x;\rho) = R(x) - (1+\rho)C(x)$; then $\frac{\partial \Delta}{\partial x} = \delta$. Note that all equilibria of (3) correspond in a one-to-one fashion to solutions to $\delta(x;\rho) = 0$, or, expressed geometrically, to the intersections of the graphs of r and $(1+\rho)c(x)$.

The same conclusion is reached when a new co-state variable $p = y + c(x)$ is introduced. The system (3) reads in the new variables as

$$\dot{x} = c^{-1}\Big(c(x) - p\Big) - x, \qquad \dot{p} = \delta(x;\rho) + (1+\rho)p + c'(x)\left[c^{-1}(c(x)-p) - x\right].$$

Since $\dot{x} = 0$ implies $p = 0$, all equilibria are located on the line $\{p = 0\}$. If the argument of c^{-1} is developed in a first order Taylor series for small values of p, the system simplifies to

$$\dot{x} = -\frac{p}{c'(x)} + \mathsf{O}(p^2), \qquad \dot{p} = \delta(x;\rho) + \rho p + \mathsf{O}(p^2).$$

It is clear that the equilibria of this system are those points $(x_*, 0)$ for which $\delta(x_*;\rho) = 0$.

3 Quadratic costs

This section restricts attention to the following model specifications. Costs are assumed to be quadratic

$$C(u) = c_1 u + \frac{c_2}{2}u^2,$$

or, equivalently, marginal costs are assumed to be linear: $c(u) = c_1 + c_2 u$. Though the economic relevant case is $c_1, c_2 > 0$, it will be seen below that the 'non-relevant' cases carry structural information about the model. The revenue function is specified as

$$R(x) = \sqrt{x} - \lambda\frac{x}{1+x^4}; \tag{5}$$

this is (up to scalings) the same as used by Haunschmied *et al.* (2003).

3.1 Bifurcation at infinity. Let E denote the epigraph $\{(x,y) \in \mathbb{R}^2 \,:\, y > r(x)\}$ of r; its boundary ∂E is the graph of r. Let moreover ℓ be the line $\{(x,y) \,:\, y = c_1 + c_2 x\}$.

If $M > 0$ is large enough, the part of the line ℓ for which $x > M$ is contained in E whenever $c_2 > 0$ or $c_2 = 0$ and $c_1 > 0$, while the part of ℓ for which $x < 0$ is in the complement of E. Hence ℓ can cross γ only an odd number of times (counting multiplicities). On the other hand, if $c_2 < 0$, or if $c_2 = 0$ and $c_1 \leq 0$, the line ℓ begins and ends outside E, and hence crosses γ an even number of times.

Hence, the line $\{c_2 = 0\}$ is a bifurcation curve, where one equilibrium disappears towards infinity as the parameter c crosses the curve.

If λ is sufficiently close to 0, the function r is convex, its epigraph is a convex set, and any line can have at most two intersections with its boundary. The above possibilities reduce to one intersection in the odd case, and zero or two intersections in the even case.

3.2 Saddle-node bifurcations. The simplest possible bifurcations of equilibria are the confluence and subsequent vanishing of two intersections of γ and ℓ; this corresponds to a saddle-node bifurcation of the original system.

The geometric condition for a saddle-node bifurcation of equilibria is that the line ℓ is tangent to γ; analytically, tangency of ℓ and γ at the point $(x_0, r(x_0))$ is expressed by

$$c_1 + c_2 x = \frac{1}{1+\rho}\left(r(x_0) + r'(x_0)(x - x_0)\right).$$

Hence all saddle-node bifurcation parameters are located on the curve

$$c_1(s) = \frac{1}{1+\rho}\left(r(s) - r'(s)s\right), \quad c_2(s) = \frac{1}{1+\rho} r'(s),$$

where $s \in \mathbb{R}$ parametrises the curve. Note that these expressions can be computed easily.

3.3 Cusps and self-intersections. The curve $c(s) = (c_1(s), c_2(s))$ has a classical interpretation as the transformation of the curve $(x, r(x)/(1+\rho))$ into *line* or *tangent* coordinates (see for instance van der Waerden (1939) or Arnol'd (1988)). Whereas ordinarily a curve $s \mapsto \gamma(s)$ is given as the set of points located on the curve, in tangent coordinates the curve is given as the set of its tangents; the curve itself is recovered as the envelope of this family of lines.

The point of this for the present case is that singularities of the transformed curve correspond to bifurcations of higher codimension, as the transformed curve gives the location of the saddle-node bifurcations in the space of parameters. But singularities of the transformed curve correspond to geometric properties of the original curve, and in the present case, to properties of the function $r(x)$.

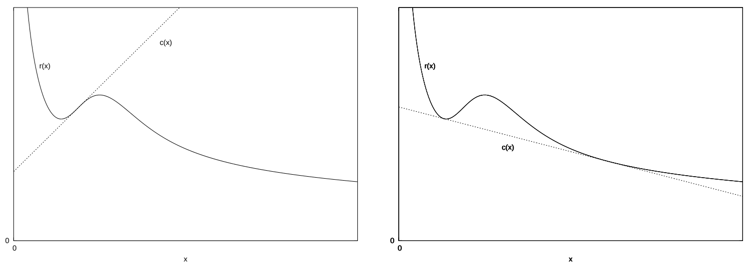


Figure 2: *Configurations of the marginal revenue function that are associated to singularities of the saddle-node bifurcation curves. Left: an inflection point gives rise to a cusp bifurcation. Right: a double tangent gives rise to a self-intersection of the saddle-node curve.*

The only two singularities which are of interest for the present case are points where r changes from convexity to concavity, that is, points s_0 where the second derivative of r has a simple zero — these correspond to cusps in the transformed curve, since $c_1'(s_0) = c_2'(s_0) = 0$ — and the situation that a line ℓ is tangent to γ at several places, giving rise to points of self–intersection of the transformed curve (see figure 2).

Note that by setting $f_1(x) = \sqrt{x}$ and $f_2(x) = x/(1+x^4)$, the function R can be written as $R = f_1(s) - \lambda f_2(s)$. Since the condition for a cusp bifurcation is that $r''(s_0) = R'''(s_0) =$

0, cusp bifurcations of the system are located on the curve

$$c_1(s) = \frac{1}{1+\rho}\left(r(s) - r'(s)s\right), \quad c_2(s) = \frac{1}{1+\rho} r'(s), \quad \lambda(s) = \frac{f_1'''(s)}{f_2'''(s)}.$$

4 Heteroclinics

In the previous section, the steady states of the state–co-state system have been determined. However, depending on the circumstances, not all of these might be economically relevant.

4.1 No discounting. This is most easily seen in the case that $\rho = 0$; since in this case the integral (2) in the utility functional usually does not converge, it cannot be used to compare investment strategies. Instead, the notion of *overtaking optimality* is needed: if $J_T = \int_0^T (R(x) - C(u))\,\mathrm{d}t$, then the strategy $u_1(t)$ overtakes the strategy $u_2(t)$ if there exists a constant T_1 such that $J_T[x_1, u_1] > J_T[x_2, u_2]$ for all $T > T_1$.

Note that for $\rho = 0$ the equilibria are determined by the solution of

$$r(x) = c(x),$$

that is, by the condition that marginal costs equal marginal production, and that there are no other possibility for the optimal investment strategy than to converge to one of these equilibria.

Assume that (x_A, y_A) and (x_B, y_B) are two saddle equilibria of the state–co-state system. In equilibrium $y_i + c(x_i) = 0$, $i \in \{A, B\}$, and

$$h_i = H(x_i, 0) = C(x_i) - R(x_i),$$

that is, h_i is the difference between cost and production for the equilibrium level of capital x_i; the lower h_i is, the better.

If $h_A < h_B$ and the future is not discounted at all, then it is clear that any investment strategy which leads to x_A as asymptotic capital stock level eventually overtakes any strategy which leads to x_B instead. Hence, for $\rho = 0$, the optimal investment always leads to the equilibrium that yields the lowest value of H, if there is a unique equilibrium with this property. Since this property is satisfied by an open and dense set of systems, this is almost always the case.

4.2 Positive discounting. Yet the case that $h_A = h_B$ is not devoid of interest; for imagine for the moment that the discount rate ρ is positive but small. If the initial level of capital x_0 is close to x_A, it is certainly better to converge to x_A than to work all the way towards x_B; even if h_B is a little below h_A, it may still be optimal to remain close to x_A: the distant winnings may be so much discounted that they do not compensate for the losses which are incurred by going from a neighbourhood of x_A to a neighbourhood of x_B.

These heuristic considerations lead to the question whether there is a unique optimal long-run steady state, or whether there may be several. It has been shown (Wagener (2003)) that the set of parameters for which there may be several optimal long-run steady states is bounded by saddle-node and heteroclinic bifurcation curves of the state-co-state system.

To analyse the situation, it is convenient to pass to the variable $p = y + c_1 + c_2 x$ as in subsection 2.2. The state–co-state equations take the form

$$\dot{x} = -\frac{p}{c_2}, \qquad \dot{p} = \delta(x; \rho) + \rho p. \tag{6}$$

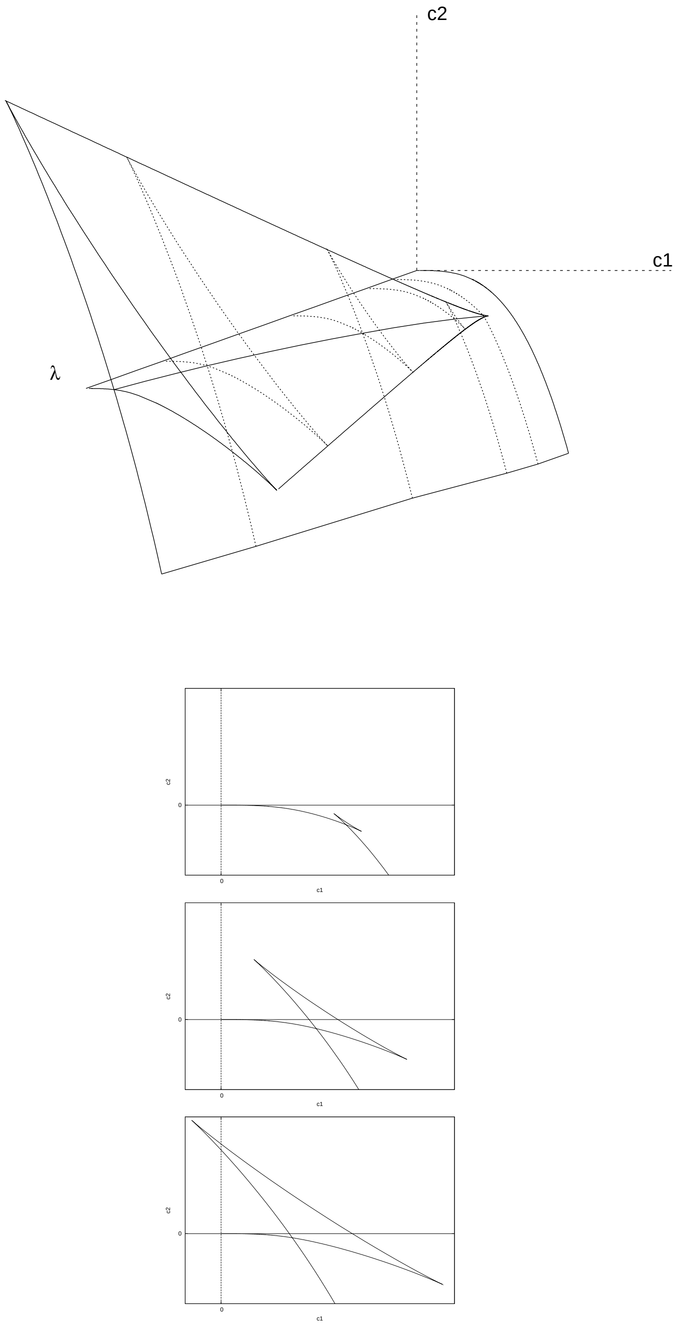


Figure 3: *Left: the three-dimensional bifurcation diagram of equilibria in (λ, c_1, c_2)-space. The surface of saddle-node bifurcations displays a swallow-tail singularity. Right: the bifurcation diagrams of equilibria for $\lambda = 0.1$, 0.3 and 0.45 respectively, from top to bottom. Shown are the curve of saddle-node bifurcations and the curve $c_2 = 0$ of bifurcations at infinity (both solid). The dashed line $c_1 = 0$ indicates the left-hand boundary of the economic relevant region $c_1, c_2 > 0$.*

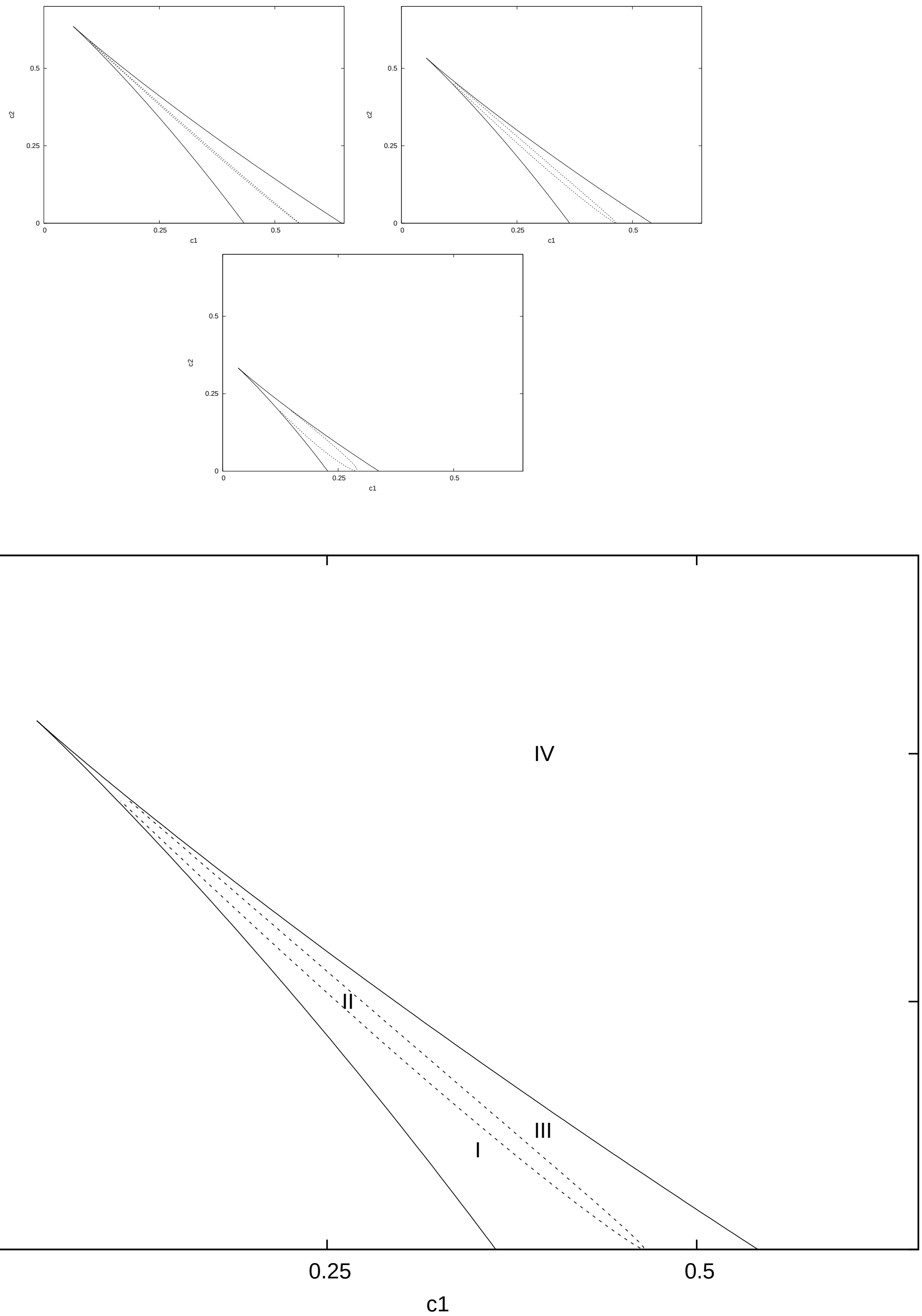
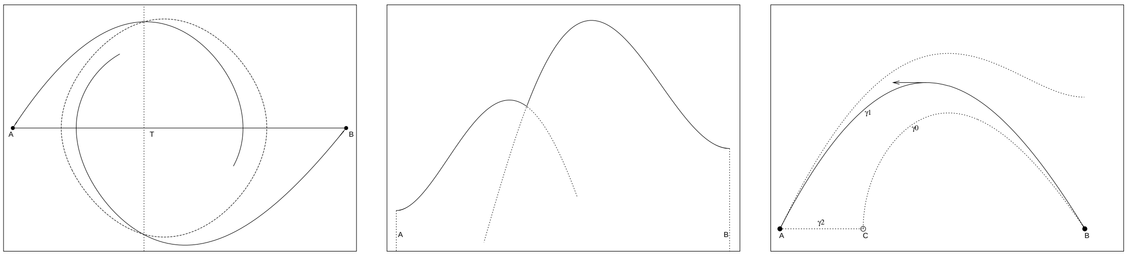


Figure 4: *Full bifurcation diagrams for $\lambda = 0.35$ and $\rho = 0.05$, $\rho = 0.25$ and $\rho = 1$ (first row, from left to right), and enlargement of the diagram for $\rho = 0.25$. The size of the set for which threshold points exist grows as ρ increases. For parameters (c_1, c_2) in region* I *of the enlargement, the equilibrium with the highest capital stock is always optimal, in* II *the long-term steady state is determined by the initial capital stock $x(0)$, whereas in* III *the lowest capital stock is always the optimal long-term outcome. For parameters in region* IV *there is only one long term steady state for the system.*



(a) *(b)* *(c)*

Figure 5: *Occurrence of a threshold point (a), the corresponding value function (b), and a heteroclinic connection between saddle points (c). In the phase diagrams (a) and (c), level curves of the Hamiltonian are shown by broken lines, while phase curves of the reduced system are solid; the arrow indicates the direction of the phase flow. In (b), the graph of the value function is drawn, while the broken lines represent the values of the functional J continued along the stable manifolds of A and B.*

The variable transformation $\Psi(x, y) = (x, y + c(x))$ is symplectic; this means roughly that the variable p is a 'good' co-state variable. In the (x, p)–coordinates, the Hamiltonian takes the form $H(x, p) = -p^2/c_2 - \Delta(x; \rho)$. Note that after a time reflection $t \to -t$, the Hamiltonian changes sign, reading as $H = \frac{p^2}{c_2} + \Delta(x)$, which is the standard form of a mechanical system in one degree of freedom (see Arnol'd (1989)). In this picture, the term $\rho p \to -\rho p$ corresponds to friction that is linear in the velocity. Also note that all equilibria are on the axis $p = 0$.

4.3 Melnikov's method. For small values of ρ, the approximate location of the heteroclinic bifurcation curves can be determined by Melnikov perturbation theory (cf. Guckenheimer and Holmes (1986)). To see this in a general way, assume that besides depending on ρ, the state–co-state system depends on another, multi-dimensional, parameter $\mu \in \Sigma \subset \mathbb{R}^q$. Let μ_0 be a value of this parameter such that all for (μ, ρ) close to $(\mu_0, 0)$, there are two saddle equilibria $(x_A, 0)$ and $(x_B, 0)$. Assume also that

$$h_A(\mu, \rho) = H(x_A(\mu, \rho), 0; \mu, \rho) \leq H(x_B(\mu, \rho), 0; \mu, \rho) = h_B(\mu, \rho);$$

this last assumption can always be achieved by relabeling the equilibria.

Let $s \in \mathbb{R}$ parametrise a curve of heteroclinic bifurcation parameters $s \to \bar{\mu}(s; \rho)$ in the parameter space Σ, such that the heteroclinic connection associated to $\mu = \bar{\mu}(s; \rho)$ is contained in the upper half plane $\{p \geq 0\}$. Melnikov's method shall be applied to find an approximation, to first order in ρ, of the quantity $\bar{\mu}(s; \rho) - \bar{\mu}(s; 0)$.

Fix s at s_1 and μ at $\mu_1 = \bar{\mu}(s_1; \rho)$. For $\mu = \mu_1$, let γ_0 denote that part of the level curve of the Hamiltonian $\{H = h_B\}$ of height $h_B(\mu_1, \rho)$ that is in the upper half plane; γ_1 the heteroclinic connection between the saddle points A and B; γ_2 that segment of the line $\{p = 0\}$ that connects the intersection C of γ_0 and $\{p = 0\}$ with the point A. See figure 5c. Denote the region bounded by γ_0, γ_1 and γ_2 by S. The curves will be assumed to be oriented such that the region S is always to the left of them.

Introduce the differential 1-form

$$\omega = (H_x - \rho p)\,\mathrm{d}x + H_p\,\mathrm{d}p = \mathrm{d}H - \rho p\,\mathrm{d}x,$$

associated to the system of differential equations (6). Note that ω vanishes by definition on any vector tangent to an integral curve of (6); hence $\int_{\gamma_1}\omega = 0$. Likewise $\int_{\gamma_0}\mathrm{d}H = 0$, since H is constant along γ_0. Taking this into account, it follows that the integral of ω over the boundary ∂S of S equals

$$\int_{\partial S}\omega = -\rho\int_{\gamma_0} p\,\mathrm{d}x + \int_{\gamma_2}\mathrm{d}H;$$

Denoting by $|S|$ the area of S, it follows by Stokes' theorem (see for instance Spivak (1965)) that

$$\int_{\partial S}\omega = \int_S \mathrm{d}\omega = \rho\int_S \mathrm{d}x\wedge\mathrm{d}p = \rho|S|.$$

Since γ_0 and γ_1 have a mutual distance of order ρ, the area enclosed by these two curves and the axis $\{p = 0\}$ is of order $\mathsf{O}(\rho)$, and the term $\rho|S|$ is of the order $\mathsf{O}(\rho^2)$. The final result that is obtained is that if γ_1 is a heteroclinic connection between the two saddles A and B in the upper half plane $\{p \geq 0\}$, then

$$h_B - h_A - \rho\int_{x_A}^{x_C}\sqrt{-c_2\Delta}\,\mathrm{d}x = 0 + \mathsf{O}(\rho^2).$$

4.4 Implications. By the envelope theorem, partial derivatives of $h_i(\mu,\rho)$, with respect to either μ or ρ are equal to partial derivatives of $H(x_i, 0; \mu, \rho)$ with respect to these variables modulo terms of order $\mathsf{O}(\rho^2)$, since the equilibria $(x_i, 0)$ have a distance of order $\mathsf{O}(\rho)$ to stationary points of H. Also note that for $\rho = 0$, the curves γ_0 and γ_1 coincide, as do the points x_C and x_B. Moreover $h_B(\mu, 0) = h_A(\mu, 0)$. Introducing $D = h_B - h_A$ and $\bar{\mu}^0 = \bar{\mu}(s_0; 0)$, expanding all terms to first order with respect to ρ, and dividing by ρ yields that

$$\frac{\partial D}{\partial\mu}(\bar{\mu}^0, 0)\frac{\partial\bar{\mu}}{\partial\rho}(s_0; 0) + \frac{\partial D}{\partial\rho}(\bar{\mu}^0, 0) = \int_{x_A}^{x_B}\sqrt{-c_2\Delta}\,\mathrm{d}x + \mathsf{O}(\rho).$$

Hence

$$\bar{\mu}(s_0,\rho) = \bar{\mu}^0 + \frac{-\dfrac{\partial D}{\partial\rho} + \displaystyle\int_{x_A}^{x_B}\sqrt{-c_2\Delta}\,\mathrm{d}x}{\dfrac{\partial D}{\partial\mu}}\rho + \mathsf{O}(\rho^2).$$

Completely analogously it follows for the heteroclinic connections in the lower half plane $\{p \leq 0\}$ that

$$\underline{\mu}(s_0,\rho) = \underline{\mu}^0 + \frac{-\dfrac{\partial D}{\partial\rho} - \displaystyle\int_{x_A}^{x_B}\sqrt{-c_2\Delta}\,\mathrm{d}x}{\dfrac{\partial D}{\partial\mu}}\rho + \mathsf{O}(\rho^2);$$

here $\underline{\mu}(s,\rho) \in P$ is the corresponding bifurcation manifold.

In particular, note that as $c_2 \to 0$, the difference between the heteroclinic bifurcation values $\bar{\mu}$ and $\underline{\mu}$ tends to 0 as well; taking $\mu = c_1$, the difference reads as

$$\bar{\mu}(s_0, \rho) - \underline{\mu}(s_0, \rho) = \frac{2\rho}{x_B - x_A} \int_{x_A}^{x_B} \sqrt{-c_2 \Delta}\, \mathrm{d}x + \mathsf{O}(\rho^2).$$

This effect is illustrated by figure 4, where the heteroclinic bifurcation curves bend and approach each other as c_2 approaches 0.

This can be interpreted in economic terms as follows: if investment costs are almost linear, almost no investment problems of the type considered here will have threshold points.

5 Conclusions

The present article has investigated a simple cost-production model of a firm in the situation that the revenue function fails to be concave in some region of firm size; this may arise due to increasing returns to scale of production or to some other reason as *e.g.* network effects (cf. Haunschmied *et al.* (2003) and references therein). The dependence on the problem parameters of the optimal investment strategies have been analysed, and it has turned out that the properties of these strategies depend quite sensitively on the parameters.

To picture the different types of properties, bifurcation diagrams have been drawn; in the model system, they turned out to be organised by a swallow-tail singularity of the equilibria, from which the existence of heteroclinic connections between saddle equilibria followed; using results obtained previously (Wagener (2003)), the parameter region for which threshold points can be expected in the system has been characterised as a region in parameter space bounded by manifolds of heteroclinic and saddle-node bifurcations; these regions are nonempty for positive discount factors.

The saddle-node manifolds have been obtained analytically; the heteroclinic bifurcation manifolds were obtained numerically. However, for small values of the discount factor, an analytic estimate of the size of the region in parameter space for which threshold points exist has been obtained by the Melnikov method. This led in particular to the result that for close-to-linear cost functions, almost no threshold parameters exist.

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