A Dynamic Model of Endogenous Interest Group Sizes and Policymaking

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Abstract

We present a dynamic model of endogenous interest group sizes and policymaking. Our model integrates 'top-down' (policy) and 'bottom-up' (behavioral) in‡uences on the development of interest groups. We show that, for example an increase in the contribution by members of an interest group need not induce larger subsidies to that group, even though it would in case of …xed interest group sizes. This is due to a political participation e¤ect, next to a redistribution e¤ect. On the other hand, the dynamic analysis of the model shows that reliance on equilibrium results such as these can be misleading since equilibria may not be stable. In fact, complicated dynamics may emerge leading to erratic and path dependent time patterns for policy and interest group sizes. We demonstrate that our model can endogenously generate the types of spurts and declines in organizational density that are observed in empirical studies.

Keywords: Interest groups, Aspiration level, Endogenous ‡uctuations

JEL classification: D23; D72; D78; E32; H30

1 Introduction

Interest groups play an important role in economic policymaking. Many empirical studies show this for Europe and the U.S. (Richardson 1994, Potters and Sloof 1996).

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According to Richardson (p 11-12) the following observation by the Norwegian political scientist Stein Rokkan captures the practical essence of European democracy:

> the crucial decisions on economic policy are rarely taken in the parties or in Parliament: the central area is the bargaining table where the government authorities meet directly with the trade union leaders, the representatives of the farmers, the smallholder and the …shermen, and the delegates of the Employers' Association. These yearly rounds of negotiations have in fact come to mean more in the lives of rank-and-…le citizens than formal elections.

Theoretically, the importance of this phenomenon is re‡ected in the studies of Mancur Olson on collective action (Olson 1965, 1982) and the upsurge of endogenous economic policy models concentrating on the interaction between interest groups and economic policymakers (van Winden 1999). Although these models have provided interesting new insights into the determinants of economic policies, their relevance is restricted in several ways. By focusing on equilibria of a properly de…ned game with interest groups of …xed size and the government as (essentially completely informed and fully rational) players, they do not provide an explanation of the development of interest groups, nor do they look into the dynamics of the interaction between the players and allow for incomplete information and (boundedly rational) adaptive behavior in a complex environment. In reality, the relations between government and interest groups are inherently dynamic, as testi…ed by the country studies collected by Richardson (1994). Timely examples are provided by the increasing participation of environmentalists and health groups in the development of agricultural policies, the changing political landscape concerning tobacco, and the recent upsurge in NGOs that are increasingly being co-opted into policymaking (The Economist 1999). On a more aggregate level, the ‡uctuations in the percentage of unionized workers in the U.S. may serve as an illustration. According to Freeman (1997, p. 8) the sudden spurts in union density shown by Figure 1 are not only characteristic for the U.S. but also for other countries.

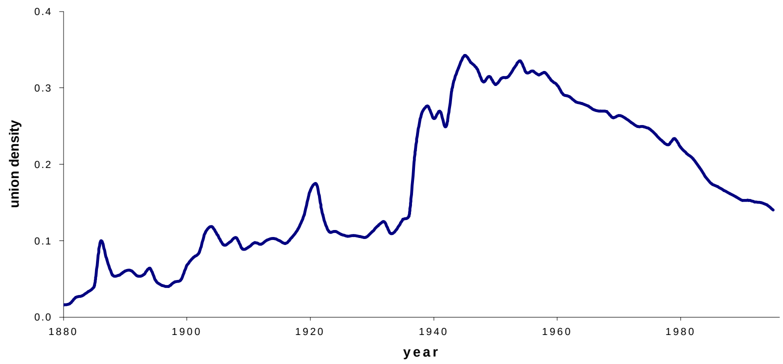

Figure 1: Time series of the density of union membership in the U.S., 1880-1995

On the other hand, the time-series Freeman (1988, p. 69) presents regarding the development of union densities in different countries show that the pattern of these fluctuations over time is very diverse, with some countries facing increases while others are experiencing declines. He concludes that this constitutes powerful evidence against, or at least casts doubt on, broad explanations (such as unions having become obsolete in modern market economies), structuralist arguments (pointing at changes in the composition of the work force), or general macroeconomic explanations (referring to the oil shock, for instance). Freeman (1997) distinguishes two types of models that can generate spurts in union growth. The first are standard comparative statics linear models in which exogenous shocks (usually generated by political forces, like laws) generate responses in otherwise stable union membership. The second are models in which the growth process creates non-linearities producing 'phase transitions' when certain conditions are met (models of self-organized complexity). Without denying the importance of political 'top-down' changes as triggers for the growth process, Freeman's study of the development of union density in the U.S. focuses on and argues in favor of the second type of ('bottom-up') models; these models stress "the underlying process by which organization occurs and the cumulative behavior of individual workers, unions, and firms. (...) the behavior of thousands or millions of individuals acting in response to one another" (p. 9). The above examples concerning agriculture, the tobacco industry, and NGOs suggest that this 'bottom-up' approach is also important for an analysis of the development and influence of other interest groups.

In this paper we take a first shot at explicitly modeling the dynamics of interest group size and the interaction between interest groups and governmental policymaking, focusing on redistribution. The advantage of a theoretical model is that one can focus on important aspects without having to bother about data limitations or violations of ceteris paribus assumptions that empirical analyses are generally plagued with (see e.g. Neumann and Rissman 1984). For tractability, we develop a simple model consisting of three essential parts: one determining the individual propensity for collective action, another determining the organizational density of an interest group, and a third generating government policy. Because the (redistribution) policy feeds back into the other two parts of the model, the 'top-down' and 'bottom-up' approaches distinguished above are integrated in one model. Both the dynamics and the comparative statics of this model are investigated.

One of the main results is that the process of interest group development may inhibit the occurrence of a stable political economic equilibrium, leading to complicated dynamics in the interaction between the organization of social groups and governmental policymaking. Different types of fluctuations in the organizational densities of the interest groups, as well as in the tax rate for redistribution, are observed for different choices of the behavioral parameter values. Regular fluctuations of short or long length, or short fluctuations superimposed on long ones, are obtained, where densities 'mirror' or 'follow' each other. But also (highly) irregular patterns show up.

Time series like exhibited by Figure 1 - with sudden spurts and sharp declines - may occur. The model offers an endogenous mechanism causing these time patterns, in which both ‘top-down’ and ‘bottom-up’ factors play a role. Furthermore, it is noted that our model is not restricted to the interaction between workers/unions and employers. It, more generally, refers to social groups with conflicting economic interests and with a potential influence on government policies (like workers versus capitalists, age-groups, industries within an economic sector, and so on).

Our analysis clearly shows the restrictiveness of the common assumption of fixed sized interest groups in endogenous policy models. It turns out that the innocence of such an assumption very much depends on the nature and state of the behavioral mechanisms (think of the occurrence of sudden spurts). The comparative-statics analysis, furthermore, helps explain why union leaders are critical of income inequality and why they may have reservations concerning social welfare policies (cf. Neumann and Rissman 1984). This analysis also addresses the impact of demographic and sectorial shifts.

The rest of the paper is organized as follows. Section 2 presents the model. Comparative statics are addressed in Section 3, while Section 4 goes into the dynamic features of the model. A concluding discussion is offered in Section 5.

2 The model

For convenience, our model focuses on two economic sectors, A and B, each employing a large number of agents. All individuals in sector $i (= A; B)$ are endowed with an income w_i. There is no mobility between sectors and the number of agents in each sector is exogenously given as m_i. Furthermore, all individuals are assumed to have the same indirect utility function $V(y)$, for which we make the following standard assumptions: $V(y) \geq 0$, $V(0) = 0$, $V'(y) > 0$, $V''(y) < 0$ and $\lim_{y \to 0} V'(y) = \infty$.

We assume that the government can redistribute income by levying a, possibly negative, lump-sum tax of τ_A on the individuals in sector A; which implies a lump-sum subsidy to the individuals in sector B equal to $\tau_B = -\frac{m_A}{m_B}\tau_A$, in order to balance the government budget.

Individuals in each sector can organize into an interest group which entails a given contribution c_i per individual. The contribution fee leads to a reservation value $r(c_i)$, with $r'(c_i) > 0$ and $r(0) = 0$. This reservation value can be seen as the monetary equivalent of the effort expended in the collective action of the interest group. For individual j ($j = 1; \ldots; m_i$) in sector i indirect utility equals $V(w_i - \tau_i - r(c_j))$, where $c_j = c_i$ ($i = A; B$) for interest group members and $c_j = 0$ for the non-members. Collective action of the interest groups consists of lobbying for a tax schedule that favors both the members and the non-members in the respective sector. The group-specific public good (bad) nature of the tax schedule introduces a free-riding problem, which is characteristic for many types of interest groups.

We first provide a simple model for the development of interest groups (a microfoundation is provided in Appendix A). The model consists of two submodels: one determining the individual propensity to join an interest group and another determining the size (membership) of the interest group. Assuming bounded rationality, the propensity of an individual to join is related to the gap between actual utility and a reference utility level. Because the cost of a contribution will be mentally traded off against the threat of a positive tax, it seems natural to take as reference utility level $V(w_i - r(c_i))$, which materializes if participation in collective action leads to the absence of a tax. This implies that people are expected to be more inclined to join the interest group the more the reservation value (i.e. the required effort cost) falls short of the tax they have to pay under the current regime. More specifically, for each of the individuals of sector i the probability of joining that sector's interest group is assumed to be given by a function Λ of this gap, that is

$$\Pr[\text{joining}]_i = \Lambda(\beta_i [V(w_i - \tau_i) - V(w_i - r(c_i))]), \qquad i = A, B,$$

where $\Lambda : \mathbb{R} \to (0,1)$ and $\Lambda' < 0$. The parameter $\beta_i > 0$ measures the behavioral sensitivity to a gap between actual and reference utility. This sensitivity may be related to cultural factors specific to the social group considered (e.g. a tradition of collective action) or to limited information on how government policies impact utility, in which case it resembles the quantal response model of McKelvey and Palfrey (1995). Empirical evidence for the assumption that dissatisfaction with government policies is a determinant of political participation is provided by empirical models of voter behavior in large-scale elections where the probability of voting for an opposing party (which can be considered as an interest group in itself) is related to the dissatisfaction of the voter with the economic situation under the incumbent government (see e.g. Paldam 1997).

Whether the propensity to join materializes into actual organization, or to staying a member, will depend on several factors. Legal rights to organize, for example, have historically played an important role in the organization of unions. Another important factor concerns the ability of interest group leaders (political entrepreneurs) to mobilize discontent or to maintain membership (cf. Rothemberg 1988). Here, we assume a simple partial adjustment process for the evolution of the size of an interest group. With probability λ_i the propensity to join is assumed to lead to actual membership, while with probability $1 - \lambda_i$ the individual stays put. Given that there are a large number of individuals in each sector, the sizes of the interest groups (n_i) evolve deterministically as

$$n_{i,t+1} = (1 - \lambda_i) n_{i,t} + \lambda_i m_i \Lambda(\beta_i [V(w_i - \tau_i) - V(w_i - r(c_i))]), \qquad i = A, B. \quad (1)$$

Given the contribution level, the size of the interest group next determines the total resources available for collective action[1].

We now turn to the government. In line with the literature concerning endogenous policy models, it is assumed that policymakers are interested in contributions from interest groups and that policies are adjusted to secure these contributions (see e.g. Becker 1983, Baron 1994, Nitzan 1994, Dixit et al. 1997; a survey is provided by van Winden 1999). Policymakers may be motivated in this respect by, for instance, political survival (think of campaign contributions), a need for policy relevant information (contributions in the form of effort), or greed (corruption). As a consequence, contributions are taken to influence the extent to which the interests of the social groups are taken into account. Since our focus is not on the precise mechanism relating interest group activity to government policy, we take a reduced-form approach by assuming that redistribution policy is in line with the maximization of the following interest function

$$G(\tau) = \frac{C_A}{C_A + C_B} m_A V(w_A - \tau) + \frac{C_B}{C_A + C_B} m_B V\left(w_B + \frac{m_A}{m_B}\tau\right);$$

where the weights attached to the interests of the individuals in the different sectors are determined by the respective total contributions of the interest groups, $C_i = c_i n_i$.[2] For given levels of the individual contributions c_i this implies that the sizes of the interest groups n_i are determinant. The tax rate that will be selected by the government follows from the following first-order condition (the second-order condition being satisfied)

$$C_A V'(w_A - \tau) = C_B V'\left(w_B + \frac{m_A}{m_B}\tau\right). \qquad (2)$$

Notice that if total contributions per sector are the same ($C_A = C_B$) after-tax income will be equalized across sectors.

An equilibrium $(n_A; n_B; \tau)$ of our model is implicitly given by eqs. (1) and (2). We have

Proposition 1 For functions $V(\cdot)$, $r(\cdot)$ and $\Lambda_i(\cdot)$ that satisfy the assumptions of our model, an equilibrium $(n_A^*; n_B^*; \tau^*)$ of the model specified by (1) and (2) exists and is unique.

See Appendix B for proof.

[1] An alternative interpretation of the model would be that people do not decide upon whether to join or not, but that the decision is about contributing or not contributing. Eq. (1) would then determine the total number of contributors, and thereby the total resources available. This interpretation would be relevant for fund raising drives, for example.

[2] We can also interpret this equation as the bargaining solution to a bargaining process. (cf Rees 1977).

3 Comparative statics

In this section we investigate the equilibrium e¤ects of changes in the contribution level (c), the size of a sector (m), the income level in a sector (w), and the behavioral sensitivity parameter (¯). Note that changes in the partial adjustment parameter (¸) have no e¤ect on an equilibrium as it drops out of eq. (1) in an equilibrium. For convenience, we will focus on parameter changes holding for sector A (similar e¤ects would be obtained for sector B). Proofs of the results and explicit expressions for the critical values mentioned in the propositions can be found in Appendix B. Before we go into the comparative statics of the full model (as given by eqs. (1) and (2)), let us brie‡y describe the comparative statics e¤ects of the model with …xed group sizes, $n_A = \overline{n}_A$ and $n_B = \overline{n}_B$. The model is then completely speci…ed by equation (2), which determines the optimal tax rate ¿, for given sizes of the interest groups.

Proposition 2 Consider the model with …xed group sizes. An increase in the (…xed) size of the interest group in sector A leads to a decrease in the tax rate ¿. Furthermore, an increase in the contribution fee in sector A (c_A) leads to a decrease in the tax rate ¿, while an increase in the income in sector A (w_A) leads to an increase in ¿ as wel as net of tax income. Finally, an increase in the size of sector A (m_A) leads to a decrease in the absolute value of the tax rate ¿.

There are two separate e¤ects playing a role in these comparative statics results, which we denote the political in‡uence e¤ect and the redistribution e¤ect, respectively. The political in‡uence e¤ect refers to the fact that an increase in political in‡uence by one of the interest groups (as captured by its total contribution C_i) will tilt the tax rate in favor of the sector it represents. The redistribution e¤ect sets in because, given contribution levels, the government has a tendency to redistribute income. The comparative statics e¤ect of a change in the group size n_A or n_B or of a change in the contribution fee c_A, are completely due to the political in‡uence e¤ect, whereas the comparative statics e¤ect of a change in the size of the sector (m_A) or the income in a sector (w_A) are due to the redistribution e¤ect. These comparative statics e¤ects seem to be quite plausible. In the remainder of this section we will see that, when we account for endogeneity of group sizes – that is, an additional participation e¤ect – the comparative statics e¤ects may become ambiguous: for many comparative statics e¤ects there are two regimes, one where the comparative static e¤ect is positive, and one where it is negative.

3.1 Contribution level

The following proposition summarizes the e¤ects of an increase in the contribution to the interest group in sector A, when group sizes are endogenous.

Proposition 3 A higher contribution in sector A (c_A) generates a lower equilibrium value of n_A. Furthermore, there exists a critical value $c^* > 0$ such that for $c_A > c^*$ an increase in c_A leads to an increase in τ and a decrease in n_B; while for $c_A < c^*$ a (marginal) increase in c_A leads to a decrease in τ and an increase in n_B.

Note from the proposition that the size of an interest group is always negatively affected by an increase in the contribution level. If this were not the case, the increased size of the interest group (n_A in this case) should be accompanied by a higher tax rate for the sector involved (see eq.(2)), which leads to a contradiction, because in the case at hand C_A would increase whereas C_B decreases (because n_B is negatively affected by the tax increase).

The tax rate, on the other hand, may be lower (higher) for this sector joint with a bigger (smaller) interest group in the other sector. The driving force here is the effect of the contribution, and the consequent effect on the interest group size, on the total contribution level of the group ($C_A = c_A n_A$) which may be positively but also negatively affected. In particular, total contributions increase when

$$\frac{\partial C_A}{\partial c_A} = n_A + c_A \frac{\partial n_A}{\partial c_A} > 0.$$

Notice that, by the argument given above, the second term in this expression is negative, hence if c_A is larger than $c^* = -\frac{n_A}{\partial n_A / \partial c_A}$, an increase in the contribution fee will lead to a decrease in total contributions from sector A. It is easily checked that as long as the total resources for political influence are decreased the tax rate must go up. The increase in the tax rate will lead to a decrease in n_B. If $c_A < c^*$ total contributions will increase with an increase in the contribution fee c_A and the opposite results follow. Although it is beyond the scope of this paper to endogenize the contribution level, it may be interesting to point at a potential dilemma for the leaders of an interest group in this context. If their main interest is in the size of the group they may want to opt for a low contribution fee. However, if their main concern would be the welfare of the members a higher contribution level may be warranted, with a lower tax rate but a smaller group size. We leave this issue of collective decision making - where also conjectures about the behavior of other interest groups may come into play - for future research.

3.2 Size of sector

In the previous case redistribution is caused by a change in political influence due to the political participation effect of the increased contribution level. When the size of a sector changes, however, there is an immediate redistribution effect with in addition participation and influence effects. The reason is that, in contrast with the contribution level, the size of a sector plays an explicit role in the interest function that is maximized by the government. The next proposition summarizes the effects of an increase in the size of sector A.

Proposition 4 There exist τ^a, τ^b and τ^c with $\tau^a < 0 < \tau^b < \tau^c$ such that τ increases (decreases) with an increase in m_A if and only if $\tau < \tau^a (\tau > \tau^a)$, n_B increases (decreases) with an increase in m_A if and only if $\tau < \tau^b$ $(\tau > \tau^b)$ and n_A increases (decreases) with an increase in m_A if and only if $\tau < \tau^c (\tau > \tau^c)$.

First consider the effect of an increase in m_A on the equilibrium tax rate τ. We know from Proposition 2, that for fixed group sizes the redistribution effect will decrease the absolute value of the tax rate and hence drive the tax rate to 0. If $\tau_A = \tau$ is positive, the increased size of sector A leads to a larger tax base which makes it possible to increase the after-tax welfare of both social groups by decreasing τ_A as well as $\tau_B = -\frac{m_A}{m_B}\tau$. If τ_A is negative, this tax (and thereby τ_B) goes up to equalize weighted after-tax welfare, because the bigger size of sector A puts a larger burden on sector B in that case. With endogenous group sizes the situation is a little more complicated because of the political influence effect. An increase in the size of sector A might increase the size of the interest group in sector A, which might lead to a decrease in the tax rate, even if it is already negative. Hence the tax rate will be driven in the direction of $\tau^a < 0$.

Now consider the effect on the group sizes. First consider the group in sector B. The only effect of m_A on the size of this group goes through the effect of the tax rate. In fact, if total taxes from group A, $m_A\tau$, increase, then after-tax income in group B will increase and hence the group size will decrease. This happens for $\tau > \tau^b$, where τ^b is the unique solution to

$$\frac{\partial (m_A\tau)}{\partial m_A} = \tau + m_A \frac{\partial \tau}{\partial m_A} = 0.$$

Obviously, $\tau^b \in (\tau^a, 0)$ and in fact $\tau^b > 0$. Finally, consider the size of the group in sector A. Here, there are two effects. The direct effect on group A is that a larger sector leads to a bigger interest group, through the participation function, $n_A = m_A \text{¤}_A$. There is also an indirect effect via the tax rate τ. That is, the total effect can be described by

$$\frac{\partial n_A}{\partial m_A} = \text{¤}_A + m_A \frac{\partial \text{¤}_A}{\partial \tau}\frac{\partial \tau}{\partial m_A},$$

where the first component corresponds to the direct effect, which is always positive, and the second component corresponds to the indirect effect, the sign of which is equal to the sign of $\frac{\partial \tau}{\partial m_A}$ and hence ambiguous. If the indirect effect is negative and sufficiently strong the group in sector A might indeed decrease as the sector increases. This happens for $\tau > \tau^c$. Thus, a change in the size of a sector - e.g. because of technological developments, migration, or changes in the age structure of the population - may have very different effects dependent on the initial distribution of the tax burden and the political influence effects.

3.3 Income level

An increase in the income level of a sector – due to technological or international economic developments, for example – induces redistribution from that sector to the other sector, for given political influence weights. However, it also affects the political participation of that sector, and thereby its political influence. The results summarized in the following proposition depend on the net outcome of these two forces.

Proposition 5 Let $\Delta V_A' \equiv V'(w_A - \tau) - V'(w_A - r(c_A))$. There exist $v^a < 0$ and $v^b > 0$ such that (i) for $\Delta V_A' > (<) v^a$ an increase in w_A leads to an increase (decrease) in the equilibrium value of τ and a(n) decrease (increase) in the equilibrium value of n_B; and (ii) for $\Delta V_A' < (>) v^b$ an increase in w_A leads to a(n) decrease (increase) in the equilibrium value of n_A. Net of tax income always increases.

In contrast to the redistribution effect, the sign of the participation effect is ambiguous. Participation is determined by the income differential $V(w_A - \tau) - V(w_A - r(c_A))$. If the income differential $V(w_A - \tau) - V(w_A - r(c_A))$ increases (decreases) with an increase in income, participation will decrease (increase). Clearly, the income differential increases if and only if $\tau > r(c_A)$. There is also an indirect effect on participation through the change in the tax rate.

Now we consider two cases. If $\tau > r(c_A)$ the direct effect of an increase in income is a decrease in participation. In this case the redistribution effect and the political influence effect work in the same direction and the tax rate will increase. The effect on n_A is indeterminate: if the increase in the income differential is large enough (larger than $v^b > 0$) the indirect effect on participation via the increased tax rate outweighs the direct effect and n_A may even increase.

If $\tau < r(c_A)$, the redistribution effect and the (direct) participation effect work in opposite directions. Only if the (absolute value of the) increase in the income differential is high enough, the latter will dominate the former and taxes will decrease. However, n_A will always increase. Finally the group size in sector B always moves in the other direction than the tax rate.

An interesting application concerns the political impact of a declining industrial sector. With a fixed interest group size (and, thus, fixed political influence weight) our model would predict a lower tax or higher subsidy for the sector, because of the redistribution effect. However, the political participation of individuals in this sector will also be affected. Dependent on the tax levied before the decline sets in the proposition suggests that this may lead to a larger interest group size, reinforcing the negative redistribution effect on the sector's tax, but possibly also to a smaller size accompanied by a larger instead of smaller tax.

3.4 Behavioral sensitivity

The sensitivity of individuals to a gap between actual and reference utility, represented by the parameter β, determines their propensity of joining an interest group. In the previous section we noted that informational as well as cultural factors may play a role here. Although such factors are likely to affect all sectors, for generality we also consider the impact of a sector specific parameter (β_i). As should be clear from eq. (1), the effects of a change in this type of behavioral sensitivity are driven by the condition whether τ_i is larger or smaller than $r(c_i)$. The next proposition summarizes the effects of a change in β_A.

Proposition 6 Everything else the same, an increase in the behavioral sensitivity to a gap between actual and reference utility in sector A (β_A) generates an equilibrium with smaller (larger) interest groups in both sectors and a higher (lower) tax for sector A if $\tau < r(c_A)$ $(\tau > r(c_A))$.

If β_A increases and actual utility is larger than reference utility ($\tau_A < r(c_A)$) then the size of the interest group in sector A decreases (see eq.(1)) inducing a higher tax for this sector. This leads in turn to a decrease also in the size of the interest group in sector B. A similar reasoning applies if the alternative condition holds. Results become more complex in case of a general change in β . The reason is that now τ_B versus $r(c_B)$ also starts to play a role, which leads to more complicated effects on n_A and n_B. The results of a general change in behavioral sensitivity (with $\beta_A = \beta_B$) are presented in the following proposition.

Proposition 7 Everything else the same, an across sectors change in behavioral sensitivity (β) generates an equilibrium with (i) if $\tau < -\frac{m_A}{m_B} r(c_B)$ an increase in the tax rate; (ii) if $-\frac{m_A}{m_B} r(c_B) < \tau < r(c_A)$ a decrease in both group sizes and an ambiguous effect on the tax rate; and (iii) if $\tau > r(c_A)$ a decrease in the tax rate.

To provide some further intuition, note from eq. (2) that on impact both interest groups become smaller when $\frac{-m_B}{m_A} r(c_B) < \tau_A < r(c_A)$, which explains result (ii). Outside this interval the effects on n_A and n_B are ambiguous . The proposition shows that larger behavioral sensitivity - e.g. due to better information or less inertia in political participation - can produce very different outcomes dependent on the size and distribution of the tax burden.

4 Dynamics

An important issue that we are interested in in this paper concerns the dynamics of the model consisting of (1) and (2). It is by now well-known that nonlinear dynamical systems like our model can give rise to complicated dynamical phenomena such as periodic cycles and irregular fluctuations. In fact, periodic and chaotic behavior seem to be the rule rather than the exception in many nonlinear dynamical models. Examples of erratic fluctuations arising naturally in economic dynamic models can be found in the literature on endogenous business cycle theory (e.g. Grandmont 1985, de Vilder 1996, Tuinstra 2000).

As will be shown in this section, also in the present model equilibria need not be stable and complicated dynamic patterns may emerge under the slightest perturbation of the parameters of the model. Crucial in this respect are the values of the political participation parameters β and λ. Instability arises if, for a given value of one parameter, the other parameter becomes sufficiently large. We have the following general result.

Proposition 8 Consider the model given by eqs. (1) and (2). There exists $\lambda^f > 0$ such that the equilibrium (n_A^*, n_B^*) of the model is locally stable for $\lambda < \lambda^f$ and unstable for $\lambda > \lambda^f$. If $\lambda^f < 1$ a period-doubling bifurcation occurs at $\lambda = \lambda^f$.

At a bifurcation there is a qualitative change of the behavior of the dynamical system. In particular, at a period doubling bifurcation the locally stable equilibrium becomes unstable and trajectories of the dynamical system are attracted to a period two orbit, where fractions keep on fluctuating between two values. That is, in even periods the system is in state $(n_A, n_B) = (n_I, n_{II})$ whereas in odd periods the system is in state $(n_A, n_B) = (n_{III}, n_{IV})$, with $n_{III} \neq n_I$ and $n_{IV} \neq n_{II}$. More complicated time series might also obtain. In order to be able to look at the possible dynamical features of the model in more detail we have to specify the model. With respect to the indirect utility function, we assume $V(x) = \frac{1}{1-\alpha}x^{1-\alpha}$, with $0 < \alpha < 1$. The first order condition (2) then leads to the following tax rule

$$\tau(C_A, C_B) = \frac{C_B^{\frac{1}{\alpha}} w_A - C_A^{\frac{1}{\alpha}} w_B}{C_A^{\frac{1}{\alpha}} \frac{m_A}{m_B} + C_B^{\frac{1}{\alpha}}}. \tag{3}$$

Furthermore, we assume that $r(c_i) = c_i$ and

$$\pi(\beta_i [V(w_i - \tau_i) - V(w_i - c_i)]) = \frac{1}{1 + \exp \beta_i [V(w_i - \tau_i) - V(w_i - c_i)]} \tag{4}$$

(see Appendix A).

Moreover, we consider a symmetric version of the model with $m_A = m_B = 1$ (thus n_i can be interpreted as the fraction of people organized in sector i), $c_A = c_B = c$, $w_A = w_B = w$ and $\beta_A = \beta_B = \beta$. For this (sector) symmetric model a unique equilibrium exists with $\tau = 0$ and $n_A = n_B = n^* = \frac{1}{1+\exp \beta [V(w) - V(w-c)]}$. Our stability result then looks as follows.

Proposition 9 Consider the symmetric model specified above. There exists a $\beta^* > 0$ such that for $\beta < \beta^*$, the symmetric equilibrium $(n_A, n_B) = (n^*, n^*)$ is locally stable for all $\lambda \in (0,1)$. Furthermore, for $\beta > \beta^*$ the symmetric equilibrium is locally stable for $\lambda < \lambda^f$ and unstable for $\lambda > \lambda^f$, where λ^f is given by

$$\lambda^f = 2\frac{1+W}{1+\left(1+\frac{\beta}{\alpha}w^{1-\alpha}\right)W},$$

with $W = \exp\left(\frac{-\beta}{1-\alpha}\left(w^{1-\alpha} - (w-c)^{1-\alpha}\right)\right)$. For $\beta > \beta^*$, the system undergoes a period doubling bifurcation at $\lambda = \lambda^f$. At this period doubling bifurcation a symmetric period two orbit of the form $\left(\left(n', n''\right), \left(n'', n'\right)\right)$ with $n' < n^* < n''$, emerges.

To illustrate, we consider some simulations with $w = 10$, $c = 1$ and $\alpha = \frac{1}{2}$. For low values of β, that is, when people are not very likely to join or leave an interest group on the basis of the economic situation, the equilibrium is stable. However, if β sufficiently increases the equilibrium becomes unstable. This is illustrated in Figure 2. The graph shown in this figure divides the (β, λ)-space into a region with stable equilibria (below the curve) and unstable equilibria (above the curve).[3]

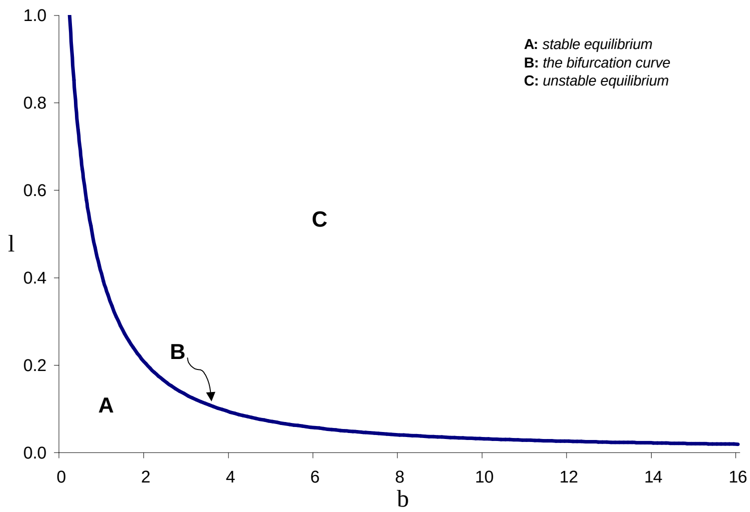

Figure 2: Regions in the $(\beta, \lambda) -$ plane with stable and unstable equilibrium and the bifurcation curve along which a period-2 cycle emerges :

[3] Actually, for the numerical example we have chosen the relationship between λ and β is given by

$$\lambda = \frac{2\left(1+\exp\left(2\beta\times\left(\sqrt{10}-3\right)\right)\right)}{1+\left(1+2\beta\sqrt{10}\right)\exp\left(2\beta\times\left(\sqrt{10}-3\right)\right)}.$$

Furthermore, $\beta^* \approx 0.3015$.

We will now fix $\lambda = \frac{1}{10}$ and investigate how the dynamics of interest group sizes and the tax rate evolve as the behavioral sensitivity parameter β varies. For $\lambda = \frac{1}{10}$ the period-doubling bifurcation described in Proposition 9 occurs at $\beta^f \approx 3.86$. At this value of β the equilibrium becomes unstable and a period two cycle emerges. For β close to, but larger than β^f almost all orbits of the dynamical system are attracted to this period two cycle.[4] The resulting period two cycle corresponds to the situation where in one period interest group A is 'large' and interest group B is 'small', and the latter group is taxed to the benefit of people in sector A, while in the next period the situation is reversed. For higher values of β more complicated dynamic patterns emerge. The panels in Figure 3 illustrate the occurrence of strange attractors and the corresponding complicated time series for different values of β. The intuition for these time series is the following. An increase in the size of one of the interest groups leads to a new tax, which is more beneficial to this interest group. This leads to an increase in the size of the other interest group which induces a tax rate more beneficial to this interest group. In this fashion the sizes of the interest groups keep on increasing until the process loses momentum, due to a diminishing effect on the tax schedule, and is eventually reversed. With larger β the reverse process becomes dominated by the organizational inertia parameter λ which causes the 'following' type of behavior in the decline of the interest groups illustrated by the bottom panel in Figure 3.

Our analysis shows that focusing on equilibria can be very misleading because they may be unstable and, therefore, extremely unlikely to be obtained. Instead, complicated dynamics may emerge. Whereas for the symmetric cases examined in Figure 3 it holds that the patterns are still regular in some sense, more irregular time series are obtained once asymmetry is allowed. To illustrate, the top panel in Figure 4 shows the dynamics of the model in case that: $w_A = 4; w_B = 10$; and $\beta = 10$ (keeping $c_i/w_i = 0.1$). The left figure in the bottom panel shows the corresponding time series for a particular time interval. When compared with the right figure in this panel - which reproduces Figure 1 - the resemblance of these two figures is striking. By letting one sector represent workers and the other sector owners or managers, it shows that the internal dynamics of our model alone can generate fluctuations in organizational density that are similar to the unionization of workers in the U.S. that Figure 1 refers to. No exogenous shocks are needed. Of course, we are not claiming here that we provide an explanation of this particular historical development. To do so would require, for instance, to allow for changes in many parameters over time (like income growth) in an appropriate way. Moreover, as argued by Freeman (1988), the redistribution conflict between workers and managers at the firm level should then also be taken into account, which could perhaps be done by linking up our model with a similar kind of political economic model that would be relevant for the wage policies of firms. The only claim we want to make is to have shown in a rigorous way

[4] Due to symmetry of the dynamical system all orbits with $n_{A0} = n_{B0}$ will converge to the equilibrium, even if it is unstable.

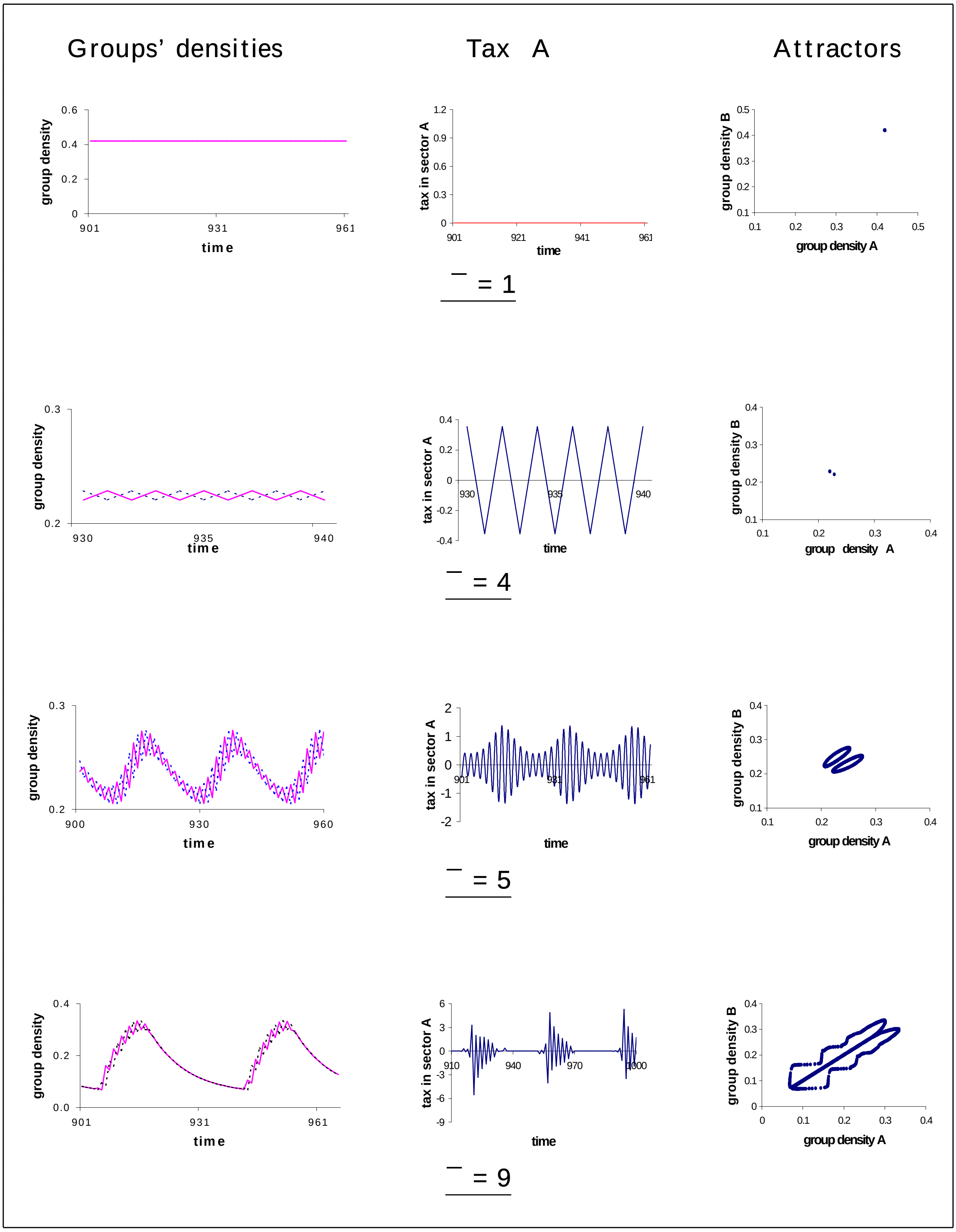

Figure 3: Top to bottom panels correspond to different values of β for the symmetric model with parameters: $c = 1$, $w = 10$, $m = 1$, $\alpha = 0.5$ and $\lambda = 0.1$ The first column shows the time series of the fraction of people organized in sector A (solid-line) and sector B (dot-line) ; second column shows the time series of tax in sector A; third column shows the attractors.

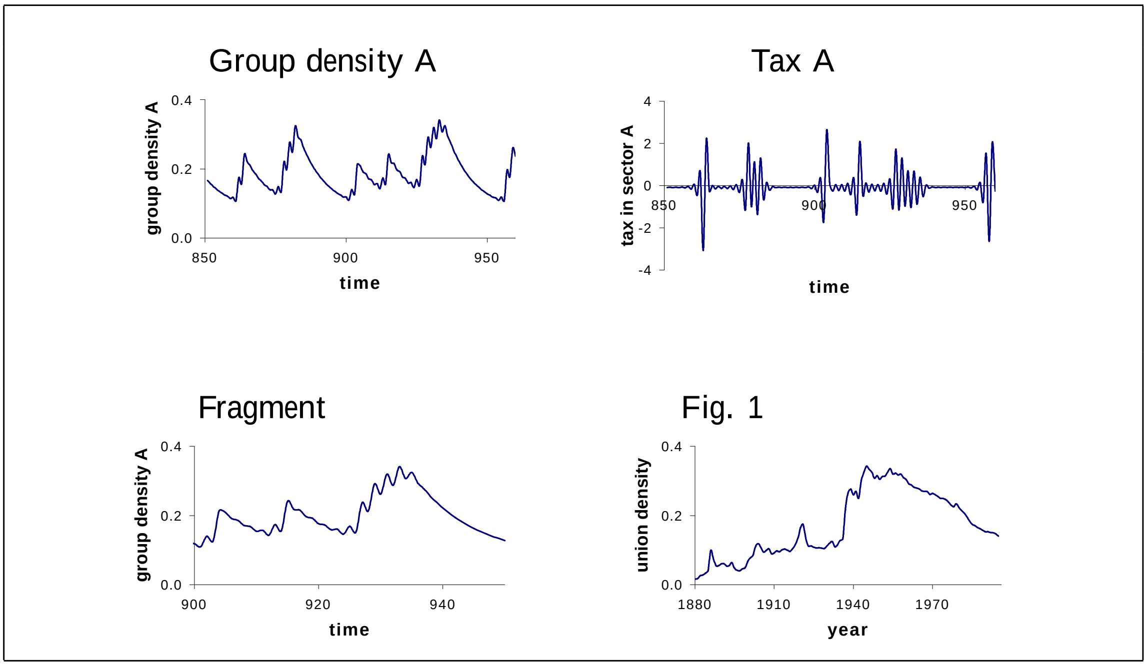

Figure 4: Top panel: time series of the fraction of individuals organized in sector A and of the tax on individuals in that sector for the model with the following asymmetric parameters: $w_A = 4$, $w_B = 10$, $c_A=w_A = c_B=w_B = 0{:}1$, $m_A=m_B = 1$, ® = 0:5 and ¸ = 0:1. Bottom panel: left …gure shows a fragment from the left …gure in the top panel, right …gure reproduces Figure 1.

that by integrating 'top-down' (policy) and 'bottom-up' (behavioral) factors spurts and declines in the organizational density of interest groups as observed in practice can be endogenously generated, without any reliance on exogenous shocks.

5 Concluding remarks

In this paper we have presented a dynamic model of endogenous interest group sizes and policymaking, focusing on redistribution. It integrates 'top-down' (policy) and 'bottom-up' (behavioral) in‡uences on the development of interest groups. Our model shows the restrictiveness of, on the one hand, the common assumption of …xed interest group sizes and, on the other hand, the concentration of attention on equilibria in the literature. For example, due to the endogeneity of the size of an interest group an increase in the contribution by its members need no longer induce lower taxes for (or larger subsidies to) this group, even though it would in case of …xed sizes. Incidentally, this may help explain the mixed results obtained by empirical political economic models using the (relative) numerical strengths of social groups as a proxy for political in‡uence (see Hettich and Winer 1999, p. 203). Similarly, an increase in the size of a social group - say, the number of retired - need not lead to smaller subsidies to individuals of this group; instead, subsidies may even increase because of an increased interest group size. On the other hand, the dynamic analysis of the model has shown that reliance on equilibrium results such as these can be very misleading. The reason is that equilibria may not be stable. For our relatively simple model we have been able to parameterize in a rigorous way the conditions for instability, which are related to the behavioral mechanism underlying the development of interest groups. If these conditions hold, complicated dynamics can emerge. Dependent on the initial situation very di¤erent time patterns for policy and interest group sizes show up in that case (path dependency). Moreover, the model can generate by itself - that is, without the help of any exogenous shocks - the types of spurts and declines in organizational density that are observed in reality. All in all, the results obtained from the comparative-statics and dynamic analysis seem interesting and realistic enough to warrant further theoretical and empirical investigation.

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Appendix A: microfoundation of interest group model

Assume that there is continuum of individuals in sector i with mass m_i. The probability of joining the interest group in sector i for individual j is determined by the following difference

$$U_{ij} = V(w_i - r(c_i)) - V(w_i - \tau_i) + \frac{1}{\beta_i}\varepsilon_j$$

where ε_j is voter j's individual "preference" or inclination for joining an interest group. This inclination ε_t is distributed according to a distribution function F. Individual j will join the interest group if $U_{ij} \geq 0$, that is, if

$$\varepsilon_j \geq \beta_i [V(w_i - \tau_i) - V(w_i - r(c_i))].$$

Now assume that in each period the individual decides with probability λ_i to reconsider his membership. Then the fraction of sector i that organizes becomes

$$n_{i,t+1} = (1 - \lambda_i) n_{i,t} + \lambda_i m_i \Phi(\beta_i [V(w_i - \tau_i) - V(w_i - r(c_i))]); \qquad i = A, B.$$

where

$$\Phi(\beta_i [V(w_i - \tau_i) - V(w_i - r(c_i))]) = 1 - F(\beta_i [V(w_i - \tau_i) - V(w_i - r(c_i))]).$$

If we assume that F is the logistic distribution (see Anderson et al. 1992) we have

$$\Phi(\beta_i [V(w_i - \tau_i) - V(w_i - r(c_i))]) = \frac{1}{1 + \exp(\beta_i [V(w_i - \tau_i) - V(w_i - r(c_i))])},$$

as used in Section 4.

Appendix B: proofs

The equilibrium (n_A^*, n_B^*, τ^*) of the model is implicitly defined as a solution to

$$n_A = m_A \Phi(\beta_A [V(w_A - \tau) - V(w_A - r(c_A))]) \tag{5}$$

$$n_B = m_B \Phi\left(\beta_B \left[V\left(w_B + \frac{m_A}{m_B}\tau\right) - V(w_B - r(c_B))\right]\right) \tag{6}$$

and

$$c_A n_A V'(w_A - \tau) = c_B n_B V'\left(w_B + \frac{m_A}{m_B}\tau\right); \tag{7}$$

where $r(c_i) \in [0; w_i]$ is the reservation value with respect to interest group membership and $r'(c_i) > 0$. The indirect utility function $V(y)$ is positive, monotonically increasing and strictly concave, i.e. $V(y) > 0$, $V'(y) > 0$ and $V''(y) < 0$. Furthermore $V(0) = 0$ and $\lim_{y \to 0} V'(y) = +\infty$. Finally, the participation function $\alpha(y) \in (0; 1)$ is decreasing in its argument: $\alpha'(y) < 0$.

To simplify the notation we introduce the following: $r_i \equiv r(c_i)$, $V_{ir} \equiv V(w_i - r(c_i))$, $V_{A\tau} \equiv V(w_A - \tau)$, $V_{B\tau} \equiv V\left(w_B + \frac{m_A}{m_B}\tau\right)$, $\Delta V_i \equiv V_{i\tau} - V_{ir}$, $\alpha_i = \alpha(\beta_i \Delta V_i)$ and derivatives are denoted in a similar fashion. The three equilibrium equations now reduce to

$$n_A = m_A \alpha_A; \; n_B = m_B \alpha_B \text{ and } c_A n_A V'_{A\tau} = c_B n_B V'_{B\tau}:$$

Let us define the function

$$f(\tau) \equiv c_A m_A \alpha_A V'_{A\tau} - c_B m_B \alpha_B V'_{B\tau}:$$

The equilibrium value of τ corresponds to a zero of $f(:)$. The associated equilibrium values of n_A and n_B then follow from the other two equilibrium conditions.

Proof of Proposition 1 (existence and uniqueness of equilibrium)
First observe that the assumption $\lim_{y \to 0} V'(y) = +\infty$ implies that $\lim_{\tau \to -\frac{m_B}{m_A} w_B} f(\tau) = -\infty$ and $\lim_{\tau \to w_A} f(\tau) = \infty$. Since f is a continuous function on a connected set this implies that there exists at least one $\tau^* \in \left(-\frac{m_B}{m_A} w_B; w_A\right)$ such that $f(\tau^*) = 0$. Now since

$$\frac{\partial f}{\partial \tau} = -c_A m_A \beta_A \alpha'_A (V'_{A\tau})^2 - c_A m_A \alpha_A V''_{A\tau} - c_B m_A \beta_B \alpha'_B (V'_{B\tau})^2 - c_B m_A \alpha_B V''_{B\tau} > 0 \tag{8}$$

this equilibrium is unique. ■

Proof of Proposition 2 (comparative statics for fixed interest group sizes)
The equilibrium value of τ solves

$$\begin{aligned} f(\tau) &= c_A n_A V'_{A\tau} - c_B n_B V'_{B\tau} \\ &= c_A n_A V'(w_A - \tau) - c_B n_B V'\left(w_B + \frac{m_A}{m_B}\tau\right) \end{aligned}$$

Differentiating $f(\tau)$ with respect to all parameters gives

$$f_\tau d\tau + f_{c_A} dc_A + f_{w_A} dw_A + f_{m_A} dm_A + f_{n_A} dn_A + f_{n_B} dn_B = 0;$$

where

$$
\begin{aligned}
f_\tau &= -c_A n_A V''_{A\tau} - c_B n_B \frac{m_A}{m_B} V''_{B\tau} > 0;\;\; f_{n_A} = c_A V'_{A\tau} > 0; \\
f_{n_B} &= -c_B V'_{B\tau} < 0;\;\; f_{c_A} = n_A V'_{A\tau} > 0; \\
f_{w_A} &= c_A n_A V''_{A\tau} < 0 \text{ and } f_{m_A} = -c_B n_B \frac{1}{m_B} V''_{B\tau}\tau.
\end{aligned}
$$

The comparative statics effects are now given by

$$
\begin{aligned}
\frac{d\tau}{dn_A} &= -\frac{f_{n_A}}{f_\tau} < 0;\;\; \frac{d\tau}{dn_B} = -\frac{f_{n_B}}{f_\tau} > 0; \\
\frac{d\tau}{dc_A} &= -\frac{f_{c_A}}{f_\tau} < 0;\;\; \frac{d\tau}{dw_A} = -\frac{f_{w_A}}{f_\tau} > 0;
\end{aligned}
$$

and

$$
\frac{d\tau}{dm_A} = -\frac{f_{m_A}}{f_\tau} < (>)\, 0 \text{ if } \tau > (<)\, 0. \square
$$

Comparative statics – endogenous group sizes

In order to study the comparative statics of the full model we take the total differential of f with respect to τ, c_A, m_A, w_A and β_A. This gives

$$
f_\tau d\tau + f_{c_A} dc_A + f_{m_A} dm_A + f_{w_A} dw_A + f_{\beta_A} d\beta_A = 0
$$

with $f_\tau = \frac{\partial f}{\partial \tau}$ given by (8) and

$$
\begin{aligned}
f_{c_A} &= m_A V'_{A\tau} \left(\Lambda_A + c_A \beta_A r' \Lambda'_A V'_{Ar}\right) \\
f_{m_A} &= c_A \Lambda_A V'_{A\tau} - \left(c_B \beta_B \Lambda'_B (V'_{B\tau})^2 + c_B \Lambda_B V''_{B\tau}\right)\tau \\
f_{w_A} &= c_A m_A \beta_A \Lambda'_A V'_{A\tau} \Delta V'_A + c_A m_A \Lambda_A V''_{A\tau} \\
f_{\beta_A} &= c_A m_A \Lambda'_A V'_{A\tau} \Delta V_A.
\end{aligned}
$$

Furthermore we have

$$
dn_A = A_\tau d\tau + A_{c_A} dc_A + A_{m_A} dm_A + A_{w_A} dw_A + A_{\beta_A} d\beta_A
$$

with

$$
\begin{aligned}
A_\tau &= -m_A \beta_A \Lambda'_A V'_{A\tau} > 0;\;\; A_{c_A} = m_A \beta_A \Lambda'_A r' V'_{Ar} < 0; \\
A_{m_A} &= \Lambda_A > 0;\;\; A_{w_A} = m_A \beta_A \Lambda'_A \Delta V'_A \text{ and } A_{\beta_A} = m_A \Lambda'_A \Delta V_A
\end{aligned}
$$

and

$$
dn_B = B_\tau d\tau + B_{c_A} dc_A + B_{m_A} dm_A + B_{w_A} dw_A + B_{\beta_A} d\beta_A
$$

with

$$B_\tau = m_A \beta_B \alpha'_B V'_{B\tau} < 0;\ B_{c_A} = 0$$
$$B_{m_A} = \beta_B \alpha'_B V'_{B\tau}\tau < 0,\ B_{w_A} = 0 \text{ and } B_{\beta_A} = 0.$$

Proof of Proposition 3 (effects of a change in c_A).
We have

$$\frac{d\tau}{dc_A} = -\frac{f_{c_A}}{f_\tau} = -\frac{1}{f_\tau} m_A V'_{A\tau}\left(\alpha_A + c_A \beta_A r' \alpha'_A V'_{Ar}\right).$$

This is positive if and only if $\alpha_A + c_A \beta_A r' \alpha'_A V'_{Ar} < 0$, i.e. if and only if $c_A > -\frac{\alpha_A}{\beta_A r' \alpha'_A V'_{Ar}} = c^* > 0$. With respect to n_A we find

$$\frac{dn_A}{dc_A} = A_{c_A} + A_\tau \frac{d\tau}{dc_A} = m_A \beta_A \alpha'_A \left(r' V'_{Ar} - V'_{A\tau}\frac{d\tau}{dc_A}\right)$$

which is negative for $r' V'_{Ar} - V'_{A\tau}\frac{d\tau}{dc_A} > 0$. This inequality can be rewritten as

$$V'_{Ar} > 0 > \frac{\left(V'_{A\tau}\right)^2 \alpha_A}{c_A \alpha_A r' V''_{A\tau} + c_B \beta_B r' \alpha'_B \left(V'_{B\tau}\right)^2 + c_B r' \alpha_B V''_{B\tau}}$$

and is therefore always satisfied. Finally

$$\frac{dn_B}{dc_A} = B_\tau \frac{d\tau}{dc_A};$$

and n_B therefore moves in the opposite direction of τ when c_A changes. ■

Proof of Proposition 4 (effects of a change in m_A).
Denote

$$\tau^a = \frac{c_A \alpha_A V'_{A\tau}}{c_B\left(\beta_B \alpha'_B \left(V'_{B\tau}\right)^2 + \alpha_B V''_{B\tau}\right)},\ \tau^b = \frac{\alpha_A V'_{A\tau}}{-\beta_A \left(V'_{A\tau}\right)^2 \alpha'_A - \alpha_A V''_{A\tau}}$$
$$\text{and } \tau^c = \frac{\left(\tau^a V''_{A\tau} + V'_{A\tau}\right)\alpha_A}{-\beta_A \left(V'_{A\tau}\right)^2 \alpha'_A}.$$

We have $\tau^a < 0 < \tau^b < \tau^c$. The latter inequality follows from the fact that the numerator of τ^c is larger than the numerator of τ^b whereas the denominator of τ^c is smaller than the denominator of τ^b.

$$\frac{d\tau}{dm_A} = -\frac{f_{m_A}}{f_\tau} = -\frac{1}{f_\tau}\left[c_A \alpha_A V'_{A\tau} - c_B\left(\beta_B \alpha'_B \left(V'_{B\tau}\right)^2 + \alpha_B V''_{B\tau}\right)\tau\right].$$

Hence it follows immediately that $\frac{d\tau}{dm_A} > 0$ $\left(\frac{d\tau}{dm_A} < 0\right)$ if and only if $\tau < \tau^a$ $(\tau > \tau^a)$. Furthermore, we have

$$\frac{dn_A}{dm_A} = \delta_A - m_A \beta_A \delta'_A V'_{A\tau} \frac{d\tau}{dm_A}$$

after some manipulation we find that this is positive if and only if $\tau < \tau^c$. Finally, we have

$$\frac{dn_B}{dm_A} = \beta_B \delta'_B V'_{B\tau} \left(m_A \frac{d\tau}{dm_A} + \tau \right);$$

hence n_B increases with an increase in m_A when $\frac{d\tau}{dm_A} < -\frac{\tau}{m_A}$ which is equivalent with $\tau < \tau^b$. $\square$

Proof of Proposition 5 (effects of a change in w_A).
We have

$$\frac{d\tau}{dw_A} = -\frac{f_{w_A}}{f_\tau} = -\frac{m_A c_A}{f_\tau} \left(\beta_A \delta'_A V'_{A\tau} \Delta V'_A + \delta_A V''_{A\tau}\right)$$

which is positive if and only if $\beta_A \delta'_A V'_{A\tau} \Delta V'_A + \delta_A V''_{A\tau} < 0$, i.e.

$$\Delta V'_A > -\frac{\delta_A V''_{A\tau}}{\beta_A \delta'_A V'_{A\tau}} = v^a:$$

Now consider the effect on n_A. We have

$$\frac{dn_A}{dw_A} = A_{w_A} + A_\tau \frac{d\tau}{dw_A} = m_A \beta_A \delta'_A \left(\Delta V'_A - V'_{A\tau} \frac{d\tau}{dw_A} \right);$$

which is positive when $\Delta V'_A < V'_{A\tau} \frac{d\tau}{dw_A}$, i.e.

$$\Delta V'_A < \frac{c_A \delta_A V''_{A\tau} V'_{Ar}}{c_B \beta_B \delta'_B (V'_{B\tau})^2 + c_B \delta_B V''_{B\tau}} = v^b:$$

Furthermore, we have

$$\frac{dn_B}{dw_A} = m_A \beta_B \delta'_B V'_{B\tau} \frac{d\tau}{dw_A}$$

and hence $\frac{dn_B}{dw_A} > 0$ if and only if $\frac{d\tau}{dw_A} < 0$. $\square$

Proof of Proposition 6 (effects of a change in β_A).

We have

$$\frac{d\tau}{d\beta_A} = -\frac{f_{\beta_A}}{f_\tau} = -\frac{1}{f_\tau} c_A m_A \alpha'_A V'_{A\tau} \Delta V_A$$

and hence $\frac{d\tau}{d\beta_A}$ has the same sign as ΔV_A and is therefore positive if and only if $\tau < r(c_A)$. For the group sizes we obtain

$$\begin{aligned} \frac{dn_A}{d\beta_A} &= m_A \alpha'_A \left(\Delta V_A - \beta_A V'_{A\tau} \frac{d\tau}{d\beta_A} \right) \\ &= m_A \alpha'_A \Delta V_A \left(1 + \frac{1}{f_\tau} c_A m_A \beta_A \alpha'_A (V'_{A\tau}) \right). \end{aligned}$$

Since the term between brackets is always positive, the sign of $\frac{dn_A}{d\beta_A}$ is always opposite the sign of ΔV_A and $\frac{d\tau}{d\beta_A}$. Finally, we have

$$\frac{dn_B}{d\beta_A} = m_A \beta_B \alpha'_B V'_{B\tau} \frac{d\tau}{d\beta_A};$$

and hence the sign of $\frac{dn_B}{d\beta_A}$ is opposite to the sign of ΔV_A and $\frac{d\tau}{d\beta_A}$. $\blacksquare$

Proof of Proposition 7 (effects of a simultaneous change in β_A and β_B). The next step is to look at some comparative statics when $\beta_A = \beta_B$ change simultaneously. The only thing changing is f_β, which becomes

$$f_\beta = c_A m_A \alpha'_A V'_{A\tau} \Delta V_A - c_B m_B \alpha'_B V'_{B\tau} \Delta V_B$$

and B_β, which becomes

$$B_\beta = m_B \alpha'_B \Delta V_B.$$

We have

$$\frac{d\tau}{d\beta} = -\frac{f_\beta}{f_\tau} = -\frac{1}{f_\tau} \left[c_A m_A \alpha'_A V'_{A\tau} \Delta V_A - c_B m_B \alpha'_B V'_{B\tau} \Delta V_B \right].$$

Now consider the change in group sizes. We have

$$\frac{dn_A}{d\beta} = m_A \alpha'_A \left(\Delta V_A - \beta V'_{A\tau} \frac{d\tau}{d\beta} \right) \text{ and } \frac{dn_B}{d\beta} = m_B \alpha'_B \left(\Delta V_B + \beta \frac{m_A}{m_B} V'_{B\tau} \frac{d\tau}{d\beta} \right)$$

Using the definition of f_τ, it follows that the signs of $\frac{dn_A}{d\beta}$ and $\frac{dn_B}{d\beta}$ are opposite to

$$\left(-c_A \alpha_A V''_{A\tau} - c_B \beta \alpha'_B (V'_{B\tau})^2 - c_B \alpha_B V''_{B\tau} \right) \Delta V_A - \beta c_B \frac{m_B}{m_A} \alpha'_B V'_{A\tau} V'_{B\tau} \Delta V_B$$

and

$$\left(-c_A \beta \alpha'_A (V'_{A\tau})^2 - c_A \alpha_A V''_{A\tau} - c_B \alpha_B V''_{B\tau} \right) \Delta V_B - \beta \frac{m_A}{m_B} c_A \alpha'_A V'_{A\tau} V'_{B\tau} \Delta V_A;$$

respectively. We then have

1. $\tau < -\frac{m_A}{m_B} r(c_B)$ (implying $\Delta V_A > 0$, $\Delta V_B < 0$). In this regime we have $\frac{d\tau}{d\beta} > 0$ and the signs of $\frac{dn_A}{d\beta}$ and $\frac{dn_B}{d\beta}$ are ambiguous.
2. $-\frac{m_A}{m_B} r(c_B) < \tau < r(c_A)$ (implying $\Delta V_A, \Delta V_B > 0$). In this regime the sign on $\frac{d\tau}{d\beta}$ is ambiguous and $\frac{dn_A}{d\beta} < 0$ and $\frac{dn_B}{d\beta} < 0$.
3. $\tau > r(c_A)$ (implying $\Delta V_A < 0$, $\Delta V_B > 0$). In this regime we have $\frac{d\tau}{d\beta} < 0$ and the signs of $\frac{dn_A}{d\beta}$ and $\frac{dn_B}{d\beta}$ are ambiguous.

Proof of Proposition 8 (stability of equilibrium).
The dynamic system is given by

$$n_{A,t+1} = (1-\lambda) n_{At} + \lambda m_A \Lambda\left(\beta_A \left[V_A(w_A - \tau(n_{At}, n_{Bt})) - V_B(w_B - r(c_A))\right]\right)$$
$$n_{B,t+1} = (1-\lambda) n_{Bt} + \lambda m_B \Lambda\left(\beta_B \left[V_B(w_B + \tau(n_{At}, n_{Bt}) m_A/m_B) - V_B(w_B - r(c_B))\right]\right)$$

where $\tau(n_{At}, n_{Bt})$ is implicitly defined by (2). The Jacobian at the equilibrium point, is given by

$$J = \begin{pmatrix} (1-\lambda) - \lambda m_A \Lambda'_A \beta_A V'_{A\tau} \frac{\partial\tau}{\partial n_A} & -\lambda m_A \Lambda'_A \beta_A V'_{A\tau} \frac{\partial\tau}{\partial n_B} \\ \lambda m_A \Lambda'_B \beta_B V'_{B\tau} \frac{\partial\tau}{\partial n_A} & (1-\lambda) + \lambda m_A \Lambda'_B \beta_B V'_{B\tau} \frac{\partial\tau}{\partial n_B} \end{pmatrix}, \tag{9}$$

where $\frac{\partial\tau}{\partial n_A}$ and $\frac{\partial\tau}{\partial n_B}$ can be found by differentiating (2) totally. This gives

$$\frac{\partial\tau}{\partial n_A} = \frac{c_A V'_A}{m_A c_A \Lambda_A V''_A + m_A c_B \Lambda_B V''_B} < 0 \text{ and } \frac{\partial\tau}{\partial n_B} = -\frac{c_B V'_B}{m_A c_A \Lambda_A V''_A + m_A c_B \Lambda_B V''_B} > 0$$

The eigenvalues of the Jacobian matrix in (9) are $\mu_1 = 1-\lambda$ and $\mu_2 = 1-\lambda-\lambda m_A(\gamma_A - \gamma_B)$, where $\gamma_i \equiv \Lambda'_i \beta_i V'_i \frac{\partial\tau}{\partial n_i}$, for $i = A, B$. The associated eigenvectors are given by $v_1 = \left[-\frac{\partial\tau}{\partial n_B} \quad \frac{\partial\tau}{\partial n_A}\right]'$ and $v_2 = \left[-\beta_A \Lambda'_A V'_A \quad \beta_B \Lambda'_B V'_B\right]'$.

Notice that $0 \le \mu_1 \le 1$ and that the second eigenvalue μ_2 goes through -1 when

$$\lambda = \frac{2}{1 + (\gamma_A - \gamma_B) m_A} = \frac{2C}{C+D} (\equiv \lambda^f);$$

where

$$C = c_A \Lambda_A V''(w_A - \tau) + c_B \Lambda_B V''\left(w_B + \frac{m_A}{m_B}\tau\right) < 0$$
$$D = c_A \beta_A \Lambda'(\beta_A \Delta V_A)(V'(w_A - \tau))^2 + c_B \beta_B \Lambda'(\beta_B \Delta V_B)\left(V'\left(w_B + \frac{m_A}{m_B}\tau\right)\right)^2 < 0.$$

Summarizing, if $\gamma_A = \gamma_B$ then $|\mu_2| = |\mu_1| = 1 - \lambda < 1$ and therefore, the equilibrium (n_A^*, n_B^*) is locally stable for all $\lambda \in (0,1)$. If $\gamma_A \neq \gamma_B$ and $\lambda^f < 1$ then $|\mu_2| < 1$

($|\mu_2| > 1$) for $\lambda \in (0; \lambda^f)$ ($\lambda > \lambda^f$). Therefore, the equilibrium $(n_A^*; n_B^*)$ is locally stable (unstable) for $\lambda \in (0; \lambda^f)$ ($\lambda > \lambda^f$): A period doubling bifurcation occurs at $\lambda = \lambda^f$ since $\mu_2 = -1$: If $\gamma_A \neq \gamma_B$ and $\lambda^f \geq 1$ then the equilibrium $(n_A^*; n_B^*)$ is locally stable for all $\lambda \in (0; 1)$(for the theory on period doubling bifurcations see e.g. Kuznetsov 1995). $\blacksquare$

Proof of Proposition 9 (stability of equilibrium in the symmetric specified model).

We use the above proof for the symmetric specified model with $w_A = w_B = w$; $m_A = m_B = 1$; $\beta_A = \beta_B = \beta$ and $c_A = c_B = c$. For this case, the eigenvalues are $\mu_1 = 1 - \lambda \in (0; 1)$ with eigenvector $v_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}'$ and $\mu_2 = 1 - \lambda - \lambda\gamma$ with eigenvector $v_2 = \begin{pmatrix} -1 & 1 \end{pmatrix}'$; where $\gamma \equiv -\frac{\beta\Lambda'(\beta \cdot V)(V'(w))^2}{\Lambda(\beta \cdot V)V''(w)}$. A period doubling bifurcation occurs at

$$\lambda^f = 2\frac{\Lambda(\beta \cdot V)V''(w)}{\Lambda(\beta \cdot V)V''(w) + \beta\Lambda'(\beta \cdot V)(V'(w))^2}:$$

For our example we have $V(y) = \frac{1}{1-\alpha}y^{1-\alpha}$ and $\Lambda(x) = \frac{1}{1+\exp x}$. This gives

$$\lambda^f = 2\frac{1 + W}{1 + \left(1 + \frac{\beta}{\alpha}w^{1-\alpha}\right)W};$$

where $W = \exp\left(\frac{\beta}{1-\alpha}\left(w^{1-\alpha} - (w-1)^{1-\alpha}\right)\right)$.

Now let β^* be the solution to $F(\beta) = 0$

$$F(\beta) = \left(\frac{\beta}{\alpha}w^{1-\alpha} - 1\right)\exp\left(\frac{\beta}{1-\alpha}\left(w^{1-\alpha} - (w-1)^{1-\alpha}\right)\right) - 1:$$

Since $F(0) = -2$, $\lim_{\beta \to \infty} F(\beta) = \infty$ and $F'(\beta) > 0$, such a solution exists and is unique. It is easily checked that for $\beta > \beta^*$ we have $\lambda^f < 1$. Finally, for the numerical example with $w = 10$, $c = 1$ and $\alpha = \frac{1}{2}$ we find $\lambda^f = 2\frac{1+\exp\left(2\beta\left(\sqrt{10}-3\right)\right)}{1+\left(1+2\beta\sqrt{10}\right)\exp\left(2\beta\left(\sqrt{10}-3\right)\right)}$ and $\beta^* \approx 0.3015$. $\blacksquare$