Interest Rate Rules with Heterogeneous Expectations

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Abstract

Recent macroeconomic literature stressed the importance of expectations heterogeneity in the formulation of monetary policy. We use a stylized macro model of Howitt (1992) to investigate the dynamical consequences of alternative interest rate rules when agents have heterogeneous expectations and update their beliefs over time along the lines of Brock and Hommes (1997). We find that the outcome of different monetary policies in terms of stability crucially depends on the ecology of forecasting rules and on the intensity of choice among different predictors. We also show that, when agents have heterogeneous expectations, an interest rate rule that obeys the Taylor principle does not always lead the system to converge to the rational expectations equilibrium but multiple equilibria may persist.

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1 Introduction

The rational representative agent approach is still the core assumption in macroeconomics. However in the last decade agents' heterogeneity is playing an increasingly important role in macro economics and monetary policy debates.

In contrast, in behavioral finance, bounded rationality and heterogeneous agent models have been developed as a concrete alternative to the standard rational representative agent approach, as discussed e.g. in the extensive surveys of LeBaron (2006) and Hommes (2006). Although bounded rationality and adaptive learning have become increasingly important in macroeconomics, most models still assume a representative agent who is learning about the economy. Only few models take seriously into account the role of *heterogeneous expectations* and how monetary policy should be conducted when heterogeneity is explicitly introduced in a macro and monetary policy framework. See e.g. Evans and Honkapohja (2001) for an extensive survey on learning in macro economics. Some examples of macro models with heterogeneous expectations include Brock and de Fontnouvelle (2000), Evans and Honkapohja (2003, 2006), Branch and Evans (2006), Honkapohja and Mitra (2006), Branch and McGough (2006, 2008), Berardi (2007), Tuinstra and Wagener (2007), Brazier, Harrison, King, and Yates (2008) and De Grauwe (2008).

The aim of our paper is to investigate how heterogeneity in agents' expectations affects macro fluctuations and stability of the economy and how different monetary policy rules can enhance stability when agents have heterogeneous expectations about future inflation. Recently, Branch (2004), Santoro and Pfajfar (2006) and Pfajfar (2008) provided empirical evidence on heterogeneous expectations using survey data on inflation expectations.

We employ the simple macro-monetary policy model of Howitt (1992, 2006) to investigate the dynamic consequences of a monetary policy aimed at pegging the interest rate, when agents have heterogeneous expectations about inflation rate. We adopt a conventional IS-LM model in which agents form expectations about the future rate of inflation using different forecasting rules, and where different beliefs are aggregated linearly. In addition, we employ an Adaptive Belief System introduced in Brock and Hommes (1997), so that agents switch from one forecasting rule to another on the basis of past performances of these rules. We analyze the inflation dynamics in such a model, using theoretical and numerical tools, for different types of forecasting rules.

Our paper contributes to the debate about the feasibility of a policy of interest rate pegging. According to Friedman (1968), controlling interest rates tightly is not a feasible monetary policy. Friedman argues that if the real interest rate in the economy does not coincide with a hypothetical ("natural") level corresponding to full employment, then inflation will follow a cumulative process. Consider an example where the Central Bank pegs the nominal interest rate too low, i.e. given the expected rate of inflation the real interest rate is below its natural level. Excess aggregate demand will then cause inflation to rise more than expected because of an expectations-augmented Phillips curve. In response to an unexpectedly high increase of inflation, people will adjust their expectations upwards, and the Fisher effect will put upward pressure on the interest rate. A monetary expansion will thus be required to maintain the peg and this will cause inflation to accelerate even further until the policy is abandoned. Likewise, if the interest rate is pegged too high, deflation will accelerate until the policy is abandoned. The cumulative process argument disappeared from the literature after the rational expectations revolution. However, Howitt (1992) pointed out that in an economy in which people try to acquire rational expectations through learning, a monetary policy aimed at controlling tightly the interest rate will lead inevitably to the cumulative process. Indeed, in a world in which any departure of expected inflation from its equilibrium level causes an overreaction of the actual inflation rate and generates a misleading signal for the agents, a forecasting rule that tries to learn from past mistakes will lead the economy away from equilibrium causing inflation/deflation to accelerate until the interest rate pegging policy is abandoned. Howitt (1992) shows that the cumulative process arises for any plausible adaptive learning rule. He also shows that if the interest rate pegging monetary policy is abandoned in favor of a Taylor rule the cumulative process is stabilized. The aim of our paper is to investigate the potentially destabilizing effect of interest rate pegging and the potentially stabilizing effect of a Taylor rule in a world with heterogeneous expectations. As we will see, the answers will depend on the ecology of forecasting rules.

The analysis performed in our paper shows that Howitt's results will not always hold in a world with heterogeneous agents. In particular, the cumulative process is not always arising when the monetary authority pegs the nominal interest rate. As an example, when there is a perfectly rational agent type in the market, even if the implemented policy leads to the cumulative process, this process is not permanent, and dynamics converge to a complex attractor with phases of inflation and/or deflation. Along the inflation/deflation paths, forecasting errors of non-rational agents will increase and the majority switches to rational expectations, forcing the inflation rate back close to its natural level. We also investigate the dynamics in an economy characterized by a *continuum* of constant forecasting rules by means of the notion of Large Type Limit (LTL hereafter) introduced in Brock, Hommes, and Wagener (2005) and we consider the impact of alternative monetary policy rules.

The rest of the paper is organized as follows. In Section 2 we briefly recall the ideas behind the cumulative process and the benchmark IS-LM model as described in Howitt (1992, 2006). The model with heterogeneous expectations is introduced in Section 3, where an example of rational versus naive agents is analyzed. In particular, we compare the policy of the nominal interest rate pegging with the Taylor rule when the nominal interest rate is set in response on the inflation level. Sections 4 and 5 present the case of an ecology of constant forecasting rules in the case of interest rate pegging as well as in the case of a Taylor rule. We study both the case when the number of forecasting rules is small (e.g. 3 or 5) and the case of an arbitrarily large number of rules. Section 6 concludes.

2 Interest Rate Rules and Cumulative Process

In this section we recall the formalization developed by Howitt (2006) of the instability problem implied by the Wicksellian cumulative process. Consider the following system of equations that describes a simple IS-LM model:

$$y_t = -\sigma(i_t - \pi_t^e - r^*)\,, \tag{2.1}$$

$$\pi_t = \pi_t^e + \varphi y_t\,, \tag{2.2}$$

where y_t is the output gap, i_t is the nominal interest rate, π_t and π_t^e are respectively the actual and expected inflation rates, r^* is the natural rate of interest, and σ and φ are positive coefficients. Equation (2.1) is the usual IS curve in which the real interest rate $i_t - \pi_t^e$ must equal the natural rate in order for output to equal its "full employment" capacity, here normalized to zero. Equation (2.2) is the expectations-augmented Phillips curve expressed in terms of inflation and output.

Let us assume that the monetary authority decides to peg the nominal interest rate at level $\bar{\imath}$. Under rational expectations the expected rate of inflation coincides with the actual inflation, and, according to (2.2), the economy is in the state of full employment, $y^* = 0$. From (2.1) the rate of inflation in the RE equilibrium depends positively on the pegged nominal interest rate:

$$\pi^* = \bar{\imath} - r^*\,.$$

Thus, assuming rational expectations the interest rate pegging is a feasible monetary policy: accelerating or decelerating inflation will not arise because the system will immediately reach the equilibrium level.

However, the policy implications change dramatically when the rational expectations assumption is relaxed, and expectations are revised in an adaptive, boundedly rational way. To illustrate the failure of the interest rate pegging policy, let us assume that the nominal interest rate is pegged too low, so that the real interest rate $\bar{\imath} - \pi_t^e$ is below its natural level r^*. In that case, inflation expectations will be higher than the equilibrium inflation π^*. Actual inflation will be even higher than expected inflation because of the expectations augmented Phillips curve:

$$\pi_t = \pi_t^e + \varphi\sigma(\pi_t^e - \pi^*)\,.$$

This means that the signal that the agents receive from the market is misleading. Even though inflation was overestimated with respect to the equilibrium level ($\pi_t^e > \pi^*$), realized inflation suggests that agents underestimated it, i.e. ($\pi_t > \pi_t^e$). Any reasonable rule that tries to learn from past mistakes will then lead agents to expect even higher inflation, causing a cumulative process of accelerating inflation. Similarly, pegging the interest rate too high will lead to a cumulative process of accelerating deflation. It implies that interest pegging is not a feasible monetary policy.

The actual dynamics depends, of course, on the forecasting rule that agents use to form their expectations. As an illustrative example, consider the case of naive expectations, when agents expect that current inflation will persist the next period, $\pi_t^e = \pi_{t-1}$. In deviations from the RE steady state, the model (2.1)–(2.2) becomes

$$\begin{aligned} y_t &= \sigma x_t^e\,, \\ x_t &= x_t^e + \varphi y_t\,, \end{aligned}$$

where $x_t^e = \pi_t^e - \pi^*$ and $x_t = \pi_t - \pi^*$ are respectively the *deviations* of the expected and actual inflation from the RE steady state. The dynamics under naive expectations is described by the following linear equation

$$x_t = (1 + \varphi\sigma)\,x_{t-1}\,, \tag{2.3}$$

whose unique steady-state corresponds to the RE equilibrium, $x^* = 0$. This steady-state is, however, unstable, and thus pegging the interest rate at a non-equilibrium level, will lead to a cumulative process.

So far we have discussed a simple IS-LM model considering a monetary institution that follows a nominal interest rate pegging monetary policy rule. Howitt (1992) proposed an alternative strategy to model monetary policy in order to stabilize inflation under adaptive learning dynamics, i.e. under the assumption that agents are not rational. He showed that the cumulative process can be avoided when the Central Bank adopts a monetary policy rule that makes the nominal interest rate respond to the rate of inflation more than point for point. This monetary policy rule has become known as the "Taylor" principle, after Taylor (1993).

Suppose that a simple Taylor rule is used in the example with naive expectations discussed above. Assume that announcing the nominal interest rate the Central Bank responds to the inflation rate according to the following relation:

$$i_t = \phi_\pi \pi_t \,, \qquad \text{where} \quad \phi_\pi > 1 \,. \tag{2.4}$$

The coefficient ϕ_π measures the response of the nominal interest rate to changes in the inflation rate π_t. A Taylor rule with $\phi_\pi > 1$ reflects an important idea: the nominal interest rate should be changed by more than one percentage point for each percentage point change in inflation. Under the Taylor rule (2.4) and naive expectations, the dynamics is described by

$$x_t = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} x_{t-1} \,,$$

which differs from (2.3) only in the slope coefficient. It is immediately clear that for a Taylor rule (2.4) with $\phi_\pi > 1$, the RE equilibrium is globally stable and the cumulative process will not arise.

3 Rational versus Naive

Will the cumulative process arise in an economy where agents have *heterogeneous expectations* about the future level of the inflation rate? To address this question we employ the framework of Adaptive Belief Systems proposed in Brock and Hommes (1997) to model heterogeneous expectations. Assume that agents can form expectations choosing from H different forecasting rules. We denote by $x^e_{h,t}$ the forecast of the deviation of inflation from its RE equilibrium level given by rule h. The fraction of agents using forecasting rule h at time t is denoted by $n_{h,t}$. Assuming that individual expectations can be aggregated linearly[1], actual inflation in the model (2.1)–(2.2) under the interest rate pegging is given by

$$x_t = (1 + \varphi\sigma) \sum_{h=1}^{H} n_{h,t} x^e_{h,t} \,. \tag{3.1}$$

The evolutionary part of the model describes the updating of beliefs over time. Fractions are updated according to an *evolutionary fitness* measure. The fitness measures of all strategies are publicly available, but subject to noise. Fitness is derived from a random utility model and given by

$$\widetilde{U}_{h,t} = U_{h,t} + \varepsilon_{h,t} \,,$$

where $U_{h,t}$ is the deterministic part of the fitness measure and $\varepsilon_{h,t}$ represent IID noise across $h = 1, \ldots, H$. As proposed in Brock and Hommes (1997), in order to obtain analytical expressions for the probabilities or fractions, we will assume that the noise $\varepsilon_{h,t}$ is drawn from a double exponential distribution. In that case, in the limit as the number of agents goes to infinity, the probability that an agent chooses strategy h is given by the well known discrete choice model (see Manski

[1]The model we are considering doesn't have an explicit microfoundation and it is composed by linear aggregate equations. Hence substituting expectations terms with a convex combination of different subjective expectations is the most natural way to proceed. Recent papers, such as Adam (2007), Arifovic, Bullard, and Kostyshyna (2007), Brazier, Harrison, King, and Yates (2008), De Grauwe (2008) follow the same approach.

and McFadden (1981) for details):

$$n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}} \,. \tag{3.2}$$

Note that the higher the fitness of a forecasting rule h, the higher the probability that an agent will select the strategy h. The parameter β is called *intensity of choice* and it reflects the sensitivity of the mass of agents to selecting the optimal prediction strategy. The intensity of choice β is inversely related to the variance of the noise term. The case $\beta = 0$ corresponds to the situation of infinite variance in which differences in fitness can not be observed, so agents do not switch between strategies and all fractions are constant and equal to $1/H$. The case $\beta = \infty$ corresponds to the situation without noise in which the deterministic part of the fitness can be observed perfectly and in every period all agents choose the best predictor. We use as a performance measure past squared forecast errors

$$U_{h,t-1} = -(x_{t-1} - x^e_{h,t-1})^2 - C_h \,, \tag{3.3}$$

where C_h is the cost of predictor h.

As a first application of the ABS in this section we consider the case in which there are two groups of agents, one with rational expectations (perfect foresight), i.e. $x^e_{1,t} = x_t$, and one with naive expectations, i.e. $x^e_{2,t} = x_{t-1}$. In a world with heterogeneous expectations perfect foresight requires knowledge about the predictions of all other agents in the population. Therefore we assume that in order to obtain the perfect foresight forecast agents will have to pay information gathering costs $C \geq 0$ per period, whereas the naive forecast is available for free. We investigate and compare two possible monetary policy rules, interest rate pegging and the Taylor rule, in the case of rational versus naive expectations.

3.1 Interest Rate Pegging

Under interest rate pegging the dynamics of the model with rational versus naive agents is described by

$$x_t = \frac{(1+\varphi\sigma)(1-n_{1,t})}{1-n_{1,t}(1+\varphi\sigma)} x_{t-1}\,, \tag{3.4}$$

where the fraction of agents with perfect foresight is evolving according to

$$n_{1,t} = \frac{e^{-\beta C}}{e^{-\beta C} + e^{-\beta(x_{t-1}-x_{t-2})^2}}\,.$$

The following result describes the steady state properties of this two-dimensional system:

Proposition 3.1. *The dynamics given by* (3.2) *and* (3.4) *has a unique steady-state with* $x^* = 0$ *and* $n_1^* = \frac{e^{-\beta C}}{1+e^{-\beta C}} \leq \frac{1}{2}$. *This "Rational Expectations" steady state is unstable for all costs* $C \geq 0$.

Proof. See Appendix A. □

The RE equilibrium with full employment is the only steady-state of the model. In this steady-state both types of agents give the same correct forecast. The population, however, is split unequally and dominated by naive agents, because of the costs of the perfect foresight predictor. The RE equilibrium is a locally unstable steady-state, which suggests that interest rate pegging is not a feasible policy, not even when the information gathering costs $C = 0$.

In order to get some intuition for the dynamics of the model, in Fig. 1 we plot the graph of the slope of (3.4) as a function of the fraction $n_{1,t}$ of rational agents. In this way one can interpret the behavior of rational agents given their knowledge about the distribution of agents over the two types. Recall that if all agents are naive, i.e. in the case $n_{1,t} = 0$ labeled N in Fig. 1, the cumulative process in (2.3) arises. At the other extreme when all agents have perfect foresight, labeled RE in

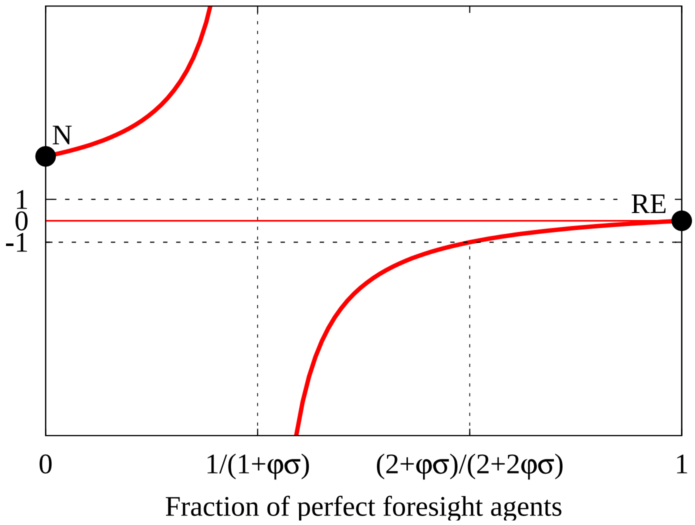

Figure 1: Function $\frac{(1+\varphi\sigma)(1-n_{1,t})}{1-n_{1,t}(1+\varphi\sigma)}$, representing the slope of the map in Eq. (3.4) as a function of $n_{1,t}$. In this figure parameters are such that $\varphi\sigma = 2$.

Fig. 1, the system immediately jumps to the RE steady-state. Fig. 1 shows that in the intermediate case, when agents have heterogeneous expectations, the perfect foresight agents can either reinforce (the left part of the curve) or counterbalance (the right part of the curve) the cumulative process, depending on the relative weight of rational agents in the population. When the fraction of rational agents is relatively low, i.e. $n_{1,t} < 1/(1+\varphi\sigma)$, the cumulative process is reinforced with accelerating inflation or deflation even stronger than under naive expectations. When the fraction of rational agents is relatively high, i.e. $n_{1,t} > 1/(1+\varphi\sigma)$, rational agents counterbalance and reverse the cumulative process. But only when the fraction of rational agents is sufficiently large, i.e. $n_{1,t} > (2+\varphi\sigma)/(2+2\varphi\sigma)$ the counterbalancing of rational agents leads to a stable process. Notice that at the steady state $n_1^* \leq 1/2 \leq (2+\varphi\sigma)/(2+2\varphi\sigma)$, so that at the steady state the counterbalancing effort of rational agents leads to an unstable process.

Thus, only when the naive agents dominate the population, the cumulative process can start. However, along such a process, the naive agents make larger and larger prediction errors. When these errors overcome the costs of the perfect foresight, the majority of agents will switch to choose the rational predictor. However,

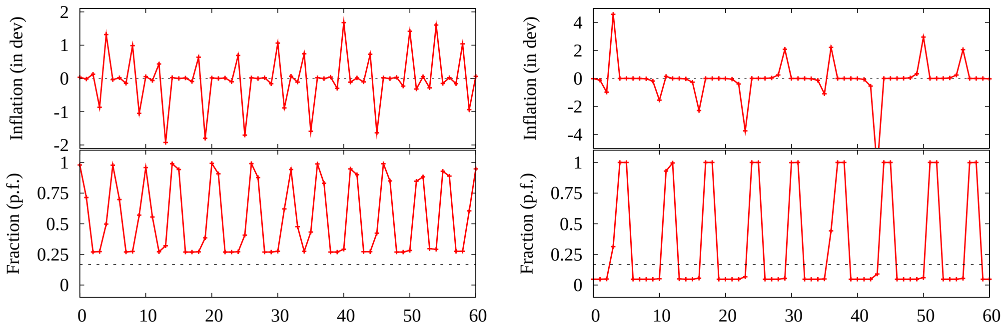

Figure 2: Dynamics of the evolutionary model with rational vs. naive agents. Deviations of inflation from the RE level (upper parts) is shown against an evolution of the fraction of the perfect foresight agents (lower parts). The threshold value $1/(1+\varphi\sigma)$ is shown by the dotted line. **Left panel:** $\beta = 1$. **Right panel:** $\beta = 3$.

as Proposition 3.1 suggests, this is not enough to stabilize the RE. Notice from Fig. 1 that the deviation of inflation from its RE value will decrease only when the fraction of the perfect foresight agents $n_{1,t} > (2+\varphi\sigma)/(2+2\varphi\sigma) > 1/2$. But close to the equilibrium value, when both forecasting rules give similar errors, the fraction of rational agents falls below $1/2$. Thus, when the inflation starts to approach the equilibrium level, more and more agents will switch back to the less-costly naive predictor.

This mechanism is illustrated in Fig. 2, where the dynamics of the actual deviation of inflation from the RE steady state and the evolution of the fraction of perfect foresight agents are shown for two levels of the intensity of choice β. In both cases we observe phases in which the actual deviation of inflation from the RE steady state is relatively small and phases in which this deviation is relatively high. As explained above, during the phases with small deviations from the steady state, the economy is dominated by naive agents, while the phases with high deviations always end up by massive switching to the perfect foresight predictor. There is an important difference between two cases. When the intensity of choice is low (the left panel), the fraction of the rational agents never falls below the threshold value $1/(1+\varphi\sigma)$. The cumulative process never starts in this case because the fraction of rational agents is sufficiently large to counterbalance the cumulative process.

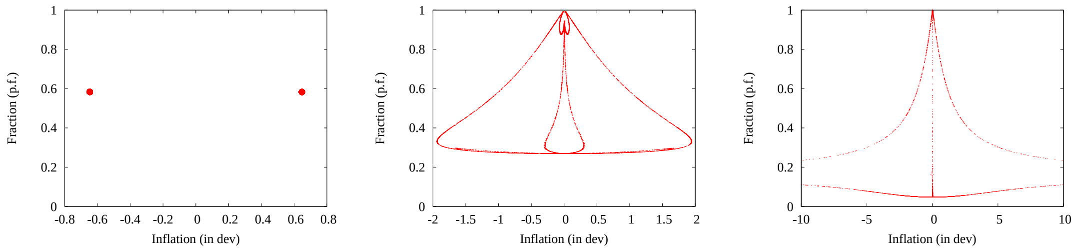

Figure 3: Phase diagram, $(x_t, n_{1,t})$, in the evolutionary model with perfect foresighting and naive agents. **Left panel:** $\beta = 0.5$. **Middle panel:** $\beta = 1$. **Right panel:** $\beta = 3$.

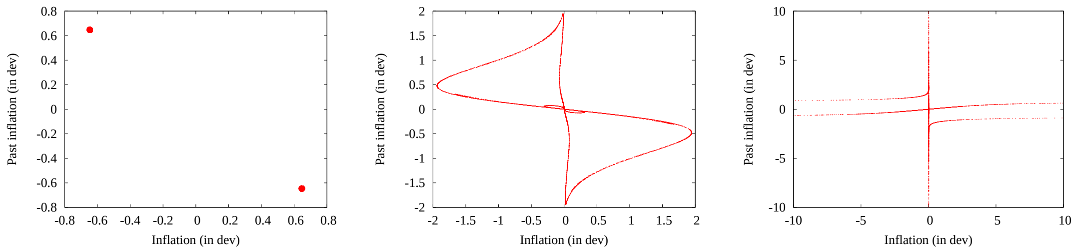

Figure 4: Delay plot, (x_t, x_{t-1}), in the evolutionary model with perfect foresighting and naive agents. **Left panel:** $\beta = 0.5$. **Middle panel:** $\beta = 1$. **Right panel:** $\beta = 3$.

When the intensity of choice is high (the right panel), the cumulative process occurs when the fraction of the rational agents falls below the threshold value. As the cumulative process evolves, forecasting errors of naive agents increase and at some point, for a large intensity of choice, almost all agents will switch to rational expectations, thus stabilizing the cumulative process and forcing inflation close to its RE steady state. With inflation close to steady state, the fraction of agents using the cheap naive forecast increases, and a new cumulative process may arise.

Figs. 3 and 4 compare the phase diagrams and delay plots for three different values of the intensity of choice. We observe that the system converges to a two-cycle for small values of β, but as soon as the intensity of choice increases the strange attractors and chaotical behavior occur. Indeed a *rational route of randomness* in inflation rates, that is, a bifurcation route to complicated dynamics, arises when the intensity of choice becomes large[2].

[2] A difference with the rational route to randomness in the cobweb model of Brock and Hommes (1997) is that in our macro model it starts off from a stable 2-cycle (the steady state is always

Our analysis shows that in an economy with heterogeneous agents with perfect foresight versus naive expectations about future inflation, a monetary policy that pegs the nominal interest rate will not lead to an ever accelerating cumulative process. However, the inflation is not stable, but rather switches irregularly between an unstable phase of a temporary cumulative process or unstable counterbalancing when the fraction of rational agents is relatively small and phases of stable counterbalancing when the fraction of rational agents is relatively large.

3.2 Taylor Rule

In this section we consider a Central Bank that responds to the inflation rate by means of a simple Taylor rule as defined in equation (2.4). Under a Taylor rule, the dynamics of the model is described by

$$x_t = \frac{k(1-n_{1,t})}{1-kn_{1,t}} x_{t-1}\,, \tag{3.5}$$

where the constant $k \equiv \frac{1+\varphi\sigma}{1+\varphi\sigma\phi_\pi}$. The fraction of agents with perfect foresight evolves according to

$$n_{1,t} = \frac{e^{-\beta C}}{e^{-\beta C} + e^{-\beta(x_{t-1}-x_{t-2})^2}}\,. \tag{3.6}$$

Under a Taylor rule with $\phi_\pi > 1$, the coefficient k belongs to the interval $(0,1)$. It is then obvious that for any $n_{1,t}$ the map (3.5) is a contraction. It leads to

Proposition 3.2. *The dynamics* (3.5)–(3.6) *under a monetary policy Taylor rule has a unique, globally stable RE steady-state with* $x^* = 0$ *and* $n_1^* = \frac{e^{-\beta C}}{1+e^{-\beta C}}$.

Hence, in an economy with rational versus naive agents, for any costs of the rational forecast the Taylor rule stabilizes inflation dynamics.

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unstable), while in the cobweb model it starts off from a stable steady state.

4 Interest Rate Pegging with Fundamentalists and Biased Beliefs

In this section we consider an interest rate pegging monetary policy rule when agents can choose between different constant "steady state" predictors to forecast future inflation. This case can be interpreted as a situation in which agents roughly know the fundamental steady state of the economy, but agents are boundedly rational in the sense that they have distorted perceptions of equilibrium values. We can thus assume that forecasting correctly the RE equilibrium value of inflation $x^* = 0$ requires some cognitive efforts and information gathering costs, which will be incorporated in the cost term $C \geq 0$. Alternatively we can think about this case as a situation in which all agents have access to the same information about the fundamentals of the economy, at zero cost for example, but, nevertheless they may decide differently about forecasting future inflation. Realized inflation and expectations will co-evolve over time and evolutionary selection based on reinforcement learning will decide which kind of forecasting rule performs better and will survive in the evolutionary environment. An important question at this point is: what kind of monetary policy should a central bank implement to stabilize inflation in such an environment? In this section we investigate the dynamic effects of a monetary policy aimed at pegging the nominal interest rate in a world of heterogeneous boundedly rational agents. The class of constant forecasts is extremely simple, but may include any potential steady state level. For this simple ecology of rules it will be possible to obtain analytical results, in examples with only a few rules as well as examples with a large number or even a continuum of constant rules.

4.1 Evolutionary Dynamics with Few Constant Beliefs Types

We start our analysis by investigating the case in which there are three different forecasting rules:

$$
\begin{aligned}
x^e_{1,t} &= 0\,, \\
x^e_{2,t} &= b\,, \\
x^e_{3,t} &= -b\,,
\end{aligned}
$$

with bias parameter $b > 0$. Type 1 agents believe that the inflation rate will be always at its RE level and so their expected deviation will be zero. Type 2 agents have a positive bias, expecting that inflation will be above the fundamental level, while type 3 agents have a negative bias, expecting an inflation level below the fundamental value. Assuming that the equilibrium predictor is available at cost $C \geq 0$ and substituting the forecasting rules of the three groups of agents into (3.1) we get

$$x_t = (1 + \varphi\sigma)(n_{2,t}b - n_{3,t}b) = f_\beta(x_{t-1})\,, \tag{4.1}$$

where fractions are updated according to the discrete choice model (3.2), that is,

$$
\begin{aligned}
n_{2,t} &= \frac{e^{-\beta(x_{t-1}-b)^2}}{Z_{t-1}}\,, \\
n_{3,t} &= \frac{e^{-\beta(x_{t-1}+b)^2}}{Z_{t-1}}\,,
\end{aligned}
$$

and

$$Z_{t-1} = e^{-\beta(x^2_{t-1}+C)} + e^{-\beta(x_{t-1}-b)^2} + e^{-\beta(x_{t-1}+b)^2}\,.$$

In what follows we will fix the parameters φ, σ, and b and consider the intensity of choice β as bifurcation parameters[3]. We will make a distinction between the case

[3]Changes in the product of the IS slope σ and the Phillips curve's slope φ, as well as changes in the bias parameter b will only affect the steady state values of non-RE steady states and the

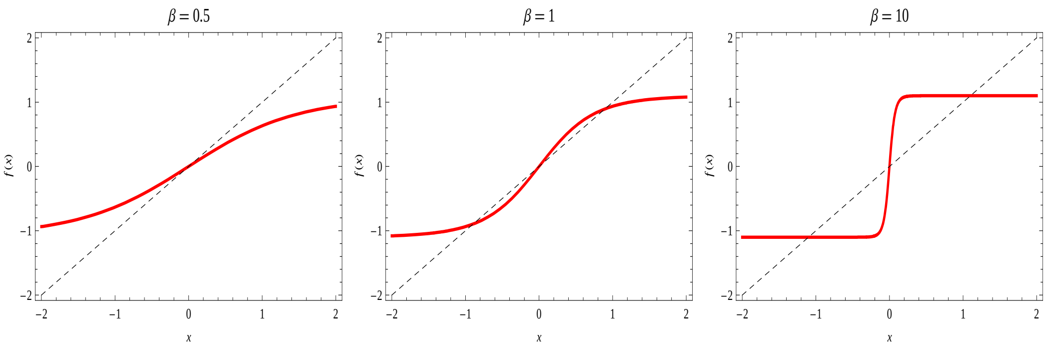

Figure 5: Maps with 3 types of beliefs and high cost C. Parameter values are $\varphi\sigma = 0.1$, $b = 1$ and $C = 1$.

in which the equilibrium predictor is available at a relatively high cost and the case in which it is freely available (or available at a relatively low cost). Notice that the dynamics in (4.1) is described by a 1-dimensional map f_β, and a straightforward computation shows that f_β is increasing.

Let us start with the case in which the fundamental predictor has relatively high cost. Fig. 5 shows the maps f_β for small, medium and high values of the intensity of choice parameter β. When the intensity of choice is relatively low, there exists only one steady state, the RE steady state, which is globally stable. For low intensity of choice agents are more or less evenly distributed over the different forecasting rules and as a result realized inflation will remain relatively close to the fundamental steady state. As the intensity of choice increases, the RE steady state loses stability in a (supercritical) pitchfork bifurcation and two new stable non-fundamental steady states are created. The economic intuition behind the fact that non-fundamental steady states exist for high intensity of choice is as follows. Suppose that the intensity of choice is high and that at time t, the deviation x_t is close to the optimistic belief, that is, $x_t \approx b$. The optimistic belief forecast will then perform better than the pessimistic and the fundamental belief and therefore, when the intensity of choice is high, almost all agents will use the optimistic belief, i.e. $n_{2,t+1} \approx 1$, implying that $x_{t+1} \approx b(1 + \varphi\sigma)$. In fact, it is easily seen that for

bifurcation values at which multiple steady states appear, but they will not alter the qualitative bifurcation scenario as discussed below.

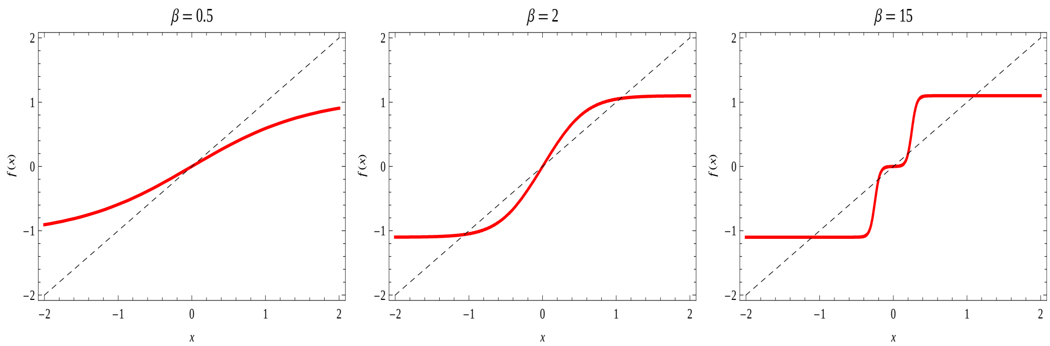

Figure 6: Maps with 3 types of beliefs and low cost $C = 0.5$.

$\beta = +\infty$ the map f_∞ has non-fundamental steady states $x^+ = b(1 + \varphi\sigma)$. The same intuition explains existence of a negative non-fundamental steady state for high intensity of choice.

Consider now the case in which the equilibrium predictor has zero (or relatively low) costs. We can think about this case as a situation in which all agents have free access to the relevant information, but agents make some computational mistakes or they just think that in a heterogeneous world not every agent will behave the same and try to anticipate deviations from RE equilibrium. Fig. 6 shows graphs of the map f_β for small, medium and high values of the intensity of choice parameter β.

As before, when the intensity of choice β is relatively low we have a unique globally stable fundamental steady state $x^* = 0$. As β increases, as before the fundamental steady state loses stability in a (supercritical) pitchfork bifurcation in which two additional stable non-fundamental steady states are created[4]. However, as β increases further, we have a second pitchfork bifurcation, this time a subcritical pitchfork bifurcation, in which the RE steady state becomes stable again and two additional unstable steady states are created. In the case of low costs for fundamentalists, we thus have three stable steady states, $x^* = 0$, $x^+ > 0$ and $x^- < 0$ for high values of the intensity of choice β. The economic intuition that, if the costs for the fundamental rule are low, the fundamental steady state will be stable

[4] Appendix B derives conditions under which the zero steady state does or does not lose stability for intermediate values of the intensity of choice β.

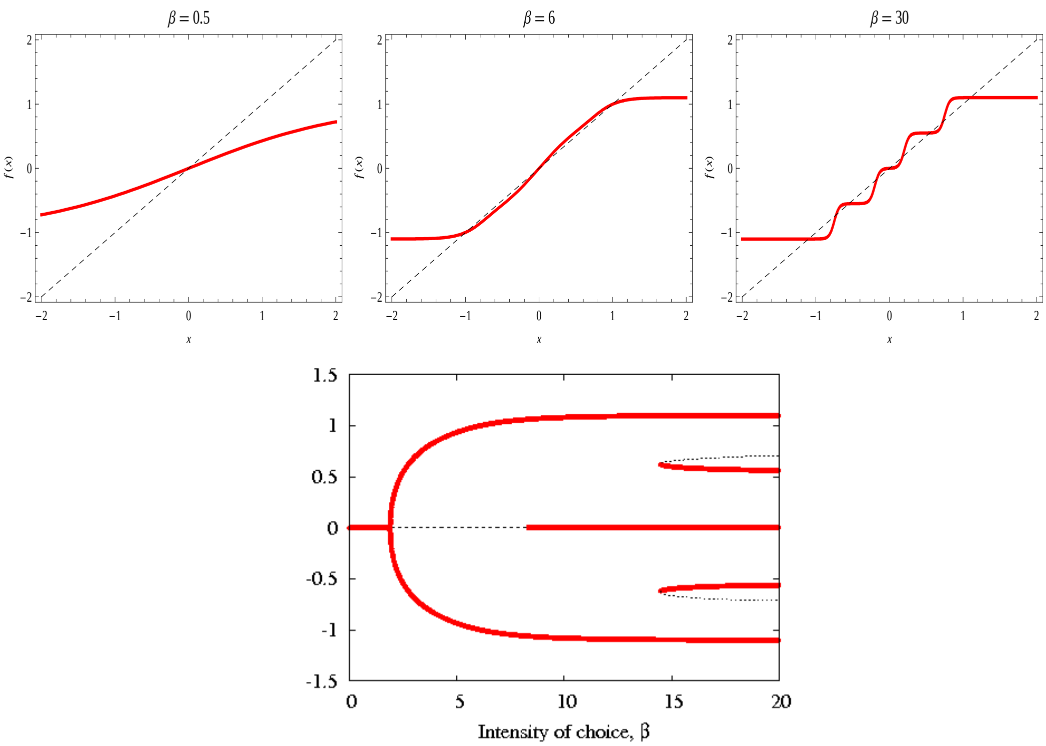

Figure 7: **Top panels:** Maps with 5 types of beliefs and low cost C for different values of β. **Lower panel:** Bifurcation diagram for 5 belief types (cost $C = 0$) with respect to the intensity of choice. Solid lines indicate stable equilibria and dashed lines unstable equilibria.

for high intensity of choice is simple: when the system is close to the fundamental steady state, a cheap fundamental rule is the best predictor, causing more agents to switch to the fundamental rule.

Fig. 7 illustrates graphs of the 1-D map when there are five strategy types $b_h \in \{-1, -1/2, 0, 1/2, 1\}$ when the cost C of the fundamental predictor are low as well as the bifurcation diagram. For small and medium values of β the bifurcation scenario is similar to the three types case. However for high values of the intensity of choice, four additional steady states, two stable and two unstable, are created via saddle-node bifurcations.

4.2 Many Belief Types

The previous analysis shows that in an economy with an ecology of 3 or 5 fundamentalists and biased beliefs, a cumulative process leading to accelerating inflation or deflation does *not* arise. For high intensity of choice, the system will rather lock in into one of the multiple steady state equilibria, with a majority of agents using the forecasting rule with the smallest mistake at that equilibrium steady state. A natural question addressed in this section is what happens when the number of constant forecasting rules increases and approaches infinity. As we will see, if agents select beliefs from a *continuum* of forecasting rules, the cumulative process will reappear.

Suppose there are H belief types b_h, all available at zero costs. The evolutionary dynamics with H belief types is given by

$$x_t = (1+\varphi\sigma)\frac{\sum_{h=1}^{H} b_h e^{-\beta(x_{t-1}-b_h)^2}}{\sum_{h=1}^{H} e^{-\beta(x_{t-1}-b_h)^2}} =: f_\beta^H(x_{t-1})\,. \tag{4.2}$$

The dynamics of the system with H belief types b_H is described by a 1-D map f_β^H. What can be said about the dynamical behavior when H is large? In general, it is difficult to obtain analytical results for systems with many belief types. We apply the concept of *Large Type Limit* (LTL) introduced in Brock, Hommes, and Wagener (2005) to approximate the evolutionary systems with many beliefs type in (4.2). Suppose that at the beginning of the economy, i.e. at period $t = 0$, all H belief types $b = b_h \in \mathbb{R}$ are drawn from a common initial distribution with density $\psi(b)$. We then can derive the LTL of the system as follows. Divide both numerator and denominator of (4.2) by H and write the "H-type system" as

$$x_t = (1+\varphi\sigma)\frac{\frac{1}{H}\sum_{h=1}^{H} b_h e^{-\beta(x_{t-1}-b_h)^2}}{\frac{1}{H}\sum_{h=1}^{H} e^{-\beta(x_{t-1}-b_h)^2}}\,.$$

The LTL is then obtained by replacing the sample mean with the population mean in both the numerator and the denominator, yielding

$$x_t = (1+\varphi\sigma)\frac{\displaystyle\int b e^{-\beta(x_{t-1}-b)^2}\psi(b)db}{\displaystyle\int e^{-\beta(x_{t-1}-b)^2}\psi(b)db} =: F_\beta(x_{t-1})\,. \tag{4.3}$$

Figure 8: LTL map for normal distribution of the initial beliefs. **Left panel:** $\beta = 1$, **Right panel:** $\beta = 1000$.

As shown in Brock, Hommes, and Wagener (2005), when the number of strategies H is sufficiently large, the LTL dynamical system (4.3) is a good approximation of the dynamical system with H belief types given by (4.2). In particular, if H is large then with high probability the steady-states and their local stability conditions as functions of β coincide for both the LTL map F_β and the H-belief system map f_β^H. In other words, properties of the evolutionary dynamical system with many types of agents can be studied using the LTL system.

For suitable distributions $\psi(b)$ of initial beliefs, the LTL (4.3) can be computed explicitly. As an illustrative example consider the case when $\psi(b)$ is a normal distribution $\psi(b) \simeq N(0, s^2)$. In Appendix C it is shown that in this case the LTL map F_β is linear, and given by

$$F_\beta(x) = (1 + \varphi\sigma)\frac{2\beta s^2}{1 + 2\beta s^2}x. \tag{4.4}$$

Fig. 8 illustrates graphs of the LTL maps for different values of the intensity of choice. For $\beta = \beta^* = \frac{1}{2s^2\varphi\sigma}$ the slope of the linear map is exactly 1. Hence, the fundamental equilibrium is globally stable for $\beta < \beta^*$ and unstable otherwise.

We thus come to the conclusion that, when initial beliefs are drawn from a normal distribution and the number of belief types is sufficiently high, an increase in the intensity of choice beyond the bifurcation value β^* leads to instability of the system. Indeed, when β is low, agents are more or less equally distributed among predictors. This means that the average expected deviation of inflation will be close to zero, hence realized inflation will be close to its fundamental value, more agents will adopt the fundamental predictor and inflation will converge to its fundamental value. However, when the intensity of choice increases and agents can switch faster to better predictors, the cumulative process arises again.

It will be instructive to look at the limiting case where $\beta = \infty$. When there is a continuum of beliefs, the best predictor in every period, according to past forecast error, will be the predictor that coincides with last period's inflation realization, $b_h = x_{t-1}$. For $\beta = \infty$, all agents will switch to the optimal predictor. Hence, for $\beta = \infty$, the economy with heterogenous agents updating their beliefs through reinforcement learning behaves exactly the same as an economy with a representative naive agent, for which we already know that a cumulative process will arise.

Finally, note that increasing the variance s^2 of the normal distribution of initial beliefs has exactly the same effect on the LTL dynamics (4.4) as increasing the intensity of choice. For $s^2 < \frac{1}{2\beta\varphi\sigma}$ the LTL map is globally stable and it is unstable otherwise. Hence, when many initial beliefs are drawn from a normal distribution with small variance, the system will be stable, while it will be unstable and a cumulative process will arise when many initial beliefs are drawn from a normal distribution with large variance. The *spread* of initial beliefs is therefore an important element for the stability of the economy.

General distribution of initial beliefs

In the previous example we have assumed that the distribution $\psi(b)$ of initial beliefs is a normal distribution. Applying the results derived in Hommes and Wagener (2003), similar results are obtained for more general distributions functions of initial beliefs.

As a first observation, note that when the beliefs distribution $\psi(b)$ is symmetric around the RE equilibrium $x^* = 0$, the latter will be a steady state for system (4.3). This immediately follows from the observation that $be^{-\beta(-b)^2}\psi(b)$ is an odd function.

Let $\psi(b)$ be a fixed continuous density function, that is, let $\psi(b) \geq 0$ for all b and $\int \psi(b)db = 1$, and consider

$$G_\beta(x) = \frac{\displaystyle\int be^{-\beta(x-b)^2}\psi(b)\mathrm{d}b}{\displaystyle\int e^{-\beta(x-b)^2}\psi(b)\mathrm{d}b} \,. \tag{4.5}$$

We recall the following result from Hommes and Wagener (2003) (Lemma 1, p. 10).

Lemma. *Let J be the interior of the support of ψ, that is, $J = int\{b|\psi(b) \geq 0\}$.*

For all $x \in J$:

$$\lim_{\beta\to\infty} G_\beta(x) = x \qquad \textit{and} \qquad \lim_{\beta\to\infty} \frac{\partial}{\partial x} G_\beta(x) = 1\,,$$

uniformly on all compact subsets K of J.

From this lemma, for example, it follows that for any strictly positive distribution function ψ describing initial beliefs, as the intensity of choice goes to infinity, the corresponding LTL-map converges to a linear map with slope $1+\varphi\sigma$. The LTL map thus exhibits a cumulative process, when the intensity of choice becomes sufficiently large. Hence, for systems with many belief types b_h and initial beliefs drawn from a fixed strictly positive distribution function, a cumulative process arises with high probability.

5 Taylor Rule with Fundamentalists and Biased Beliefs

In this section we consider the dynamic consequences of an alternative monetary policy rule as introduced in (2.4), namely

$$i_t = \phi_\pi \pi_t \,. \tag{5.1}$$

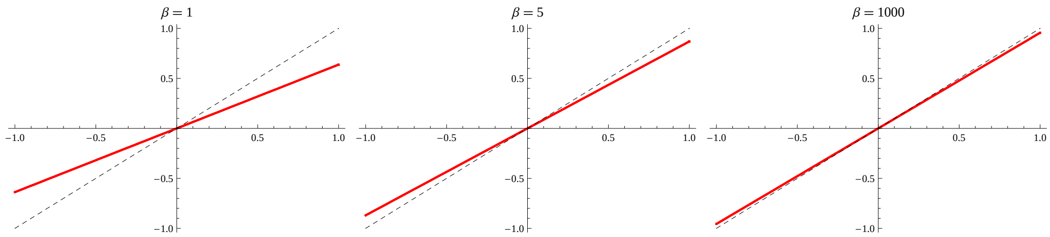

Figure 9: LTL map under Taylor rule for $\beta = 1$, $\beta = 5$, $\beta = 1000$.

Plugging equation (5.1) in the system (2.1)-(2.2) and rewriting the model in deviations from the RE equilibrium yields

$$x_t = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} \sum_{h=1}^{H} n_{h,t} x_{h,t}^e \,. \tag{5.2}$$

5.1 Many types

When the central bank implements the interest rate rule described by (5.1), the LTL of the system is given by

$$x_t = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} \frac{\displaystyle\int b e^{-\beta(x_{t-1} - b)^2} \psi(b) db}{\displaystyle\int e^{-\beta(x_{t-1} - b)^2} \psi(b) db} = F_\beta(x_{t-1}) \,. \tag{5.3}$$

Under the assumption $\psi(b) \equiv N(0, s^2)$ the LTL map (5.3) is a linear map with slope increasing in β, as shown in Fig. 9. In this case we will have that

$$\lim_{\beta \to \infty} F_\beta(x) = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} x \,, \qquad \text{and} \qquad \lim_{\beta \to \infty} \frac{\partial}{\partial x} F_\beta(x) = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} \,.$$

Hence an interest rate rule that responds aggressively to actual inflation, i.e. $\phi_\pi > 1$, will fully stabilize the system.

5.2 Few types

Now consider the case the Central Bank implements a Taylor-type interest rate rule and there are only three predictors available in the economy. The map describing

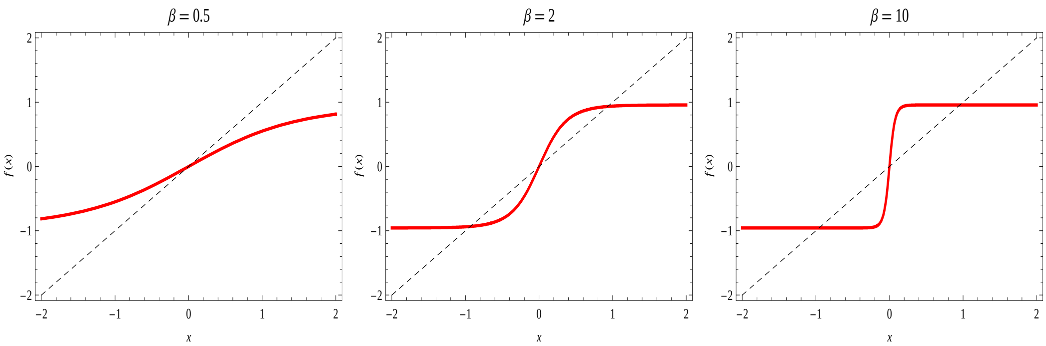

Figure 10: Maps with 3 types of beliefs and high cost C.

the dynamics of the system will be then given by

$$x_t = \frac{1+\varphi\sigma}{1+\varphi\sigma\phi_\pi}(n_{2,t}b - n_{3,t}b)\,. \tag{5.4}$$

We will again consider the two different cases in which the equilibrium predictor is respectively available at a relatively high cost C and freely available. Fig. 10 depicts the dynamics of the system using the same parameterization as Section 4.1 for the coefficient $\phi_\pi = 1.5$ and with a relatively high cost C.

In this case we observe that when the intensity of choice is relatively low the RE equilibrium is unique and globally stable. However as β increases we have that the zero steady becomes unstable after a supercritical pitchfork bifurcation. We thus observe that when agents can switch faster between different predictors, the dynamics converges to equilibria different from zero because of equilibrium predictor's relatively high costs. Fig. 11 shows instead the dynamics of the model when the equilibrium predictor is freely available.

In this case we observe that the RE equilibrium remains locally stable when the intensity of choice increases and four additional steady states, two stable and two unstable, are created via saddle node bifurcation. The analysis performed in this session shows that even if the interest rate rule followed by the central bank obeys the Taylor principle, multiple equilibria can arise when only a few predictors are available in the economy.

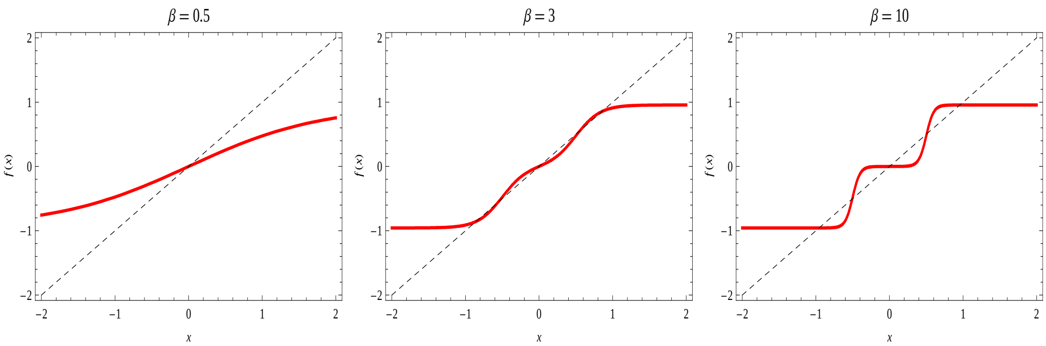

Figure 11: Maps with 3 types of beliefs and zero cost C.

We conclude that if the number of strategies is finite, e.g. because of a finite grid, a Taylor rule with $\phi_\pi > 1$ does not always stabilize inflationary dynamics and multiple steady state equilibria may arise.

6 Conclusion

The analysis performed in the paper shows that in a world with heterogeneous agents updating their beliefs over time according to an evolutionary fitness measure, the cumulative process is not always arising under an interest rate pegging policy. Whether a cumulative process will arise under interest rate pegging depends on two important features: (i) the ecology and the number of heterogeneous forecasting rules, and (ii) the magnitude of the intensity of choice measuring how quickly agents switch strategies.

In the case of rational versus naive agents the interest rate pegging will not lead to an ever accelerating cumulative process. However an increase in the intensity of choice leads to a bifurcation route to complicated dynamics. The inflation rate undergoes phases with small deviations from the RE steady state, dominated by naive agents, and phases with high deviations from the RE equilibrium which always come to an end because of agents' massive switching to the perfect foresight predictor.

In the case of heterogeneous constant forecasting rules we observed that when the intensity of choice is relatively low the cumulative process does not arise in both cases of few belief types and many belief types and the dynamics converges to the unique RE steady state. However, when the intensity of choice parameter is increasing, the system can converge to equilibria different from the RE steady state in the case of few belief types, or diverge because of the occurrence of the cumulative process in the case many belief types.

The paper also investigates the dynamical consequences of a Taylor-type interest rate rule. When the ecology of forecasting rules is composed by a perfect foresight and a naive predictor, an interest rate rule that responds aggressively to inflation fully stabilizes the system and the inflation rate converges to the RE steady state level. In the case of constant belief types we observed that when many types of predictors are available in the economy, an interest rate rule that obeys the Taylor principle always stabilizes the system and the dynamics converges to the RE equilibrium. However, when there is just a limited number of belief types, multiple equilibria can arise.

APPENDIX

A Proof of Proposition 3.1

The dynamical system describing the model with perfect foresight and naive agents is given by the difference equation of second order:

$$x_t = \frac{(1+\varphi\sigma)\left(1 - \frac{e^{-\beta C}}{e^{-\beta C}+e^{-\beta(x_{t-1}-x_{t-2})^2}}\right)}{1-(1+\varphi\sigma)\left(\frac{e^{-\beta C}}{e^{-\beta C}+e^{-\beta(x_{t-1}-x_{t-2})^2}}\right)} x_{t-1}\,.$$

We can rewrite the latter equation as a two-dimensional system by introducing $z_t = x_t$ and $w_t = x_{t-1}$

$$\begin{aligned} z_t &= \frac{(1+\varphi\sigma)\left(1 - \frac{e^{-\beta C}}{e^{-\beta C}+e^{-\beta(z_{t-1}-w_{t-1})^2}}\right)}{1-(1+\varphi\sigma)\left(\frac{e^{-\beta C}}{e^{-\beta C}+e^{-\beta(z_{t-1}-w_{t-1})^2}}\right)} z_{t-1} \\ w_t &= z_{t-1}\,. \end{aligned}$$

The Jacobian of the system computed in the RE steady-state $(0,0)$ is given by

$$J(0,0) = \begin{bmatrix} \frac{(1+\varphi\sigma)(1-n_1^*)}{1-(1+\varphi\sigma)n_1^*} & 0 \\ 1 & 0 \end{bmatrix},$$

where $n_1^* = \frac{e^{-\beta C}}{1+e^{-\beta C}}$. The eigenvalues are

$$\begin{aligned} \lambda_1 &= 0\,, \\ \lambda_2 &= \frac{(1+\varphi\sigma)(1-n_1^*)}{1-(1+\varphi\sigma)n_1^*} = \frac{1+\varphi\sigma-n_1^*-\varphi\sigma n_1^*}{1-n_1^*-\varphi\sigma n_1^*}\,. \end{aligned}$$

The numerator in expression of λ_2 is always positive since $0 < n_1^* < 1$, while the denominator is positive if $\varphi\sigma < 1/e^{-\beta C}$. In this case we have that $\lambda_2 > 1$. When $\varphi\sigma > 1/e^{-\beta C}$, the stability condition implies

$$\frac{(1+\varphi\sigma)(1-n_1^*)}{1-(1+\varphi\sigma)n_1^*} > -1 \Rightarrow (1+\varphi\sigma)(1-n_1^*) < (1+\varphi\sigma)n_1^* - 1 \Rightarrow \varphi\sigma < \frac{2(n_1^*-1)}{1-2n_1^*} \Rightarrow \varphi\sigma < \frac{-2}{1-\exp(-\beta C)}$$

Since both φ and σ are positive coefficients, the stability condition is never satisfied and thus we conclude that $|\lambda_2| > 1$. ■

B 3 types system

Consider the 3 types system described by

$$x_t = \frac{1+\varphi\sigma}{1+\varphi\sigma\phi_\pi} b \frac{e^{-\beta(x_{t-1}-b)^2} - e^{-\beta(x_{t-1}+b)^2}}{e^{-\beta(x_{t-1}^2+C)} + e^{-\beta(x_{t-1}-b)^2} + e^{-\beta(x_{t-1}+b)^2}} = f(x_{t-1}) .$$

Now define $k = \frac{1+\varphi\sigma}{1+\varphi\sigma\phi_\pi}$ and compute the derivative of the map f in the RE steady state to get

$$f'(0) = k 4\beta b^2 \frac{e^{-\beta b^2}}{2e^{-\beta b^2} + e^{-\beta C}} = k 4\beta b^2 \frac{1}{2 + e^{-\beta(C-b^2)}} .$$

The stability condition is thus given by

$$f'(0) < 1 \Rightarrow \frac{4b^2\beta}{2 + e^{-\beta(C-b^2)}} < \frac{1}{k} .$$

Now define

$$h(\beta) = \frac{4b^2\beta}{2 + e^{-\beta(C-b^2)}}$$

and consider the following two cases.

If $C > b^2$ we have that $h(\beta)$ is monotonically increasing in β. Thus, when β is higher than the bifurcation value β^* defined as

$$\beta^* : h(\beta^*) = \frac{1}{k}$$

the zero steady state looses stability, as shown in the left panel of Fig. 12.

If $C < b^2$ we have that the function $h(\beta)$ is initially increasing in β and then decreasing. We indeed have that

$$h'(\beta) = \frac{4b^2}{\left[2 + e^{-\beta(C-b^2)}\right]^2} \left[2 + e^{-\beta(C-b^2)} + \beta e^{-\beta(C-b^2)}(C-b^2)\right] .$$

Figure 12: Stability/Instability of the REE. **Left panel:** The case of high cost, $C > b^2$. **Right panel:** The case of small cost, $C < b^2$.

We have that $h'(\beta) = 0$ when

$$2 + e^{-\beta(C-b^2)} + \beta e^{-\beta(C-b^2)}(C - b^2) = 0 \, .$$

Now define $z \equiv (C - b^2)\beta$, so that the previous equation becomes

$$2 + e^{-z} + ze^{-z} = 0 \, ,$$

which can be rewritten as

$$2e^z = -z - 1 \, . \tag{B.1}$$

Now, when $C < b^2$, we have that z is a variable defined over $(-\infty, 0)$ since β is increasing from 0 to ∞. This means that there is only one solution $z^* < 0$ to the previous equation, i.e. $h(\beta)$ has only one optimum as shown in the right panel of Fig. 12.

We can find an approximate numerical solution to (B.1) which is given by $z^* \approx -1.46306$. We then have that the maximum point β^* is defined through $(C - b^2)\beta^* = z^*$. Plugging β^* in $h(\beta)$ we find the maximum value of the function, which is given by

$$h(\beta^*) \approx \frac{4b^2\beta^*}{2 + e^{1.46306}} \approx 0.926111 \frac{b^2}{b^2 - C} \, .$$

The condition for the RE steady state to remain stable when $C < b^2$ is given by

$$0.926111 \frac{b^2}{b^2 - C} < \frac{1}{k} \, .$$

This implies that given parameters b and C, the Central Bank can always implement an interest rate rule that keeps the RE steady state stable.

C Large Type Limit

To analyse the LTL map we, first, notice that the fractions $n_{h,t}$ will not be affected if a term that is independent from h is subtracted to all fitnesses $U_{h,t-1}$. Subtracting the term x_{t-1}^2, we can substitute all the fitnesses by

$$U_{h,t-1} = -(x_{t-1} - b_h)^2 - x_{t-1}^2 = 2x_{t-1} b_h - b_h^2 \, .$$

The LTL system can thus be rewritten as

$$x_t = (1 + \varphi\sigma) \frac{\displaystyle\int b e^{-\beta(b^2 - 2x_{t-1} b)} \psi(b) db}{\displaystyle\int e^{-\beta(b^2 - 2x_{t-1} b)} \psi(b) db} \, . \tag{C.1}$$

Consider now the derivative of this map, called F_β:

$$\frac{\partial}{\partial x} F_\beta(x) = (1 + \varphi\sigma) \frac{\partial}{\partial x} z(x) \, ,$$

where

$$z(x) = \frac{\displaystyle\int b e^{-\beta(b^2 - 2xb)} \psi(b) db}{\displaystyle\int e^{-\beta(b^2 - 2xb)} \psi(b) db}$$

Differentiating under the sign of integral yields

$$z'(x) = \frac{\displaystyle\int b e^{-\beta(b^2 - 2xb)} 2b\beta \, \psi(b) db \int e^{-\beta(b^2 - 2xb)} \psi(b) db - \int b e^{-\beta(b^2 - 2xb)} \psi(b) db \int e^{-\beta(b^2 - 2xb)} 2b\beta \, \psi(b) db}{\left[\displaystyle\int e^{-\beta(b^2 - 2xb)} \psi(b) db \right]^2} \, ,$$

which can be rewritten as

$$z'(x) = \frac{2\beta \displaystyle\int b^2 e^{-\beta(b^2-2xb)}\psi(b)db}{\displaystyle\int e^{-\beta(b^2-2xb)}\psi(b)db} - \frac{2\beta \left[\displaystyle\int be^{-\beta(b^2-2xb)}\psi(b)db\right]^2}{\left[\displaystyle\int e^{-\beta(b^2-2xb)}\psi(b)db\right]^2}\,.$$

Defining the density function

$$\xi_{x,\beta}(b) = \frac{e^{-\beta(b^2-2xb)}\psi(b)}{\displaystyle\int e^{-\beta(\widetilde{b}^2-2x\widetilde{b})}\psi(\widetilde{b})d\widetilde{b}}\,,$$

which is easily seen to be nonnegative and to integrate to one, we can then write

$$\begin{aligned} z'(x) &= 2\beta\left[\int b^2\,\xi_{x,\beta}(b)db - \left(\int b\,\xi_{x,\beta}(b)db\right)^2\right] \\ &= 2\beta\left[E_\xi b^2 - (E_\xi b)^2\right] \\ &= 2\beta\, Var_\xi b\,, \end{aligned}$$

where $E_\xi b$ and $Var_\xi b$ are respectively the expected value and the variance of the stochastic variable b distributed according to the probability density function ξ. The slope of the LTL-map is then

$$\frac{\partial}{\partial x}F_\beta(x) = (1+\varphi\sigma)2\beta\, Var_\xi b\,.$$

The condition for the stability of the steady state of the system (C.1) is given by

$$(1+\varphi\sigma)2\beta\, Var_\xi b < 1\,. \tag{C.2}$$

Consider now the case in which

$$\psi(b) \equiv N(0,s^2)$$

In this special case the LTL system is linear and thus the RE equilibrium is the unique steady state. The slope is given by

$$\frac{\partial}{\partial x}F_\beta(x) = (1+\varphi\sigma)\frac{2\beta s^2}{1+2\beta s^2}\,.$$

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