Optimal Diversity in Investments with Recombinant Innovation*

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Abstract

The notion of dynamic, endogenous diversity and its role in theories of investment and technological innovation is addressed. We develop a formal model of an innovation arising from the combination of two existing modules with the objective to optimize the net benefits of diversity. The model takes into account increasing returns to scale and the effect of different dimensions of diversity on the probability of emergence of a third option. We obtain analytical solutions describing the dynamic behaviour of the values of the options. Next diversity is optimized by trading off the benefits of recombinant innovation and returns to scale. We derive conditions for optimal diversity under different regimes of returns to scale. Threshold values of returns to scale and recombination probability define regions where either specialization or diversity is the best choice. In the time domain, when the investment time horizon is beyond a threshold value, a diversified investment becomes the best choice. This threshold will be larger the higher the returns to scale.

JEL classification: B52, C61, O31, Q55.

Key words: balance, modularity, recombination, returns to scale, threshold effects.

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1 Introduction

When making decisions on investment in technological innovation, implicitly or explicitly choices are made about diversity of options, strategies or technologies. Such choices should ideally consider the benefits and costs associated with a certain level of diversity and arrive at an optimal trade-off. One important benefit of diversity relates to the nature of innovation, which often results from combining existing but separate technologies or knowledge bases (Ethiraj and Levinthal, 2004). For instance, a laptop computer in essence is a combination of a desktop computer and a battery; the windmill is a combination of the water mill technology and the idea of a sail (i.e. wind turned into kinetic energy); a laser is quantum mechanics integrated into an optical device; and an optical fibre used in telecommunication is a laser applied to glass technology.

Here we propose a theoretical framework for the description of a generic innovative process resulting from the interaction of two existing but different technologies. The interaction will depend on how these two options match. Matching can occur via spillover or recombination, leading to modular innovation.[1] The model will allow addressing the problem of optimal diversity in the context of modular innovation, building upon the conceptual framework in van den Bergh (2008). The main idea is that in an investment decision problem where available options may recombine and give birth to an innovative option (technology), a certain degree of diversity of parent options can lead to higher benefits than specialization.

Usually in economics and finance, diversity is seen as conflicting with efficiency of specialization. Such efficiency is claimed on the basis of increasing returns to scale arising from fixed costs, learning, network and information externalities, technological complementarities and other self-reinforcement effects. Arthur (1989) studies the dynamics of competing technologies in cases where increasing returns cause path dependence and self-reinforcement, possibly leading to lock-in. This can be seen as a descriptive or positive approach to understand the dynamics of systems in the presence of positive feedback. Our approach instead is normative in that it studies the efficiency of the system of different options, considering total net benefits of technologies over time, including innovation-related and scale-related effects of diversity.

The positive role of diversity is recognized in option value and real option theories, which clarify when to keep different options open in the face of irreversible change and uncertain circumstances (Arrow and Fisher, 1974; Dixit and Pindyck, 1994). However, these theories treat diversity as exogenous and do not consider innovation, whereas our model treats diversity as endogenous and contributing to the value of the overall system beyond merely keeping decisions open.

The relevance of our analysis relates to myopia of economic agents, both in private (management) and public (politicians and public servants) sectors. In real world decision making short term interests often prevail, possibly since the advantages of increasing returns are perceived as more clear and certain than the advantages of diversity and recombinant innovation. Fleming (2001) argues that one reason for uncertainty in recombinant innovation is that inventors experiment with unfamiliar technologies and unexploited combination of technologies. The trade-off between short term efficiency and long term benefits from diversity resembles the exploitation versus exploration problem (March, 1991; Rivkin and Siggelkow, 2003). In fact, recombinant innovation can be regarded as

[1] What economics calls spillover corresponds to recombination or cross-over in genetics and evolutionary computation and to modular innovation in biology and technological innovation studies.

a form of exploration and search.

A model of diversity connects not only with the research on modularity but also with the approach of evolutionary economics as expressed by Nelson and Winter (1982), Dosi et al. (1988), Frenken (1999) and Potts (2000) among others. However, evolutionary economics tends to avoid the notions of optimality and efficiency in terms of maximizing a net present value function. Our approach in fact can be seen as combining diversity-innovation ideas from evolutionary economics with optimality and cost-benefit analysis of neoclassical economics. Adopting the view of an evolutionary approach we will talk of a population of parent options and an offspring to refer to the innovative option. Here we will deal with the smallest population possible, namely only two parent options, so as to keep the model simple and allow for analytical solutions.

Following Stirling (2007), we will consider three dimensions of diversity, namely variety, balance and disparity. Variety refers to the number of starting options, the elements in the parent population. Balance denotes the relative size or distribution of parent options. And disparity indicates the degree of difference between the options, representing a sort of distance in technological, organizational or institutional space.

A motivation for the proposed model is the recent attention for a socio-technological transition to a renewable energy system (Geels, 2002; van den Bergh and Bruinsma, 2008). Diversity can here be related to lock-in of an inferior or undesirable technology, such as fossil-fuel based electricity generation that contributes considerably to global warming. A diversity analysis of energy systems can provide insight into the appropriate level of diversity that should be aimed for or maintained in different phases of an energy transition (van den Heuvel and van den Bergh, 2008).

This paper is organized as follows. Section 2 presents a simple pilot model to illustrate the main concepts and their interactions. Section 3 develops a generalization of this model that includes a more elaborate structure of diversity. In section 4 we solve the model and obtain a general solution for the value of the innovative option as a function of time. Section 5 introduces a size effect into the probability of recombinant innovation. In section 6 we address the optimization problem for the different versions of the model, and present conditions under which diversity or specialization is optimal. We also study the effect of the time horizon on the optimal solution. Section 7 concludes and provides suggestions for further research.

2 A pilot model

Consider a system of two investment options that can be combined to give rise to a third one. Let I denote cumulative investment in the parent options. Investment I_3 in the new option only occurs if it emerges, which happens with probability P_E. The growth rates of parent options are proportional to investments, with shares α and $1-\alpha$. We assume no depreciation and a constant allocation of investment through time. Let O_1 and O_2 represent the values of the cumulative investment in parent options and O_3 the (expected) cumulative investment in the innovative option. The dynamics of the system can then be described by the set of differential equations:

$$\begin{aligned}\dot{O}_1 &= I_1 = \alpha I \\ \dot{O}_2 &= I_2 = (1-\alpha)I \\ \dot{O}_3 &= P_E(O_1, O_2) I_3\end{aligned} \tag{1}$$

The optimization problem that we address is how to set an α that maximizes the final total benefits of parent and innovative options.

The matching factor P_E denotes the probability of emergence of the third option through recombinant innovation. Since this is a random event, such a rate of growth is the expected value of the investment into the new option. Recombinant innovation is a binary event where the new option emerges with probability P_E and nothing happens with probability $1-P_E$. Then the expected value is simply P_E times the capital invested in the new option I_3.

The probability of emergence depends on two factors, namely the diversity of the parent options and a scaling factor π which can be interpreted as the effectiveness of the R&D process underlying the recombinant innovation:

$$P_E(O_1, O_2) = \pi B(O_1, O_2) \tag{2}$$

R&D effectiveness π can be seen to depend on learning and progress in general. Diversity is expressed as the balance B of parent options: the more equal the sizes of parent options (cumulative investment) are, the larger is the probability of emergence.[2] When one option is zero we have pure specialization. The balance function must have the following properties: $B(O_1, O_2) \in [0, 1]$, $B(O_1 = O_2) = 1$ (maximum diversity or perfect balance) and $\lim_{O_i \to 0} B(O_i, O_j)|_{O_j=const} = 0$ with $i, j = 1, 2$ and $i \neq j$. We consider the following functional specification for the balance (figure 1):

$$B(O_1, O_2) = 4\frac{O_1 O_2}{(O_1 + O_2)^2}$$

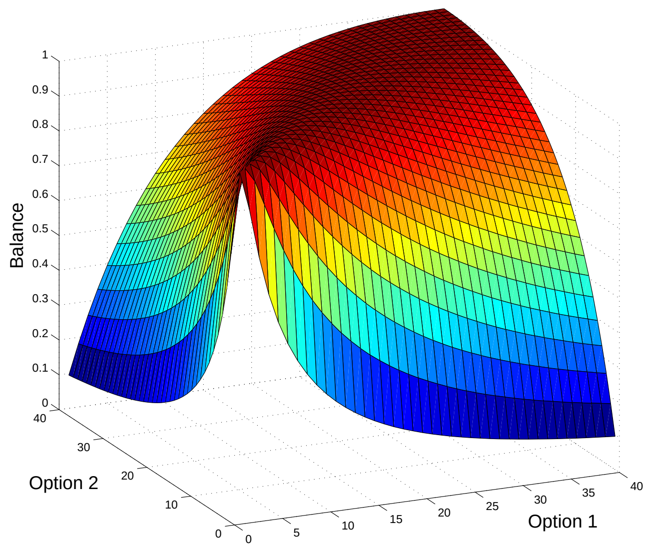

Figure 1: Graph of the diversity function with two parent options.

[2]This idea is consistent with both tacit and codified knowledge. In the case of tacit knowledge more balance can be seen as more opportunities for cooperation and exchange of information among engineers. With codified knowledge, balance can involve a single engineer combining different types of codified technological information.

Assuming that investment in parent options begins at time $t = 0$, their value at time t is simply $O_1(t) = \alpha It$ and $O_2(t) = (1-\alpha)It$. Under this assumption the balance function is independent of time: $B = 4\alpha(1-\alpha)$. Consequently the probability of emergence is constant and only depends on the initial allocation α. The innovative option grows linearly with time then:

$$O_3(t) = 4\pi I_3 \alpha(1-\alpha)t \tag{3}$$

The optimization problem of this investment decision is addressed considering the joint benefits of parents and innovative options. In order to model the trade-off between diversity and scale advantages of specialization we introduce a returns to scale parameter s. This acts on the cumulative investment in each option, in order to capture not only economies of scale but also learning over time. We can then express the overall benefits from investment as:

$$V(\alpha;T) = O_1(T;\alpha)^s + O_2(T;\alpha)^s + O_3(T;\alpha)^s \tag{4}$$

Where $t = T$ is the time horizon. According to this expression, once we substitute the expressions of options' values, the maximization problem of the investment decision can be written as

$$\max_{\alpha\in[0,1]} V(\alpha;T) = T^s I^s \big[\alpha^s + (1-\alpha)^s + C^s\alpha^s(1-\alpha)^s\big] \tag{5}$$

where $C = \frac{4\pi I_3}{I}$. This factor weights the contribution of diversity to total benefits. Such a contribution will be larger for a larger probability of recombination π. It is useful to normalize the benefits function to its value in case of specialization $V(\alpha = 0;T) = V(\alpha = 1;T) = I^sT^s$:

$$\tilde{V}(\alpha) \equiv \frac{V(\alpha;T)}{I^sT^s} = \alpha^s + (1-\alpha)^s + C^s\alpha^s(1-\alpha)^s \tag{6}$$

Depending on returns to scale s and the factor C, $\tilde{V}$ will be maximum for $\alpha = 1/2$ (maximum diversity) or for either $\alpha = 0$ or $\alpha = 1$ (specialization). It is instructive to look at some examples of the curve $\tilde{V}(\alpha)$ for different values of returns to scale s and efficiency π. Setting $I = 4I_3$ we have $C = \pi$. Figure 2 reports the normalized benefits curves in a case of increasing returns to scale ($s = 1.2$) for six different values of the factor π. Here either specialization or diversity is the best choice, depending on the efficiency of the recombinant innovation process as captured by the probability factor π. There is a threshold value $\overline{\pi}$ for this probability such that for $\pi < \overline{\pi}$ the optimal decision is specialization, while for $\pi > \overline{\pi}$ diversity is optimal. Conversely, given an intensity of recombinant innovation π one can derive the turning point $\overline{s}$ of returns to scale at which maximal diversity ($\alpha = 1/2$) becomes optimal. This is given by the threshold level $\overline{s}$ that solves the equation

$$\tilde{V}\big(\alpha = 1/2\big) = \frac{1}{2^{\overline{s}}}\left[2 + \left(\frac{C}{2}\right)^{\overline{s}}\right] = 1 \tag{7}$$

If $C = 0$ (for instance with $\pi = 0$) we have $\overline{s} = 0$. If $C = 1$ (for instance with $I = 4I_3$ and $\pi = 1$) we find $\overline{s} \simeq 1.2715$. There is no closed form solution $\overline{s}$ as function of other parameters, but we can instead solve for C. For $s > 1$ this solution is

$$\overline{C} = 2(2^s - 2)^{1/s} \tag{8}$$

Since $C = 4\pi I_3/I$, equation (8) links the ratio of investments invested and the probability of recombination to the level of returns to scale: any value higher than $\overline{C}$ causes

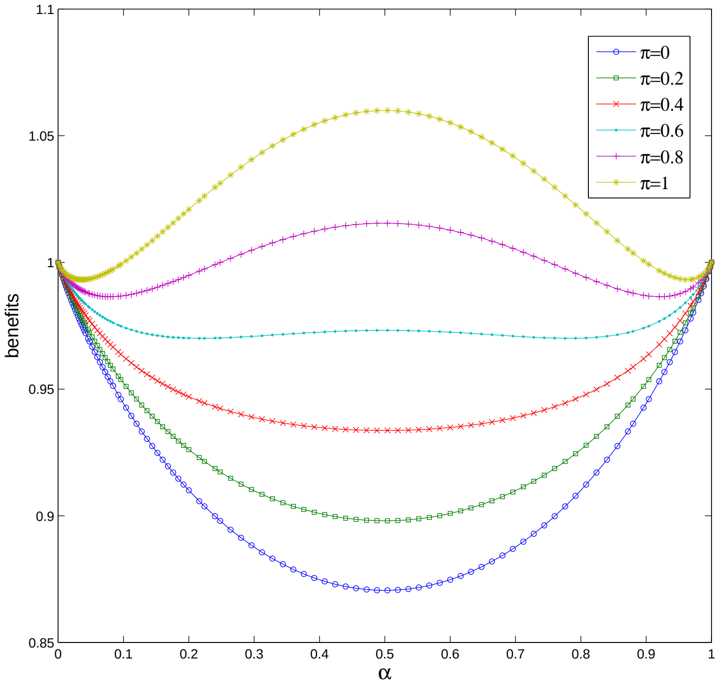

Figure 2: Normalized final benefits $\tilde{V}$ as a function of the investment share α under increasing returns to scale ($s = 1.2$) for different values of the innovation efficiency factor $\pi = 0, 0.2, 0.4, 0.6, 0.8, 1$. Here $I = 4I_3$.

diversity to be the optimal solution. Furthermore, since $\overline{C}(s)$ is increasing, concave and converging to 4, there is a sort of saturation effect: as returns to scale get larger, less and less investment is needed in the new technology to make diversified investment the best choice.[3] In the limit of infinite returns to scale, the threshold value of I_3/I approaches $1/\pi$. This leads to:

Proposition 1. *For a given positive value of the recombination probability π, if $I_3/I > 1/\pi$ benefits from diversity are larger than benefits from specialization for any value of returns to scale s.*

The reason is that the rate of growth of innovation is unbounded: with infinite investment I_3, the maximally diversified innovation system can always be rendered the optimal choice of the allocation problem, no matter how small the recombination probability $\pi > 0$ is and no matter how large the returns to scale parameter s is.

Assume the ratio of investments I_3/I is given. For $s = 1$ (constant returns to scale) we have $\tilde{V}(1/2)_{s=1} = 1 + C/4 \geq 1$, since $C \geq 0$. If a positive level of investment I_3 is devoted to the innovative technology, the following statement holds true:

Proposition 2. *The threshold $\overline{s}$ below which a diversified system is the optimal choice has the property that $\overline{s} \geq 1$ and $\overline{s} > 1$ iff $\pi > 0$.*

[3] We have $\frac{d}{ds}2(2^s-2)^{1/s} = (2^s-2)^{1/s}\left[\frac{2^s \ln 2}{s(2^s-2)} - \frac{\ln(2^s-2)}{s^2}\right]$. The first term is $\frac{2^s \ln 2}{2^s-2} \geq \frac{(2^s-2)\ln 2}{2^s-2} = \ln 2$, while $\frac{\ln(2^s-2)}{s}$ is increasing and converges to $\ln 2$ from below. This means that $\frac{d}{ds}\overline{C}(s) \geq 0\ \forall s > 1$.

Corollary 1. *For all decreasing or constant returns a maximum value of total final benefits is realized for the allocation $\alpha = 1/2$, i.e. for maximum diversity.*

This result holds true no matter what value the factor C assumes.[4] In other words, in all cases of decreasing returns to scale up to constant returns it is better to split equally the investment among the two parent options. Notice that diversity is optimal also in absence of recombinant innovation, when returns to scale are low enough. This situation is summarized in figure 3. The case of increasing returns to scale is the most

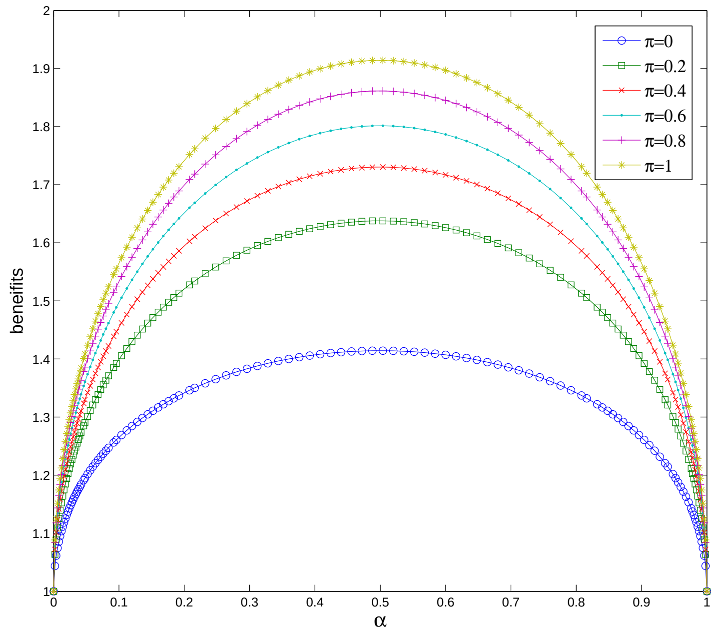

Figure 3: Normalized final benefits $\tilde{V}$ as a function of the investment share α under decreasing returns to scale ($s = 0.5$) for different values of the innovation efficiency factor $\pi = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$. Here $I = 4I_3$.

interesting and also the one that better represents real cases of technological innovation, among others, because of fixed costs and learning. In this regime we study the tradeoff between scale advantages and benefits from diversity. If the probability of recombinant innovation is insufficiently large, returns to scale may be too high for diversity to be the optimal choice. In figure 2 this holds for the bottom four curves. In general we have the following result, which completes Proposition 1:

Corollary 2. Diversity $\alpha = 1/2$ can be optimal also with increasing returns to scale ($\overline{s} > 1$) provided that the probability of recombination π is large enough.

[4]Consider the function $f(s) \equiv \big(2 + (C/2)^s\big)/2^s$. The statement is true if $f(s) \geq 1\ \forall s \in [0, 1]$. Since $f'(s) < 0\ \forall s \geq 0$, $f(s)$ is a decreasing function for fixed C. For fixed s instead f is an increasing function of C. When $C = 0$ $f(1) = 1$ and $f(s) \geq 1\ \forall s \in [0, 1]$. When $C > 0$ $f(1)|_{C>0} > f(1)|_{C=0} = 1$ and $f(s)|_{C>0} > f(s)|_{C=0} = 1\ \forall s \in [0, 1]$. This proves proposition 1.

3 A general model

Now we present a more general model of recombinant innovation which will relax some of the assumptions of the pilot model and at the same time we will enter the structure of the probability of emergence. We allow for non-zero initial values of parent options and consider a marginally diminishing effect of options' size on P_E. The optimization of diversity is addressed for the more general model then, with successive steps of increasing complexity.

3.1 Innovation probability and diversity factors

We define the probability of emergence of an innovative option P_E as depending positively on the diversity Δ of parent options and negatively on the disparity D:

$$P_E(O_1, O_2) = k\frac{\Delta(O_1, O_2)}{D^\gamma} \tag{9}$$

The factor k can be interpreted as the effectiveness of recombinant innovation. The parameter γ allows for a non-linear effect of disparity D. It can be seen to express the concept of "cognitive distance" between two technologies: it may be that two ideas are very different but historical or geographical events make the cognitive distance small, for instance through interdisciplinary research.

As observed by Stirling (2007) diversity is a multidimensional concept. In a study of innovation he indicates three dimensions: variety, disparity and balance. Diversity can be expressed as follows:

$$\Delta(O_1, O_2) = \delta NDB(O_1, O_2) \tag{10}$$

Variety N and disparity D are set exogenously, while balance B is a function of the values of the existing options. The factor δ is a scaling parameter that can be set to normalize maximum diversity to one. Variety indicates the number of parent options present (technologies, organizations, investment projects, firms, etc.). Disparity captures how "different" or how far apart in technology space the two options are. In principle D can assume any positive value since it expresses a degree of differentiation among two alternatives (sort of distance between different species, as in Weitzman, 1992). Balance expresses how (un)equally different options are present in a population, assuming that the more balanced a system is, the more diversified. The mathematical expression of P_E shows that disparity has two opposite effects on the probability of recombinant innovation, a direct and an indirect one. The overall effect will depend on the parameter γ. We regard the value of γ as an empirical issue. It is likely that γ differs between technologies and innovation processes.

3.2 The balance function

A balance function is defined in the positive octant of a n-dimensional space. A functional specification of the balance of two options x and y should have the following properties:

1. it is symmetric in its arguments $B(x, y) = B(y, x)$
2. the maximum value (normalized to one) is attained on the diagonal $B(x, x) \geq B(x, y)\ \forall x, y \geq 0$

3. the minimum value (lowest balance) is attained when one of the two options is zero: $B(x,0) = B(0,x) = 0 < B(x,y) \ \forall y > 0$

4. it is homogeneous of degree zero: $B(\lambda x, \lambda y) = B(x,y)$

The latter means that the balance of two quantities can be expressed as a function of their ratio $b = O_1/O_2$ (simply put $\lambda = 1/x$). The functional specification of the balance that we adopt is the so-called "Gini" balance:[5]

$$B(O_1, O_2) = 1 - \frac{(O_1 - O_2)^2}{(O_1 + O_2)^2} = 4\frac{O_1 O_2}{(O_1 + O_2)^2} \tag{11}$$

The main reason for such a choice is the differentiability in $O_1 = O_2$. Expressed as a function of the ratio the above specification reads $B(b) = 4\frac{b}{(1+b)^2}$.

3.3 The innovation effectiveness factor

Equation (9) contains a scaling factor, k, which must be such that $P_E \leq 1$. Diversity Δ assumes values in a compact interval $[0, \Delta_{max}]$, depending on variety N, disparity D and balance B. Variety is set to $N = 2$ (two parent options). Disparity, we restricted to two discrete values, $D = 1$ (identical options) and $D = 2$ (maximum disparity). Balance takes values in the interval $[0, 1]$. Looking at equation (10), the maximum value Δ_{max} is attained for $N = 2$ and $D = 2$. Setting $\delta = \frac{1}{4}$ we have $\Delta_{max} = 1$. Resuming, we have the following cases:

$$\begin{aligned} \Delta_{min} &= 0 \quad \text{for } N = 1,\ B = 0,\ D = 1 \\ \Delta_{max} &= 1 \quad \text{for } N = 2,\ B = 1,\ D = 2 \end{aligned}$$

If we substitute equation (10) into equation (9), the probability of emergence is given by $P_E = \pi B(O_1, O_2)$, where we define

$$\pi = \frac{k}{4} N D^{1-\gamma}$$

This is a static probability factor which tells about the nature of interacting technologies (their number is held fixed to $N = 2$ here): with a cognitive distance $\gamma > 1$ the closer technologies are to each other (lower disparity D) the more likely recombination occurs. Normalization is achieved by requiring that $\pi \leq 1$, which translates in the following condition for the efficiency factor k

$$k \leq \frac{4D^{\gamma-1}}{N} \tag{12}$$

The factor k captures all effects that influence the recombinant innovation process other than N, D and B. For instance, two recombinant innovation processes with the same number of parent options, the same disparity and the same balance may render different values of the innovation likelihood P_E due to different recombination efficiency k, possibly reflecting different levels of knowledge (education) or experience.

[5] Other specifications are possible, for instance $B(O_1, O_2) = 1 - \frac{|O_1 - O_2|}{O_1 + O_2}$ and $B(O_1, O_2) = \frac{min\{O_1, O_2\}}{max\{O_1, O_2\}}$ (see also Stirling, 2007). A detailed analysis of the latter specification is available upon request. The case $O_1 = O_2 = 0$ is excluded by all these specifications. This is a rather degenerate and irrelevant case, however, as we are only interested in systems with at least one option ($\exists\ i = 1, 2 \mid O_i > 0$). Otherwise we can always define $B(0,0) = lim_{O_1, O_2 \to 0} B(O_1, O_2) = 1$.

4 Solving the dynamic model

Our model of recombinant innovation consists of the system of equations (1) and the definitions (9) and (10). Here we relax the hypothesis of zero initial values of parent options. This introduces more complicated dynamics into the system. In this section we solve this dynamic model. The solutions will be used in section 6 to address the optimization of diversity.

Assuming for parent options a constant allocation of capital I over time $\frac{I_1}{I_2} = \frac{\alpha}{1-\alpha}$ results in a constant linear growth (accumulation) of parent options O_1 and O_2. The time pattern of the innovative option is non-linear:

$$\begin{aligned} O_1(t) &= O_{10} + I_1 t \\ O_2(t) &= O_{20} + I_2 t \\ O_3(t) &= I_3 \int_0^t P_E(s) ds \end{aligned} \tag{13}$$

The first two equations of system (13) are independent. The third equation depends on the first and the second through the probability of emergence $P_E(t) = \pi B\big(O_1(t), O_2(t)\big)$. The value of the innovative option at time t is then

$$O_3(t) = 4I_3 \int_0^t P_E(s) ds = \pi I_3 \int_0^t \frac{O_1(s)O_2(s)}{\big(O_1(s) + O_2(s)\big)^2} ds \tag{14}$$

Before computing the integral (14) we will analyse the dynamic behaviour of the balance function. If the initial value of parent options is zero ($O_{01} = O_{02} = 0$) the balance is constant and equal to $4\alpha(1-\alpha)$; this is the case of the pilot model, where also the innovative option grows linearly in time.

If we allow for positive initial values O_{10}, O_{20} we obtain the following function of time

$$B = 4\frac{(O_{10} + \alpha I t)(O_{20} + (1-\alpha) I t)}{(O_{10} + O_{20} + It)^2} \longrightarrow 4\alpha(1-\alpha) \tag{15}$$

where the last limit holds for $t >> O_{i0}/(\alpha I)$, $i = 1, 2$. In the long run the balance converges to a constant value, which depends only on the investment shares and is the same that results for zero initial values. We can state the following proposition then:

Proposition 3. *In the long run the balance converges to the constant value* $B(\alpha) = 4\alpha(1-\alpha)$, *which is independent of initial values of parent options.*

The dynamics of the balance in the transitory phase ($t \sim O_{i0}/(\alpha I)$) depends on initial conditions and on the investment share α and can be understood easily by looking at options trajectories in (O_1, O_2) space. Starting from the expression of the two options' ratio $\frac{O_1}{O_2} = \frac{O_{10}+\alpha It}{O_{20}+(1-\alpha)It}$ one can eliminate time and express one option in terms of the other:

$$O_2 = O_{20} - \frac{1-\alpha}{\alpha} O_{10} + \frac{1-\alpha}{\alpha} O_1$$

The starting point ($t = 0$) of each trajectory is determined by the initial values (O_{10}, O_{20}). The slope is the ratio of investment shares. For our recombinant innovation system we identified seven major cases, which are reported in figure 4 (for a detailed analysis of each of these cases see van den Bergh and Zeppini-Rossi, 2008). In principle the optimal condition for recombinant innovation is when the balance is constant and maximal (case 7). In general for constant balance the following condition applies:

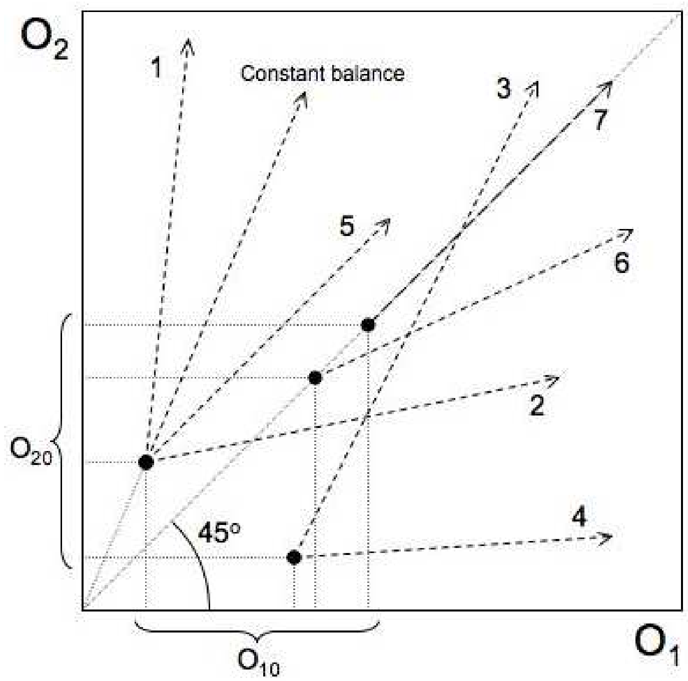

Figure 4: Trajectories of the two parent options in (O_1, O_2) space. Trajectory "1" has $O_{10} < O_{20}$ and $\alpha < 1/2$, trajectory "2" has $O_{10} < O_{20}$ and $\alpha > 1/2$, trajectory "3" has $O_{10} > O_{20}$ and $\alpha < 1/2$, trajectory "4" has $O_{10} > O_{20}$ and $\alpha > 1/2$, trajectory "5" has $O_{10} \neq O_{20}$ and $\alpha = 1/2$, trajectory "6" has $O_{10} = O_{20}$ and $\alpha < 1/2$ and trajectory "7" has $O_{10} = O_{20}$ and $\alpha = 1/2$. The trajectory of constant balance has a slope equal to the ratio of the coordinates of the starting point.

Proposition 4. *The balance is constant through time and equal to* $B(\alpha) = 4\alpha(1-\alpha)$ *iff*

$$\frac{O_{10}}{O_{20}} = \frac{\alpha}{1-\alpha} \tag{16}$$

For a proof of this proposition see appendix A. This configuration falls into cases 1, 4 and 7 of figure 4. As a function of time the balance may have a critical point t^* where it reaches its maximum value.[6] Figure 5 shows two examples of monotonic and non-monotonic dynamics. Here we have set $I = 4$, with initial values $O_{10} = 1$ and $O_{20} = 2$. In example 2 we have $\alpha/(1-\alpha) = 3$: there is a time $t^* = 1/2$ when the balance is equal to one (a perfectly similar pattern one would obtain in case 3). In example 1 the balance is monotonically decreasing, with $\alpha/(1-\alpha) = 1/4$. In general $B(t)$ is decreasing when $\frac{\alpha}{1-\alpha} < \frac{O_{10}}{O_{20}} < 1$ and increasing when $\frac{\alpha}{1-\alpha} > \frac{O_{10}}{O_{20}} > 1$, while a non-monotonic behaviour is obtained for $\frac{\alpha}{1-\alpha} < 1 < \frac{O_{10}}{O_{20}}$ or $\frac{O_{10}}{O_{20}} < 1 < \frac{\alpha}{1-\alpha}$

Now we proceed to the integration of balance, giving the value of the innovative option at time t. We assume that $k = 4D^{\gamma-1}/N$, so that $\pi = 1$ (maximal efficiency of recombinant innovation). Equation (14) becomes

$$O_3(t) = 4I_3 \int_0^t \frac{(O_{10} + \alpha Is)(O_{20} + (1-\alpha)Is)}{(O_0 + Is)^2} ds \tag{17}$$

[6] The critical time value is $t^* = (O_{20} - O_{10})/(2\alpha - 1)I$.

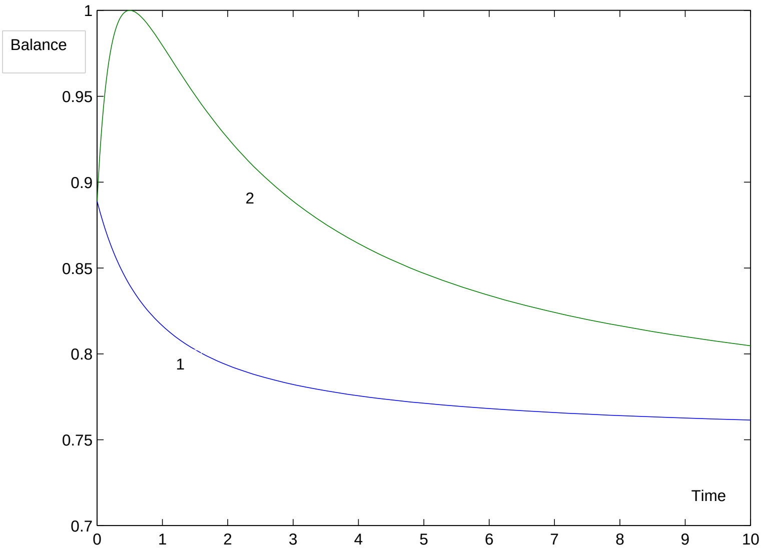

Figure 5: Two cases for the balance as a function of time ($I = 4$, $O_{10} = 1$ and $O_{20} = 2$). Case 1 has $\alpha = 1/4$. Case 2 has $\alpha = 3/4$.

The detailed solution of this integral is in appendix B. The final result is the following:

$$\begin{aligned} O_3(t) = 4\frac{I_3}{I}\Bigg[& \big(O_{10} - \alpha O_0\big)^2 \left(\frac{1}{O_0 + It} - \frac{1}{O_0}\right) + \\ + & \big(O_{10} - \alpha O_0\big)(1 - 2\alpha) \ln \frac{O_0 + It}{O_0} + \alpha(1 - \alpha)It \Bigg] \end{aligned} \tag{18}$$

If condition (16) holds, $O_{10} = \alpha O_0$ and the expression of the innovative option reduces to $O_3(t) = 4I_3\alpha(1-\alpha)t$ as in the pilot model. This linear expression of $O_3(t)$ is also valid in the early stages of innovation, namely when $It << O_0$. In the long run instead the logarithmic term can not be neglected and the value of innovation is approximately given by

$$O_3(t) \simeq 4\frac{I_3}{I}\left[\big(O_{10} - \alpha O_0\big)(1 - 2\alpha) \ln \frac{It}{O_0} + \alpha(1 - \alpha)It\right] \tag{19}$$

The coefficient of the logarithmic term will determine whether the time pattern of the innovative option will be concave (positive sign) or convex (when the sign is negative). The first case arises when $\alpha < 1/2$ and $\alpha < O_{10}/O_0$ or $\alpha > 1/2$ and $\alpha > O_{10}/O_0$. These are exactly the conditions of cases 3 ($\alpha < 1/2$ and $O_{10} > O_{20}$) and 2 ($\alpha > 1/2$ and $O_{10} < O_{20}$) in the previous list, when the balance has a critical point t^*. The convex time pattern occurs when balance does not have a critical point instead. For example take $O_0 = 3$, $O_{10} = 1$, $O_{20} = 2$, $\alpha = 2/3$. Since $O_{10}/O_{20} = 1/2 < \alpha/(1-\alpha) = 2$ we have that option 3 follows a concave time pattern, $O_3(t) = \frac{4}{3}\left[2t + \ln(1+t) - \frac{t}{1+t}\right]$.

5 A size effect

5.1 Specifying the size effect

Up to now, the probability of emergence of a third option was basically an index of diversity of two starting options and the dynamics of the system was driven by their balance. We now introduce a size effect into the probability of emergence. This is meant to capture the positive effect that a larger cumulative size has on the probability of emergence, i.e. a kind of economies of scale effect in the innovation process. If the size effect is captured by a factor $S(O_1, O_2)$, the probability of emergence of the third option can be expressed as:

$$P_E = \pi B(O_1, O_2) S(O_1, O_2) \tag{20}$$

The size effect is defined to have the following properties. First it is increasing in the size of each parent option with marginally diminishing effects. Second it must be bounded, to guarantee that the probability P_E is in the interval $[0, 1]$. In addition, it should not overlap with the balance factor, which means that only the total sum of the sizes of options matters and not their distribution. These properties can be understood as capturing increased learning subject ultimately to saturation. One attractive functional specification is the following:

$$S(O_1, O_2) = 1 - e^{-\sigma(O_1+O_2)} \tag{21}$$

Here $\partial S/\partial O_i = \partial S/\partial O = \sigma/e^{\sigma O}$, with $O = \sum_i O_i$. The parameter σ captures the sensitivity of P_E to the size when the balance is kept constant: the higher σ, the stronger the size effect.[7] After including the size factor, the probability of emergence as a function of time looks

$$P_E(t) = \pi B(t)\big(1 - e^{-\sigma(O_0+It)}\big) \tag{22}$$

Note how the effect of size on the probability of emergence does not depend on whether it comes from "old" value O_0 or from "new" investment It. This is not true for the balance.[8]

The size effect converges to one ($lim_{t\to\infty} S(t) = 1$): after a time long enough ($It >> O_0$) the effect of cumulative size on P_E vanishes.

5.2 Time pattern of P_E with constant balance

In order to understand the impact of the size of parent options on the innovation process we look at the behaviour of the probability of emergence through time for few different values of the balance in the particular setting in which the balance is constant (condition (16)). Assume that the efficiency of recombination is maximal ($\pi = 1$), so that $P_E(t; \alpha) = B(\alpha)S(t)$, with $B(\alpha) = \alpha(1 - \alpha)$. Considering the previous analysis of the balance and the specification of the size effect, in the long run we have

$$lim_{t\to\infty} P_E(t) = B(\alpha) \tag{23}$$

In the case $\alpha = 1/2$ we have the maximal balance $B(\alpha) = 1$. This means that the third option can arise with certainty only in an infinite time. The size factor $S(t)$ describes

[7]Alternatively, one could allow for heterogeneous effects with the specification $1 - e^{-\sigma_1 O_1 - \sigma_2 O_2}$. For example, this can address two different technologies operating in different sectors with different sensitivities σ_1 and σ_2.

[8]Formally, $S(t)$ is invariant to a time shift $t \to t^*$ such that $O_0 + It = O_0^* + It^*$, while $B(t)$ is not.

a saturation effect of the probability of emergence P_E. We might think of the event of innovation as occurring suddenly at a time t_E. Then we can write $P_E(t) = Prob(t_E < t)$. In cases other than the symmetric one the balance is suboptimal ($B < 1$) and $P_E(t) < 1$ $\forall t$. This can be summarized in the following proposition:

Proposition 5. *When a marginal diminishing size effect is introduced in the probability of emergence, innovation occurs almost surely iff the balance is constant and equal to its maximum value ($B = 1$).*

Table 1 helps to get an idea of how the balance and the size factor jointly determine the probability of emergence. Here the balance is constant and the dynamics is due only to the size effect. We set $\sigma = 1/O_0$ and consider the investment shares $\alpha = 1/2$, $\alpha = 1/3$, $\alpha = 1/4$ and $\alpha = 1/8$:

P_E		$It >> O_0$ ($S = 1$)	$It = 3O_0$ ($S \cong 0.98$)	$It = 2O_0$ ($S \cong 0.95$)	$It = O_0$ ($S \cong 0.87$)	$It = 0$ ($S \cong 0.63$)
$\alpha = 1/2$	$B = 1$	100%	98%	95%	87%	63%
$\alpha = 1/3$	$B = 8/9$	89%	87%	84%	77%	56%
$\alpha = 1/4$	$B = 3/4$	75%	74%	71%	65%	47%
$\alpha = 1/8$	$B = 7/16$	44%	43%	42%	38%	28%

Table 1: Probability of emergence for different values of balance B and size factor S.

In the long run the size factor is nearly one and the probability of emergence eventually reflects the balance of the two options.

5.3 Solving the dynamic model with the size factor

We now integrate the third equation of the model (1) with a full specification of the probability of emergence, taking into account the balance and the size effect together. Beforehand it is useful to write down the general expression of the probability of emergence as a function of time (again we assume $\pi = 1$):

$$P_E(t) = 4\frac{(O_{10} + \alpha It)(O_{20} + (1-\alpha)It)}{(O_{10} + O_{20} + It)^2}\left(1 - e^{-\sigma(O_0+It)}\right) \tag{24}$$

We will proceed in steps in order to better understand the effect of size in the model. First assume that the investment shares are set in a way that their ratio equals the ratio of the initial values of the parent options (condition (16)). In this case we obtain a constant balance $B = 4\alpha(1-\alpha)$ and the rate of growth P_E of the third option becomes

$$P_E(t) = B\left(1 - e^{-\sigma(O_0+It)}\right) \tag{25}$$

With this specification of the dynamics we obtain the following time pattern for option the innovative option:

$$O_3(t) = I_3 B\left[t + \frac{e^{-\sigma O_0}}{\sigma I}\left(e^{-\sigma It} - 1\right)\right] \tag{26}$$

The first term of this expression is what we have without size factor. The second term comes from the size effect. Here $\dot{O}_3(t) > 0$ and $\ddot{O}_3(t) > 0$ $\forall t \geq 0$.[9] This means the innovative option has a convex time pattern. Such a behaviour accounts for a transitory phase in which the innovation "warms up" before becoming effective. This is a stylized fact of innovation processes.

The time pattern of $O_3(t)$ tends to the asymptote $\pi I_3 B\big(t - e^{-\sigma O_0}/\sigma I\big)$: after a sufficiently long time the innovative option attains linear growth. An indication of the characteristic time interval of transitory phase is given by the intercept $\hat{t} = \frac{e^{-\sigma O_0}}{\sigma I}B$. Depending on the sensitivity parameter σ and depending on the total initial value of the parent options and their cumulative investment I, the transitory phase can last a very long time or may be very brief: the higher the sensitivity σ or the initial value O_0 or the investment rate I, the shorter the transitory phase and the faster the innovative option gets to linear growth. In figure 6 we plot an example of function $O_3(t)$: here we have set

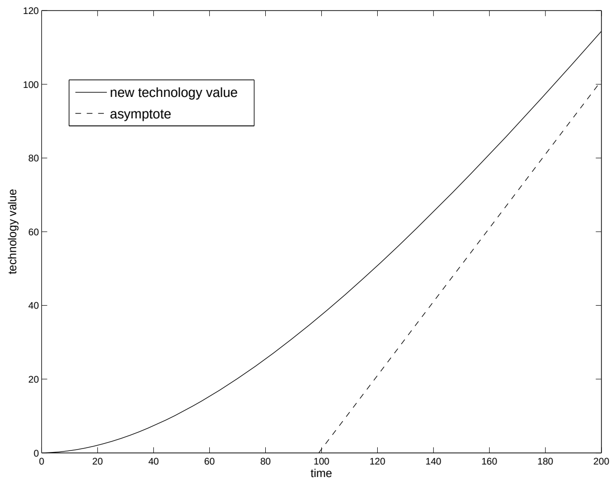

Figure 6: Value of the innovative option at time t, case of constant balance ($B = 1$, $\sigma = 1/400$, $I = 4$ and $O_{10} = O_{20} = 2$).

$\alpha = 1/2$, $\pi = 1$, $I_3 = 1$, $I = 4$, $\sigma = 1/400$, and $O_{10} = O_{20} = 2$. With these values we have $O_3(t) = t + 100e^{-0.01t}(e^{-0.01t} - 1)$ and the asymptote is $t - 100e^{-0.01}$.

Relaxing the assumption of constant balance, we have to solve the following integral:

$$O_3^\sigma(t) = 4I_3 \int_0^t \frac{(O_{10} + \alpha I s)(O_{20} + (1-\alpha)Is)}{(O_0 + Is)^2}\big(1 - e^{-\sigma(O_0 + Is)}\big)ds$$

We call this solution $O_3^\sigma(t)$ to differentiate it from the solution without size effect. Appendix B contains the detailed derivation. The result is

$$\begin{aligned} O_3^\sigma(t) &= BI_3 t + B\frac{e^{-\sigma O_0}}{\sigma I}\big(e^{-\sigma I t} - 1\big) + \frac{4I_3}{I}\sigma EG \ln\frac{O_0 + It}{O_0} + \\ &+ \frac{4I_3}{I}EG\Big[\frac{1}{O_0}\big(1 - e^{-\sigma O_0}\big) - \frac{1}{O_0 + It}\big(1 - e^{-\sigma(O_0 + It)}\big)\Big] + \\ &+ \frac{4I_3}{I}\Big[\sigma EG - (EH + FG)\Big]\Bigg[\sum_{k=1}^{\infty}\frac{\big(-\sigma(O_0 + It)\big)^k}{k \cdot k!} - \sum_{k=1}^{\infty}\frac{\big(-\sigma O_0\big)^k}{k \cdot k!}\Bigg] \end{aligned} \tag{27}$$

[9]The first derivative is $\dot{O}_3(t) = I_3 P_E(t)$ while the second derivative is $\ddot{O}_3(t) = I_3 \pi B \sigma I e^{-\sigma(O_0 + It)}$

Here $B = 4\alpha(1-\alpha)$ is the value of the balance when it does not depend on time. $E = O_{10}(1-\alpha) - \alpha O_{20}$, $F = \alpha$, $G = -E$ and $H = (1-\alpha)$. When the balance is constant we have $O_{10}(1-\alpha) = O_{20}\alpha$, and the expression of $O_3(t)$ only contains the first two terms since $E = G = 0$. When the balance is not constant the time pattern of the third option contains a logarithmic term, a negative exponential divided by a linear function and two infinite sums, one constant and the other dependent on time. As argued in appendix B, the two sums converge to negative exponentials. This means that the infinite sum which depends on time goes to zero for $It >> O_0$. In the long run the time pattern of O_3^σ is given by the following expression:

$$O_3^\sigma(t) \simeq 4\alpha(1-\alpha)I_3t - 4\frac{I_3}{I}\sigma[O_{10}(1-\alpha) - O_{20}\alpha]^2 \ln\frac{It}{O_0} \tag{28}$$

Without size effect we have (see equation (19))

$$O_3(t) \simeq 4\alpha(1-\alpha)I_3t + 4\frac{I_3}{I}[O_{10}(1-\alpha) - O_{20}\alpha](1-2\alpha)\ln\frac{It}{O_0}$$

When a size factor is present, the logarithmic term adds negatively to the value of the innovative option, producing the expected convex time pattern which tells about the diminishing marginal contribution of parent technologies. Without size effect the logarithmic term can be either positive or negative instead. This shows how a marginally diminishing size effect is important in reproducing the typical threshold effect of recombinant innovations. The contribution of the logarithmic term depends much on the value of the sensitivity σ, which should be assessed empirically for each context.

6 Optimization of diversity

Now we address the problem of optimal diversity in the general model. As in the pilot model, the objective function is the sum of final benefits from parent and innovative options, where each contribution is affected by a returns to scale parameter. The maximization problem is then as follows:

$$\max_{\alpha\in[0,1]} O_1(t;\alpha)^s + O_2(t;\alpha)^s + O_3(t;\alpha)^s \tag{29}$$

The solution will in general be a function of the time horizon, $\alpha^*(t)$. Before solving for α^* we study in some detail the first order conditions for the pilot model because many of its properties remain valid in more complex specifications. Moreover, the pilot model serves as a benchmark for the general dynamic case.

6.1 The shape of the benefits curve

Substituting the solutions $O_i(t)$ of the pilot model (see section 2), the maximization problem becomes

$$\max_{\alpha\in[0,1]} \tilde{V}(\alpha;T) = \alpha^s + (1-\alpha)^s + C^s\alpha^s(1-\alpha)^s \tag{30}$$

The first order necessary condition for maximization of final benefits is

$$\frac{\partial\tilde{V}}{\partial\alpha} = s\alpha^{s-1} - s(1-\alpha)^{s-1} + C^s s\big[\alpha(1-\alpha)\big]^{s-1}(1-2\alpha) = 0 \tag{31}$$

There may be one or three interior solutions to this equation. The symmetric solution $\alpha = 1/2$ always exists. Depending on the returns to scale parameter s two other solutions are present, $\alpha_1(s)$ and $\alpha_2(s)$. They are symmetric with respect to $\alpha = 1/2$ (the whole investment system is symmetric without initial values of parent options) so that $\alpha_1 + \alpha_2 = 1$ and if they exist they always give a minimum value of benefits, while $\alpha = 1/2$ may be either a minimum or a maximum. The transition from $\alpha = 1/2$ as a minimum to $\alpha = 1/2$ as a maximum depends on the appearance of these two roots. In general for a given value of the factor C there is a threshold level of returns to scale $\hat{s}$ at which $\alpha = 1/2$ is neither a maximum or a minimum. This threshold value is given by a tangency requirement

$$\left.\frac{\partial^2 \tilde{V}}{\partial \alpha^2}\right|_{\alpha=1/2} = 0$$

Computing the second derivative in $\alpha = 1/2$ and setting it to zero one works out the condition

$$\hat{s} = \left(\frac{C}{2}\right)^{\hat{s}} + 1 \tag{32}$$

This means that for a given probability of recombinant innovation (C given) the threshold value of returns to scale $\hat{s}$ is a fixed point of the function $f(s) = \left(\frac{C}{2}\right)^s + 1$. With $C = 1$ (for instance with $I = 4I_3$ and $\pi = 1$) we have $\hat{s} \simeq 1.3833$. Note that $\hat{s} > 1$ since $C \geq 0$. Then we have the following proposition:

Proposition 6. *A necessary condition for only 1 stationary point ($\alpha = 1/2$ a local and global minimum) is increasing returns to scale. With decreasing returns there are always 3 stationary points.*

Conversely, given a value s of returns to scale, one can compute the transition value in terms of the probability of recombination, $\hat{C} = 2(s-1)^{1/s}$. For $C > \hat{C}$ there are three stationary points.

The following figures show $\tilde{V}(\alpha)$ and its derivative[10] for two different values of returns to scale. In the first case ($s = 1.5$, figure 7) the only stationary point is $\alpha = 1/2$, a local and global minimum of final benefits. Global maxima are the corner solutions $\alpha = 0$ and $\alpha = 1$. In the second case ($s = 1.2$, figure 8) there are three stationary points: $\alpha = 1/2$ is now a local (and also global) maximum, while the two symmetric stationary points, α_1 and α_2, are local and global minima.

We can compare the transition value $\hat{s}$ with the value $\overline{s}$, i.e. the threshold between diversity and specialization as optimal solution for maximum final benefits (section 2):

Proposition 7. *In general $\hat{s} \geq \overline{s} \geq 1$ and $\hat{s} = \overline{s} = 1$ only for $\pi = 0$ (no recombinant innovation).*

This means that three different regions can be identified in the returns to scale domain, as shown in figure 9.

[10] In figures 7 and 8 we show $\tilde{V}'(\alpha)/s = \alpha^{s-1} - (1-\alpha)^{s-1} + \left[\alpha(1-\alpha)\right]^{s-1}(1-2\alpha)$, which has the same roots as $\tilde{V}'(\alpha)$.

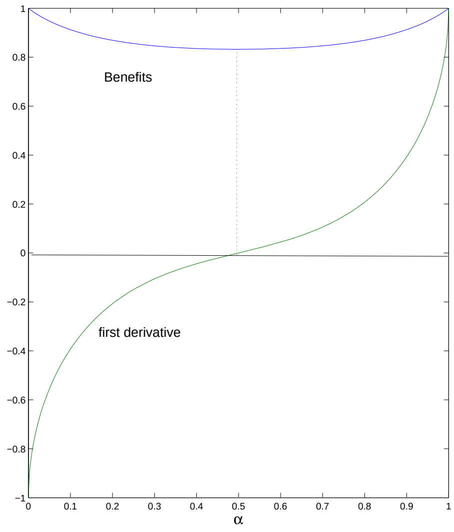

Figure 7: Normalized final benefits $\tilde{V}(\alpha)$ and its derivative. Case $s = 1.5$.

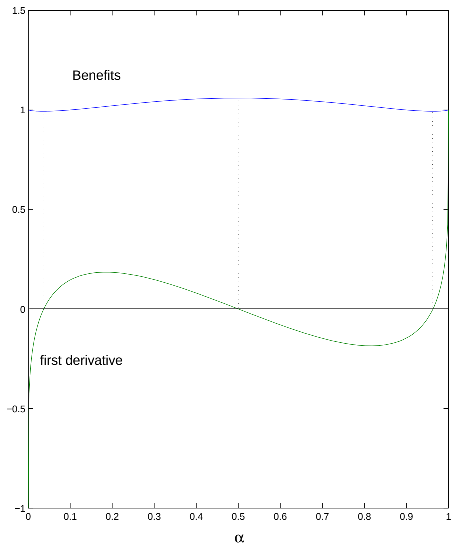

Figure 8: Normalized final benefits $\tilde{V}$ and its derivative. Case $s = 1.2$.

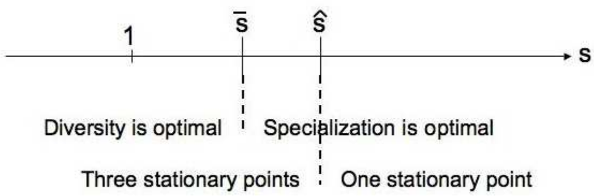

Figure 9: With a positive probability of recombinant innovation $\pi > 0$ we have $\hat{s} > \overline{s} > 1$.

6.2 Optimization with size effect and zero initial values

In this subsection we consider zero initial values for the parent options and a probability of emergence P_E containing both the balance and the size factors. Without initial values the balance is constant, but P_E depends on time because of the size effect. The expression of the innovative option is given by (26). Substituting this into the objective function of the maximization problem (29), we obtain

$$V(\alpha, t) = (\alpha I t)^s + \big((1-\alpha) I t\big)^s + \big[4\pi I_3 \alpha(1-\alpha)\big]^s \big[t + g(t)\big]^s \tag{33}$$

where $g(t) = (e^{-\sigma I t} - 1)/\sigma I$. If we normalize the objective function dividing it by $(It)^s$ (benefits from specialization) we have

$$\tilde{V}(\alpha, t) = \alpha^s + (1-\alpha)^s + C^s m(t)^s \alpha^s (1-\alpha)^s \tag{34}$$

where the constant factor is again $C = 4\pi I_3 / I$. Now a time dependent factor shows up, $m(t) = 1 + \frac{e^{-\sigma I t} - 1}{\sigma I t}$, with $m'(t) > 0$, $\lim_{t\to 0} m(t) = 0$ and $\lim_{t\to\infty} m(t) = 1$. The factor $m(t)$ monotonically modulates the contribution of innovative recombination to final benefits, being very small at early stages and converging to one as $\sigma I t >> 1$.

In the long run ($It >> O_0$) the model converges to the pilot model, where only C appears in the expression of final benefits. One can incorporate $m(t)$ into C defining a function $C(t) = Cm(t)$. Final benefits with size effect (34) are formally the same as in the pilot model (6): only difference is that constant C now depends on time. This consideration is maximally important for the optimization of diversity. Even if the size effect makes the investment system dynamic, still the optimal solution will be either $\alpha = 0, 1$ or $\alpha = 1/2$. The optimal diversity now is time dependent but it can be just one of these values. This is better understood by looking at figures 2 and 3. Given I, I_3 and π, as time flows the factor $C(t)$ increases and the benefits curve goes from the lower curve $\pi = 0$ (representing $C = 0$) to the upper curve $\pi = 1$ (which stands for $C = 1$).

The first order necessary condition for optimization of diversity in this dynamic setting is the following:

$$s\alpha^{s-1} - s(1-\alpha)^{s-1} + C(t)^s \big[\alpha(1-\alpha)\big]^{s-1}(1-2\alpha) = 0 \tag{35}$$

The analysis of section 6.1 can be repeated by substituting the constant factor C with the function $C(t)$. In particular the transition value $\hat{s}$ where $\alpha = 1/2$ becomes a (local) maximum of benefits is given by

$$\hat{s}(t) = \left(\frac{C(t)}{2}\right)^{\hat{s}(t)} + 1 \tag{36}$$

Now the transition value is a function of time. It may also be interesting to think in terms of transition time $\hat{t}$: for a given value of returns to scale s one computes the factor C that satisfies the equation above:

$$C(\hat{t}) = 2(s-1)^{1/s} \tag{37}$$

Similarly to the transition from one to three stationary points, also the threshold analysis for optimal diversity is formally the same as in the pilot model. We define the threshold value $\overline{s}(t)$ as the returns to scale level at which, for a given time horizon t, the benefits with $\alpha = 1/2$ are the same as the benefits from specialization ($\alpha = 0, 1$):

$$\tilde{V}\big(\alpha = 1/2\big) = \frac{1}{2^{\overline{s}(t)}}\left[2 + \left(\frac{C(t)}{2}\right)^{\overline{s}(t)}\right] = 1 \tag{38}$$

Proposition 8. *For a given time horizon t diversity ($\alpha = 1/2$) is optimal iff $s < \overline{s}(t)$.*

How does $\overline{s}(t)$ behave? The larger t, the larger $\overline{s}(t)$. The intuition behind this is as follows. $C(t)$ is increasing, which means that time works in favour of recombinant innovation. As time goes by, the region of returns to scale where diversity is optimal enlarges. The threshold $\overline{s}(t)$ converges to the value $\overline{s}$ of the pilot model (see figure 10). It is important to observe that even with $\pi = 1$ diversity may never become the optimal solution if returns to scale are too high ($\overline{s} < s$). But if investment I_3 is large enough, diversity will always become the optimal choice. This is consistent with proposition 1: given returns to scale s, if one has infinite disposal of investment I_3, threshold $\overline{s}$ can always be made such that $\overline{s} > s$, so that at some time t one will see $\overline{s}(t) > s$.

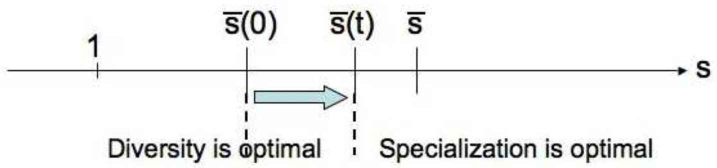

Figure 10: As time goes by, the region of returns to scale where diversity is optimal becomes larger.

Alternatively one can define a threshold time horizon $\overline{t}$ such that for $t < \overline{t}$ specialization is optimal, while for $t \geq \overline{t}$ diversity is the best choice.

$$C(\overline{t}) = 2\big(2^s - 2\big)^{1/s} \tag{39}$$

We want to understand how such a threshold time depends on returns to scale. The function $C(t)$ is monotonically increasing: the inverse $C^{-1}(\cdot)$ can be defined (increasing as well) and a unique solution $\overline{t}$ exists. The right hand side of (39) is increasing[11] in s. We then have the following result:

Proposition 9. *For higher returns to scale s the threshold time horizon $\overline{t}$ is larger and it takes a longer time for diversity ($\alpha = 1/2$) to become the optimal decision.*

Concluding, the size effect introduces a dynamical scale effect into the system. The optimal solution may change through time, but in this case it can only switch from $\alpha = 0, 1$ to $\alpha = 1/2$. This happens if and only if the probability of recombination π is large enough (see corollary 2 in section 2).

Finally, in the limit of infinite time ($It >> O_0$) the size effect saturates ($\lim_{t\to\infty} S(t) = 1$). This means that if one faces a time horizon long enough the size factor can be discarded in the probability of emergence of recombinant innovation. Not considering the transitory phase, the solution for optimal diversity at time $t >> O_0/I$ is approximated by the solution of the static pilot model.

6.3 The effect of non-zero initial values on the optimal strategy

Now we want to see what happens if we consider the initial value of parent options in the optimization of final benefits. Equation (18) shows the value of the innovative option in this case:

$$O_3(t) = C\Big[f(\alpha, t) + \alpha(1-\alpha)It\Big] \tag{40}$$

[11] We have $\frac{d}{ds}2^{s+1}\big(2^{s-1} - 1\big) = 2^{s+1}\ln 2(2^s - 1) > 0$ since $s > 0$.

where $C = 4\pi I_3/I$. Comparing this with the expression that we used in the model of section 2 we have one more term:

$$f(\alpha,t) = \big(O_{10} - \alpha O_0\big)^2 \Big(\frac{1}{O_0 + It} - \frac{1}{O_0}\Big) + \big(O_{10} - \alpha O_0\big)(1-2\alpha)\ln\frac{O_0 + It}{O_0}$$

This is the sum of two terms: one is hyperbolic and converges to a negative value as time goes to infinity. The other is logarithmic and monotonically increasing or decreasing depending on the factor $\big(O_{10} - \alpha O_0\big)(1-2\alpha)$. The objective function for maximization of final benefits is

$$V(\alpha,t) = \big(O_{10} + \alpha It\big)^s + \big(O_{20} + (1-\alpha)It\big)^s + C^s\Big[f(\alpha,t) + \alpha(1-\alpha)It\Big]^s \tag{41}$$

Normalizing this function to $(It)^s$ as done before is less meaningful since with non-zero initial values $(It)^s$ does no longer represent the value of benefits with specialization. Nevertheless this normalization leaves us with an adimensional function and allows to compare the results with other versions of the model. The normalized benefits are

$$\tilde{V}(\alpha,t) = \left(\frac{O_{10}}{It} + \alpha\right)^s + \left(\frac{O_{20}}{It} + 1 - \alpha\right)^s + C^s\left[\frac{f(\alpha,t)}{It} + \alpha(1-\alpha)\right]^s \tag{42}$$

The first order necessary condition for a maximum is

$$\begin{aligned} &\left(\frac{O_{10}}{It} + \alpha\right)^{s-1} - \left(\frac{O_{20}}{It} + 1 - \alpha\right)^{s-1} + \\ + \quad & C^s\left[\frac{f(\alpha,t)}{It} + \alpha(1-\alpha)\right]^{s-1}\left(\frac{1}{It}\frac{\partial f(\alpha,t)}{\partial\alpha} + 1 - 2\alpha\right) = 0 \end{aligned} \tag{43}$$

The solution to this equation is rather complicated. The main result is a reduction of symmetry in the system (unless $O_{10} = O_{20}$). Note that $\alpha = 1/2$ is not a solution to the above equation in general.[12] Optimal diversity is represented by a function of time $\alpha^*(t)$. In figure 11 we report the graph of benefits for five different times. The optimal share α^* is seen to shift with time. Moreover, there is an "overshooting" effect during the transitory phase: if at some time t_1 the optimal solution is $\alpha^*(t_1) < 1/2$, the system will first experience a period of time during which the optimal share is larger than 1/2 and then go back to the symmetric allocation. In the long run, when $t >> O_0/I$, symmetry is restored. The effect of the initial values of capital stocks has then dissipated and we are back in the situation of the pilot model.

6.4 Optimization in the general case

In this last section we address the optimization of the more general model, with a size factor and initial values different from zero. The value of the innovative option at time t is given by (27). With such solution the value of final benefits from the overall investment is as follows:

$$\begin{aligned} V(\alpha,t) \;&=\; \big(O_{10} + \alpha It\big)^s + \big(O_{20} + (1-\alpha)It\big)^s + \\ &+\; C^s\Big[B(\alpha)It + B(\alpha)\frac{e^{-\sigma O_0}}{\sigma}\Big(e^{-\sigma It} - 1\Big) + h(\alpha,t)\Big]^s \end{aligned} \tag{44}$$

[12]The symmetric allocation is still a solution in the particular case of equal initial values $O_{10} = O_{20}$.

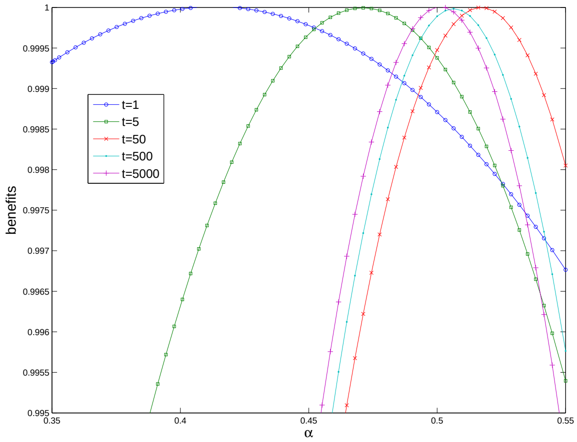

Figure 11: Final benefits with positive initial values and no size effect. Here we have $O_{10} = 1$, $O_{20} = 10$, $s = 1.2$, $\pi = 1$ and $I = 4I_3 = 1$. The five time horizons are in units of $1/I$.

where $h(\alpha, t)$ collects all terms in the expression of O_3 but the first two. Note that it is not possible to separate this expression into two factors dependent separately on t and α as we managed to do in section 6.2. The contribution of innovation (the term multiplied by C^s) consists of three terms. The first is the linear one, which appears also in the pilot model. The second is a direct effect of the size factor. The third one is due to the presence of non-zero initial values of parent options. This expression combines the effects that we have been analysing separately so far. If we normalize this expression dividing it by $I^s t^s$ we obtain

$$\tilde{V}(\alpha, t) = \left(\frac{O_{10}}{It} + \alpha\right)^s + \left(\frac{O_{20}}{It} + 1 - \alpha\right)^s + C^s \left[B(\alpha)n(t) + \frac{h(\alpha, t)}{It}\right]^s \tag{45}$$

where $n(t) = 1 + e^{-\sigma O_0}/(\sigma It)(e^{-\sigma It} - 1)$. This time factor can be expressed in terms of the factor $m(t)$ that we have introduced in section 6.2: $n(t) = e^{-\sigma O_0} m(t) + 1 - e^{-\sigma O_0}$, $n(0) \simeq 1 - e^{-\sigma O_0}$, $n'(t) = e^{-\sigma O_0} m'(t) > 0$ and $\lim_{t\to\infty} n(t) = 1$. The smaller the sum of initial values (O_0) the closer $n(t)$ is to $m(t)$. With no initial values $n(t)|_{O_0=0} = m(t)$. The effect of $n(t)$ is symmetric: the benefits curve rises from lower values where the contribution of innovation is negligible to higher values where diversity may be the optimal choice eventually. The presence of non-zero initial values brakes the symmetry of the system through the term $h(\alpha, t)$ and the ratios O_{10}/It and O_{20}/It. $\alpha = 1/2$ is not a solution to the optimization problem in general, but the benefits curve moves towards a symmetric shape around the point $\alpha = 1/2$.

In the long run ($It >> O_0$), the initial values become negligible and the size factor converges to one. In other words, if the time horizon is long enough, the general case reduces to the much simpler pilot model.

7 Conclusions and further research

This study has proposed a model of an investment allocation problem where the decision maker faces a trade-off between scale advantages and diversity benefits through recombinant innovation. We considered three different versions of the model with increasing levels of complexity. First a pilot model was developed to express the core elements of recombinant innovation. A more general model devoted attention to the detailed structure of diversity and allowed initial values of parent options to be different from zero. Finally, a third version introduced a diminishing marginal size effect in the probability of emergence of a recombinant innovation.

The initial part of the analysis consisted of deriving a solution for the model dynamics. A condition for constant diversity of the system of parent options is that the ratio of investment shares equals the ratio of initial values of parent options. When this is not the case, diversity will change over time and may be increasing, decreasing or non monotonic depending on the relative value of these two ratios. Nevertheless, in all cases diversity converges to the same constant value in the long run. The investment shares and the initial values of parent options determine the shape of the time pattern of the innovative option. In the long run only a linear and a logarithmic term count. The time pattern of innovation may be either convex or concave.

In order to account for a diminishing marginal effect of parent options in recombinant innovation, a size factor is included in the innovation probability. In the long run the value of innovation reduces again to a linear plus logarithmic term. But in this case there can only be a convex time pattern. This shape reflects the typical threshold effect of recombinant innovations.

We optimized diversity given a final benefits function, which comes down to finding an optimal balance or an optimal trade-off between the benefits of diversity due to recombinant innovation and the benefits associated with returns to scale. We derived conditions for optimal diversity under different regimes of returns to scale. Maximum diversity, expressed by a perfectly symmetric system with $\alpha = 1/2$, may be either a local maximum or a local minimum of final benefits, depending on the level of returns to scale. When diversity is a local maximum, two other stationary points of final benefits are present. We have defined two threshold values of returns to scale: the first one is the value where the system makes a transition from one to three stationary points of final benefits. The second threshold is the returns to scale level below which diversity is a global maximum of final benefits.

The presence of a size factor in the probability of emergence makes the returns to scale threshold time dependent. This suggests a threshold analysis in the time domain: for a given level of returns to scale, when the investment time horizon is beyond a critical value, the best choice becomes diversity. This threshold time horizon will be larger the higher are the returns to scale. Introducing positive initial values of parent options breaks the symmetry of the system. An investment share $\alpha = 1/2$ is no longer a general solution to the maximization problem then. In the long run symmetry is restored, that is, approximated through convergence. Maximal diversity ($\alpha = 1/2$) then will become optimal eventually if increasing returns are not too high.

Several directions for future research can be identified. Investment in the innovative option can be endogenized, i.e. made part of the allocation decision. Extending the number of parent options allows for an examination of the role of disparity (one of the dimensions of diversity), as well as for assessing the marginal effect of new options (e.g., diminishing returns) and the optimal number of options. Finally, the value of parent options can be modelled as a stochastic process, which suggests an analogy between the innovative option and a financial derivative: parent options would then play the role of underlying assets.

Appendix A Condition for constant balance

Here we give a proof of the necessary and sufficient conditions of constant balance for the "Gini" specification.

In order to prove necessity we differentiate the expression $B\big(O_1(t), O_2(t)\big)$ with respect to time and see under which conditions the derivative is equal to zero. Using the chain rule we have

$$\frac{dB}{dt} = \frac{\partial B}{\partial O_1}\frac{dO_1}{dt} + \frac{\partial B}{\partial O_2}\frac{dO_2}{dt} \tag{46}$$

where

$$\frac{\partial B}{\partial O_i} = \frac{O_j(O_j - O_i)}{(O_i + O_j)^3} \qquad i, j = 1, 2 \qquad i \neq j$$

Time derivatives are given by the specifications of the model (1). If now one substitutes the time flow of each option value, $O_1(t) = O_{10} + \alpha It$ and $O_2(t) = O_{20} + (1-\alpha)It$, the time derivative of balance becomes

$$\frac{dB}{dt} = \frac{O_{10} - O_{20} + (2\alpha - 1)It}{(O_{10} + O_{20} + It)^3}\Big[(O_{10} + \alpha It)(1-\alpha)I - (O_{20} + (1-\alpha)It)\alpha I\Big] \tag{47}$$

Setting this derivative to zero we obtain

$$(O_{10} + \alpha It)(1-\alpha) = (O_{20} + (1-\alpha)It)(\alpha I)$$

This equation must hold true for any value of t. For instance, taking $t = 1/I$ we have

$$\frac{O_{10}}{O_{20}} = \frac{\alpha}{1-\alpha}$$

which is condition (16).

This is also a sufficient condition for constant balance as one can see by direct substitution:

$$\begin{aligned} B(t) &= 4\frac{(O_{10} + \alpha It)(O_{20} + (1-\alpha)It)}{(O_{10} + O_{20} + It)^2} = 4\frac{(O_{10} + \alpha It)(O_{10}\frac{1-\alpha}{\alpha} + (1-\alpha)It)}{(O_{10} + O_{10}\frac{1-\alpha}{\alpha} + It)^2} \\ &= 4\frac{(1 + \frac{\alpha}{O_{10}}It)(\frac{1-\alpha}{\alpha} + \frac{1-\alpha}{O_{10}}It)}{(1 + \frac{1-\alpha}{\alpha} + \frac{It}{O_{10}})^2} = 4\frac{1-\alpha}{\alpha}\frac{(1 + \frac{\alpha}{O_{10}}It)^2}{(\frac{1}{\alpha} + \frac{It}{O_{10}})^2} = 4\frac{1-\alpha}{\alpha}\alpha^2 = 4\alpha(1-\alpha) \end{aligned}$$

Appendix B General solution to the dynamic model

Here we report the steps of the integration of the probability of emergence as defined in (24), that is, the integration of the third equation of the model (1) leading to the time value of the third option O_3. This computation contains the solution without size effect as a particular case. In what follows we set $I_3 = 1$ for investment in the innovative option.

$$O_3(t) = \int_0^t 4\frac{(O_{10}+\alpha Is)(O_{20}+(1-\alpha)Is)}{(O_0+Is)^2}\big(1-e^{-\sigma(O_0+Is)}\big)ds \tag{48}$$

We substitute $s = (x - O_0)/I$ and obtain

$$O_3 = \frac{4}{I}\int_{O_0}^{O_0+It} \frac{(E+Fx)(G+Hx)}{x^2}\big(1-e^{-\sigma x}\big)dx \tag{49}$$

where $E = O_{10}(1-\alpha) - \alpha O_{20}$, $F = \alpha$, $G = -E$ and $H = (1-\alpha)$. The expression above is the difference of two integrals (for ease of notation we consider indefinite integrals for the moment). The first one is

$$\begin{aligned}
\int \frac{(E+Fx)(G+Hx)}{x^2}dx &= EG\int\frac{dx}{x^2} + (EH+FG)\int\frac{dx}{x} + FH\int dx \\
&= -\frac{EG}{x} + (EH+FG)lnx + FHx
\end{aligned}$$

As for the second integral we have

$$\begin{aligned}
\int \frac{(E+Fx)(G+Hx)}{x^2}e^{-\sigma x}dx &= EG\int\frac{e^{-\sigma x}}{x^2}dx + (EH+FG)\int\frac{e^{-\sigma x}}{x}dx + \\
&+ FH\int e^{-\sigma x}dx = \\
&= -\frac{FH}{\sigma}e^{-\sigma x} - EG\frac{e^{-\sigma x}}{x} + \\
&+ [EH+FG-\sigma EG]\Big[lnx + \sum_{k=1}^{\infty}\frac{(-\sigma x)^k}{k\cdot k!}\Big]
\end{aligned}$$

When substituting the latter two results into equation (49) we obtain

$$\begin{aligned}
\int \frac{(E+Fx)(G+Hx)}{x^2}\big(1-e^{-\sigma x}\big)dx &= -\frac{EG}{x} + FHx + FH\frac{e^{-\sigma x}}{\sigma} + \\
&+ EG\frac{e^{-\sigma x}}{x} + \sigma EGlnx + \\
&+ [\sigma EG - (EH+FG)]\sum_{k=1}^{\infty}\frac{(-\sigma x)^k}{k\cdot k!}
\end{aligned}$$

It is instructive to look first at the case of constant balance. The necessary and sufficient condition can be written as $O_{10}(1-\alpha) = O_{20}\alpha$. Then $EG = 0$, $EH + FG = 0$ and $FH = \alpha(1-\alpha)$ and the integral above simplifies to

$$\int \frac{(E+Fx)(G+Hx)}{x^2}\big(1-e^{-\sigma x}\big)dx\Big|_{B=const} = \alpha(1-\alpha)\Big(x + \frac{e^{-\sigma x}}{\sigma}\Big) \tag{50}$$

The solution for the value of the third option as a function of time is then

$$O_3(t) = \frac{4}{I}\alpha(1-\alpha)\Big(x + \frac{e^{-\sigma x}}{\sigma}\Big)\Big|_{x=O_0}^{x=O_0+It} = Bt + B\frac{e^{-\sigma O_0}}{\sigma I}\Big(e^{-\sigma It} - 1\Big) \tag{51}$$

where $B = 4\alpha(1-\alpha)$. It is useful to check the "physical" dimensions of the solution just obtained. The first term Bt is *time* (balance is dimensionless). The second term is *time* again, since σ is *capital*$^{-1}$ while I is capital per unit of time. Not surprisingly O_3 has a *time* dimension, after we have set $I_3 = 1$.

Relaxing the condition of constant balance we have the following general result for the value of the innovative option at time t:

$$\begin{aligned} O_3(t) &= \frac{4}{I}\int_{x=O_0}^{x=O_0+It} \frac{(E+Fx)(G+Hx)}{x^2}\big(1-e^{-\sigma x}\big)dx = \\ &= Bt + B\frac{e^{-\sigma O_0}}{\sigma I}\big(e^{-\sigma It} - 1\big) + \frac{4}{I}\sigma EG \log\frac{O_0+It}{O_0} + \\ &+ \frac{4}{I}EG\Big[\frac{1}{O_0}\big(1-e^{-\sigma O_0}\big) - \frac{1}{O_0+It}\big(1-e^{-\sigma(O_0+It)}\big)\Big] + \\ &+ \frac{4}{I}\Big[\sigma EG - (EH+FG)\Big]\Big[\sum_{k=1}^{\infty}\frac{\big(-\sigma(O_0+It)\big)^k}{k\cdot k!} - \sum_{k=1}^{\infty}\frac{\big(-\sigma O_0\big)^k}{k\cdot k!}\Big] \end{aligned} \tag{52}$$

The first two terms are what we have with constant balance (see section 5.3). In the short run ($It << O_0$) we have $O_3(t) \simeq Bt$. A bit more complex is the analysis of the long run behaviour ($t >> O_0/I$). The part referring to constant balance will tend to a linear growth, as we have seen already in the main text. In the logarithmic term the value of the new investment It overcomes the initial option value O_0. The second part of the third term vanishes even faster than the exponential term of the part relative to constant balance, because of the presence of t in the denominator. Finally the infinite sum containing t goes to zero at least exponentially: this can be seen by noting that for even values of k we have ($O_0 + It = y$)

$$\frac{(-y)^k}{2^k\cdot k!} < \frac{(-y)^k}{k\cdot k!} < \frac{(-y)^k}{k!}$$

For odd values of k the inequalities are reversed. This means that our series is bounded between the functions $-1 + e^{-(O_0+It)}$ and $-1 + e^{-(O_0+It)/2}$, implying that it goes to zero at least exponentially:

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{\big(-\sigma(O_0+It)\big)^k}{k\cdot k!} &= -\sigma(O_0+It) + \frac{\sigma^2(O_0+It)^2}{2\cdot 2} - \frac{\sigma^3(O_0+It)^3}{3\cdot 3!} + \frac{\sigma^4(O_0+It)^4}{4\cdot 4!} - \dots \\ &< -\sigma(O_0+It) + \frac{\sigma^2(O_0+It)^2}{2} - \frac{\sigma^3(O_0+It)^3}{3!} + \frac{\sigma^4(O_0+It)^4}{4!} - \dots \\ &= -1 + e^{-\sigma(O_0+It)} \leq 0 \end{aligned}$$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{\big(-\sigma(O_0+It)\big)^k}{k\cdot k!} &= -\sigma(O_0+It)+\frac{\sigma^2(O_0+It)^2}{2\cdot 2}-\frac{\sigma^3(O_0+It)^3}{3\cdot 3!}+\frac{\sigma^4(O_0+It)^4}{4\cdot 4!}-\dots\\
&> -\frac{\sigma(O_0+It)}{2}-\frac{\sigma(O_0+It)}{2}+\frac{\sigma^2(O_0+It)^2}{2^2\cdot 2!}-\frac{\sigma^3(O_0+It)^3}{2^3\cdot 3!}+\dots\\
&= -1-\frac{\sigma(O_0+It)}{2}+e^{-\frac{\sigma(O_0+It)}{2}}
\end{aligned}
$$

Alternatively one can think that for $k >> 1$ we have $k\cdot k! \simeq ke^{k\log k-k} \simeq k!$. This means that the infinite sums in the expression of $O_3(t)$ do not differ too much from negative exponential functions. In particular the one depending on t goes to zero as time is long enough ($It >> O_0$). Consequently we are left with the following long run functional behaviour:

$$
\begin{aligned}
O_3(t) &\simeq B\Big(t-\frac{e^{-\sigma O_0}}{\sigma I}\Big)+\frac{4}{I}\sigma EG\log\frac{It}{O_0}+\\
&+ \frac{4}{I}EG\Big[\frac{1}{O_0}\big(1-e^{-\sigma O_0}\big)\Big]-\frac{4}{I}\Big[\sigma EG-(EH+FG)\Big]D(\sigma,O_0)
\end{aligned}
\tag{53}
$$

The factor $D(\sigma,O_0)=\sum_{k=1}^{\infty}\frac{(-\sigma O_0)^k}{k\cdot k!}$ only depends on parameters σ and O_0; similarly to what we have noticed for the series dependent on t we can say that such a quantity is bounded between e^{-O_0} and $e^{-O_0/2}$. In particular one can easily see that $C(\sigma,O_0)$ is finite:

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{\big(-\sigma O_0\big)^k}{k\cdot k!} &= -\sigma O_0+\frac{\sigma^2O_0^2}{2\cdot 2}-\frac{\sigma^3O_0^3}{3\cdot 3!}+\frac{\sigma^4O_0^4}{4\cdot 4!}-\dots\\
&< -\sigma O_0+\frac{\sigma^2O_0^2}{2}-\frac{\sigma^3O_0^3}{3!}+\frac{\sigma^4O_0^4}{4!}-\dots\\
&= -1+e^{-\sigma O_0}\le 0
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{\big(-\sigma O_0\big)^k}{k\cdot k!} &= -\sigma O_0+\frac{\sigma^2O_0^2}{2\cdot 2}-\frac{\sigma^3O_0^3}{3\cdot 3!}+\frac{\sigma^4O_0^4}{4\cdot 4!}-\dots\\
&> -\frac{\sigma O_0}{2}-\frac{\sigma O_0}{2}+\frac{\sigma^2O_0^2}{2^2\cdot 2!}-\frac{\sigma^3O_0^3}{2^3\cdot 3!}+\frac{\sigma^4O_0^4}{2^4\cdot 4!}-\dots\\
&= -1-\frac{\sigma O_0}{2}+e^{\frac{-\sigma O_0}{2}}
\end{aligned}
$$

Obviously the expression in (53) must be positive. The third and fourth terms are constant and since we consider long run behaviour of the system it does not really matter whether they are positive or negative. Actually the third term is negative, while the fourth can be either negative or positive depending on σ, the investment share α and the initial values O_{10} and O_{20}. The second term is negative, since $G=-E$. But in the long run the linear function overcomes the logarithmic one. Then we can be sure that what we obtain for $O_3(t)$ in the long run is a positive quantity.

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