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COMPARING THE EFFECTIVENESS OF RANK CORRELATION STATISTICS

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Abstract: Rank correlation is a fundamental tool to express dependence in cases in which the data are arranged in order. There are, by contrast, circumstances where the ordinal association is of a nonlinear type. In this paper we investigate the effectiveness of several measures of rank correlation. These measures have been divided into three classes: conventional rank correlations, weighted rank correlations, correlations of scores. Our findings suggest that none is systematically better than the other in all circumstances. However, a simply weighted version of the Kendall rank correlation coefficient provides plausible answers to many special situations where intercategory distances could not be considered on the same basis.

Keywords: ordinal data, nonlinear association, weighted rank correlation.

1 Introduction

Measuring agreement between two sets of rankings is an issue frequently encountered in research studies. Classic fields where rank data occur are market segmentation, information retrieval, biosignal analysis, priority ranking and consensus formation, sorting algorithms, scales of symptoms and feelings, risk and credit scoring.

Throughout this paper we will examine situations of the following type. Consider a fixed set of n distinct items ordered according to the different degree in which they possess two common attributes represented by X and Y. Let us suppose that each attribute consist of a host of intangibles that can be ranked but not measured and that the evaluations are expressed in terms of an ordinal scale of n ranks: $\mathbf{q} = q_1, q_2, \cdots, q_n$ for X and $\mathbf{s} = s_1, s_2, \cdots, s_n$ for Y. Here we consider only a complete linear ordering case so that s_i and q_i take on value in the set of integers $\{1, 2, \cdots, n\}$; moreover, evaluators are asked to decide on a definite rank order for each attribute so that no two items are given the same rank. In practice, the vectors $\mathbf{s}$ and $\mathbf{q}$ are elements of ${}_nP_n$, the set of all $n!$ permutations. With no essential loss of generality we may assume that s_i is the rank of y_i after $\mathbf{q}$ has been arranged in its natural order ($q_i = i, i = 1, 2, \cdots, n$) with the corresponding ranks s_i aligned beneath them.

$$\begin{array}{cccccccc} 1 & 2 & \cdots & i & \cdots & n-1 & n & \mathbf{q} \\ s_1 & s_2 & \cdots & s_i & \cdots & s_{n-1} & s_n & \mathbf{s} \end{array} \tag{1}$$

This scenario can be also invoked when one considers ranks as manifestations of an underlying absolutely continuous random variable whose observed values are transformed into a ranking. Ranking methods, in fact, are often recommended when the variables are scaled over a different range or the absolute distance among their values is unknown or cannot be measured for practical or theoretical reasons. Also, a rank transformation may be employed in order to avoid distortion because of the actual data are contaminated with error or include outliers.

A rank correlation $r(\mathbf{q}, \mathbf{s})$ is a statistic summarizing the degree of association between two arrangements $\mathbf{q}$ and $\mathbf{s}$ where $\mathbf{q}$ acts as a reference to the other. For comparability, the coefficients are usually constructed to vary between -1 and 1. Their magnitude increases as the association increases with a $+1$ (-1) value when there is perfect positive (negative) association from concordance (discordance) of all pairs. For a different choice see [11] or [37]. The value of zero is indicative of no association, but does not necessarily imply independence.

Rankings in (1) are referred to a classification of n items with 1 assigned to the most preferred item, 2 to the next-to-most preferred and so forth. If an opposite orientation of the arrangement is applied, then a rank correlation statistic that changes its sign, but not its absolute value is said to be antisymmetric under reversal

$$r(\mathbf{q}, \mathbf{s}) = -r(\mathbf{q}, \mathbf{s}^*) \tag{2}$$

where s_i^* is the antithetic ranking of s_i, that is, $s_i^* = n - s_i + 1, i = 1, \cdots, n$. The usefulness of this principle is that a classification of n items can be organized according to the types of problems that occur and thereby providing more meaningful measurement.

The inverse permutation $\mathbf{s}'$ of $\mathbf{s}$ is the ranking of $\mathbf{q}$ with respect to that of $\mathbf{s}$, that is, $s'_{s_i} = i, i = 1, \cdots, n$. A rank correlation statistic is said to be symmetric under inversion if

$$r(\mathbf{q}, \mathbf{s}) = r(\mathbf{q}, \mathbf{s}') \tag{3}$$

Conditions (2) and (3) can easily be obtained by averaging the statistic computed on the ordinary ranks with the same statistic computed on the antithetic and inverse permutation, respectively (*e.g.* [5], [13],[31]).

The main objective of this paper is to examine a selection of rank correlations and identify limitations and merits of each relatively to various situations of nonlinear type. The contents of the various sections are as follows. Section 2 presents several cases of nonlinear association between rankings. In section 3 we will concentrate mainly on analyzing the fundamental factors that affect the behavior of some conventional rank correlation statistics under a nonlinear interaction. In particular, we will show the inadequacy of standard coefficients to deal with such situations. Section 4 reviews the general formulation of weighted rank correlations in which the incorporation of a weight function allows more flexibility in the measure of agreement for permutations. The function is to be chosen so as to weigh the comparisons according to the importance attached to various subsets of ranks. In this sense, section 4 highlights the more salient features and the performance of several choices of the weight function. Section 5 reports on a class of correlation statistics obtained by computing the Pearson product-moment correlation coefficient on suitably chosen scores that replace ordinary ranks. In section 6 we obtain the critical values of the most promising coefficients to enable such statistics to be applied to real data. Finally, we conclude and point out future research direction in section 7.

2 Nonlinear association

Situations in which a coefficient of agreement/disagreement should take into account the contextual factors that affect judgment are common in real world. In this section we describe a number of tight but nonlinear relationships between two rankings.

Ceiling or floor effects. These represent cases of limited resource allocation because ascribing higher importance to one item reduces the importance of another. For example, it is more satisfactory to place the winner in a race in the first position than to place the worst contestant last. In other cases differences in low ranks would seem more critical. For example, when an admission office expunges the less qualified candidates.

Bipolarity conditions. The top-down and the bottom-up process may simultaneously affect the same attribute giving rise to a bi-directional effect. Let us consider, for example, the comparison of the final league tables with expert forecasts made before the start of the season. In league football, both the teams placed near the top (which gain promotion) and those placed near the bottom (which risk relegation) are relevant to evaluate the accuracy of the prediction. The teams placed in the middle part of the rankings have negligible influence.

Quadratic trends. Two of the most common nonlinear patterns are a U-shaped and an inverted U-shaped relationship in which the values in the ranking show an increase followed by a decrease or *vice versa*. An example of the former is the environmental Kuznets curve predicting that the environmental quality appears to deteriorate with countries' economic growth at low levels of income, and then to improve with economic growth at higher levels of income. An example of an inverted U-shaped pattern is the Yerkes-Dodson law relating the level of arousal and the expected quality of performance.

Bilinear association. Increasing degree of attribute Y are combined with increasing degree of attribute X, but in a bilinear ascending (descending) pattern the mean of the ranks to the left of the central rank of Y is significantly higher (lower) than the mean of the ranks on opposite side. These situations may occur, for instance, when the evaluators tend to separate the items under consideration into two distinct groups, but all the items in a group are considered superior, in some sense, to all the items in the other group.

In order to get a feeling as to the nature of nonlinear association, the relationships discussed above are illustrated in Table (1) with $n = 15$ fictitious rankings. The abbreviation LH (HL) indicates that the lowest (highest) points of the scale come first. The suffix A and D stand for ascending and descending respectively.

Table 1: examples of nonlinear rankings

A	Natural ordering	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
B	Inverse ordering	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
C	Floor effect	1	2	3	4	15	14	13	12	11	10	9	8	7	6	5
D	Ceiling effect	11	10	9	8	7	6	5	4	3	2	1	12	13	14	15
E	Bipolarity/A	1	2	3	4	11	10	9	8	7	6	5	12	13	14	15
F	Bipolarity/D	15	14	13	12	11	6	7	8	9	10	5	4	3	2	1
G	U-shaped/LH	8	7	6	5	4	3	2	1	9	10	11	12	13	14	15
H	U-Shaped/HL	15	14	13	12	11	10	9	1	2	3	4	5	6	7	8
I	Inverted U/LH	1	2	3	4	5	6	7	8	15	14	13	12	11	10	9
J	Inverted U/HL	9	10	11	12	13	14	15	8	7	6	5	4	3	2	1
K	Bilinear/A	9	10	11	12	13	14	15	1	2	3	4	5	6	7	8
L	Bilinear/D	8	7	6	5	4	3	2	1	15	14	13	12	11	10	9

Naturally, such nonlinearities are not exhaustive. For instance, we have neglected orderings affected by a critical region phenomenon, that is, a relationship exists within the central part of the scale of measurement, but at the extremes no relationship is observed, either by virtue of insensitivity of the measures, or through some more intrinsic characteristic of the causal relationship, or because the errors of observations are greatest at the extremes. However, we believe that the rankings in Table (1) cover some of the most interesting cases or, at least, those more often mentioned in the applied statistics literature.

3 Conventional rank correlations

In [26] a simple technique is suggested for devising a nonlinear rank correlation that does not take explicitly into account a weighting scheme for ranking comparisons. First we define a ranking $\mathbf{q}$ which shows such behavior perfectly. Given some other ranking s we measure its departure from such an ideal behavior by quantifying the nonlinear deficit by a distance $\delta\left(.\right)$ suitable for rankings

$$r\left(\mathbf{q},\mathbf{s}\right)=1-2\frac{\delta\left(\mathbf{q},\mathbf{s}\right)}{\max\limits_{\mathbf{q},\mathbf{s}\in{}_nP_n}\delta\left(\mathbf{q},\mathbf{s}\right)} \tag{4}$$

where ${}_nP_n$ is the set of all $n!$ permutations. The smaller $\delta\left(.\right)$, the more similar are the two rankings. The famous rank correlation coefficients proposed by Kendall's [21] and Spearman [35], can be expressed using (4). Another example is the ratio between distance of s from a reference permutation $\mathbf{q}$ and distance of s from the antithetic reference permutation $\mathbf{q}^*$

$$r\left(\mathbf{q},\mathbf{s}\right)=\frac{\delta\left(\mathbf{q}^*,\mathbf{s}\right)-\delta\left(\mathbf{q},\mathbf{s}\right)}{\delta\left(\mathbf{q}^*,\mathbf{q}\right)} \tag{5}$$

The Gini cograduation coefficient, [17], and the Gideon-Hollister maximal deviation, [16], belong to this class (see [5]). Table (2) reports several examples of rank correlations based either on (4) or on (5).

The indices r_1 and r_4 are well-known, but continue to be rediscovered. For example, the order statistics correlation coefficient proposed in [39] coincides with the Spearman's r_1 if the order statistics are those of samples drawn from a $[0,1]$uniform distribution. The cograduation coefficient r_2 was proposed as an improvement over the Spearman's footrule r_{14} ([36]) and it has been recently rediscovered by [32] (see also [27]). Index r_3 is linked to the Hamming distance between two permutations, that is, the number of their unmatched ranks. Here $h\left(x\right)$ denotes the indicator function that equals 1 if x is true and 0 otherwise. The statistic r_5, given by [16], originates from the principle of greatest deviation. The coefficient r_6 is based on the squared index of a permutation discussed by MacMahon [22, p 16]. Salvemini [33] described the Fechner's coefficient r_7 and introduced the rank correlation r_8. The Fechner index r_7 can also be determined by the number of "runs up" in the permutation s; in fact, r_7 coincides with the test of randomness devised by [25] and with the rank correlation statistics based on rises discussed by [31].

Let the (q,s) plane be divided into four regions by the lines $q=\left(n+1\right)/2$ and $s=\left(n+1\right)/2$. The statistics r_9, developed by [4], is based on concor-

Table 2: unweighted rank correlations

Name	Formula
Spearman	$r_1 = 1 - \dfrac{6\sum_{i=1}^{n}(i-s_i)^2}{n^3-n}$
Gini	$r_2 = 2\dfrac{\sum_{i=1}^{n}\lvert i-s_i^*\rvert - \sum_{i=1}^{n}\lvert i-s_i\rvert}{n^2-k_n};\ k_n = n \mod 2$
Hamming distance	$r_3 = \dfrac{\sum_{i=1}^{n} h(s_i = i) - \sum_{i=1}^{n} h(s_i^* = i)}{n-k_n}$
Kendall	$r_4 = \dfrac{2\sum_{i<j} sgn(s_j - s_i)}{n(n-1)}$
Gideon-Hollister	$r_5 = 2\dfrac{\max\limits_{1\le i\le n}\sum\limits_{j=1}^{i} h\left(s_j^* > i\right) - \max\limits_{1\le i\le n}\sum\limits_{j=1}^{i} h(s_j > i)}{n-k_n}$
MacMahon	$r_6 = 1 - \dfrac{12\sum_{i=1}^{n-1} i^2 h(s_i > s_{i+1})}{2(n-1)^3 3(n-1)^2 + n - 1}$
Fechner	$r_7 = \dfrac{\sum_{i=2}^{n} sgn(s_i - s_{i-1})}{n-1}$
Salvemini	$r_8 = \dfrac{\sum_{i=2}^{n}(s_i - s_{i-1})}{\sum_{i=2}^{n}\lvert s_i - s_{i-1}\rvert}$
Quadrant association	$r_9 = \dfrac{n_1 - n_2}{n_1 + n_2}$
Dallal-Hartigan	$r_{10} = \dfrac{\lambda_n - \gamma_n}{n-1}$
Average slope	$r_{11} = 2\dfrac{\sum_{i<j}\left(\dfrac{s_j - s_i}{j-i}\right)}{n(n-1)}$
Median slope	$r_{12} = median\left\{b_{ij} \middle\vert b_{ij} = \dfrac{s_j - s_i}{j-i}, 1 \le i < j \le n\right\}$
Knuth	$r_{13} = 1 - 2\sqrt{\dfrac{6\sum_{i=1}^{n} b_i^2}{2(n-1)^3 + 3(n-1)^2 + (n-1)}}$
Spearman footrule	$r_{14} = 1 - \dfrac{4\sum_{i=1}^{n}\lvert i-s_i\rvert}{(n^2-k_n)}$
Gordon	$r_{15} = 2\left(\dfrac{\lambda_n - 1}{n-1}\right) - 1$
Bhat-Nayar	$r_{16} = 1 - \dfrac{2\max_{1\le i\le n}\sum_{j=1}^{i} h\left(s'_j > i\right)}{\lfloor\dfrac{n}{2}\rfloor}$
Linear trend	$r_{17} = 1 - 1.5\dfrac{\sum_{j=2}^{n}\left[(s_{i-1} + s_i) - (2i-1)\right]^2}{n^3 - 3n^2 + 2n}$
Average determinant	$r_{18} = \prod_{i<j}\left(\dfrac{i - s_j}{i + s_j}\right)$

The symbols $k_n, n_1, n_2, \lambda_n, \gamma_n$ are explained in the text.

dance/discordance in the number of pairs n_1 belonging to the first and third quadrants compared with the number n_2 belonging to the second and fourth quadrants.

The coefficient r_{10} has been suggested by [10] as a measure of monotone cover that is nearly unaffected by outliers. The symbols λ_n and γ_n indicate the maximum length of a subsequence $\left(q_{i_j}, s_{i_j}\right), j = 1, \cdots, \lambda_n$ such that both q_{i_j} and s_{i_j} are increasing or decreasing respectively (λ_n is also known as Ulam distance). The slopes statistic r_{11} is the average pairwise slope between observation (i, s_i) and (j, s_j). An analogous concept to r_{11} is the median r_{12} of the slopes between all combinations of two points in the data (see [38]). Coefficient r_{13} derives from formula (4) applied to the Euclidean distance between the inversion table of the current ranking (see [22, p. 12]) and the inversion table of the sorted permutation $s_i = i$ for $i = 1, 2, \cdots, n$.

A similar coefficient based on the city-block distance gives the same values as the median slope statistic. The coefficient r_{15}, developed by [18], is a linear transformation of the Gower measure [19] of similarity for variables measured on an ordinal scale. Coefficient r_{16}, [2], is based on the distance between the identity permutation $\mathbf{q}$ and the inverse permutation $\mathbf{s}'$ of $\mathbf{s}$. The values of r_{17} can be used to quantify the degree of linear order because it compares the average rank between two successive terms of $\mathbf{s}$ with the average of the corresponding terms in $\mathbf{q}$. Coefficient r_{18}, recommended in [24], is the average determinant of the second order minors with constant sum of elements of each columns from a data matrix of two ordinal variables. See [1] for an alternative interpretation of r_{18}.

Coefficients in Table (2) have been computed for the rankings in Table 1 and the results are reported in Table (3).

The findings reveal that unweighted rank correlation coefficients are not well suited to measure the association in nonlinear cases. The most classical indices r_1 and r_4 obtain a high value for the bipolarities E, F and for the inverted U relationship, but the other nonlinearities turn out not to have a large impact on them. The values of the Gini's r_2 are very similar to those produced by r_1. The Hamming distance r_3, the Gideon-Hollister coefficient r_5, the Salvemini index r_8 and the Dallal-Hartigan r_{10} have low values for nearly all rankings. Coefficient r_6 describes properly the bilinear relationships K, L and the floor effect C.

The Fechner index r_7 focuses its attention on the bilinear configurations M and N. The quadrant association r_9 illuminates quadratic and bilinear relationships but the other patterns go undetected; in fact, r_9 has a large negative value for too many patterns which can be misleading. The average slope r_{11} draws attention to the dual character in E and F and to the quadratic relationships G, H, I, J. Satisfactory results have been obtained by the median slope r_{12} and by r_{17} which allow a correct

Table 3: values of unweighted correlation coefficients

	C	D	E	F	G	H	I	J	K	L
r_1	0.21	0.21	0.80	-0.93	0.70	-0.70	0.80	-0.80	-0.50	0.50
r_2	0.25	0.25	0.57	-0.79	0.71	-0.71	0.79	-0.79	-0.50	0.50
r_3	0.36	0.36	0.14	-0.43	0.50	-0.50	0.57	-0.57	-0.07	0.07
r_4	-0.05	-0.05	0.60	-0.81	0.47	-0.47	0.60	-0.60	-0.07	0.07
r_5	-0.14	-0.14	0.14	-0.43	0.43	-0.43	0.57	-0.57	-0.43	0.43
r_6	-0.94	0.24	0.30	-0.55	0.72	0.72	-0.60	-0.82	0.90	-0.87
r_7	-0.43	-0.43	0.14	-0.43	0.00	0.00	0.14	-0.14	0.86	-0.86
r_8	0.17	0.17	0.54	-0.64	0.33	-0.33	0.40	-0.40	-0.04	0.04
r_9	0.00	0.00	0.14	-0.43	0.86	-0.86	1.00	-1.00	-0.86	0.86
r_{10}	-0.43	-0.43	0.14	-0.43	0.00	0.00	0.14	-0.14	0.43	-0.43
r_{11}	0.24	0.24	0.85	-0.95	0.66	-0.66	0.75	-0.75	-0.41	0.41
r_{12}	-1.00	-1.00	1.00	-1.00	0.88	-0.88	1.00	-1.00	-0.25	0.25
r_{13}	-0.23	-0.23	0.40	-0.87	0.26	-0.38	0.40	-0.91	-0.24	0.05
r_{14}	-0.07	-0.07	0.57	-0.79	0.43	-1.00	0.57	-1.00	-1.00	0.00
r_{15}	-0.43	-0.43	0.14	-0.43	0.00	0.00	0.14	-0.14	0.00	-0.86
r_{16}	-0.43	-0.43	0.14	-0.43	-0.14	-1.00	0.14	-1.00	-1.00	-0.14
r_{17}	0.22	0.22	0.81	-0.89	0.73	-0.77	0.83	-0.88	-0.60	0.60
r_{18}	0.53	0.12	1.00	-1.00	0.25	-0.25	0.78	-0.78	-0.05	0.05

evaluation of most of the effects (the first, however, fails to characterize the bilinear condition in K and L and the last misses ceiling and floor effects); more importantly, both the indices have the same magnitude, but opposite sign for the patterns (G, H) and (I, J). The Spearman's footrule r_{14} understates ceiling, floor, and bilinear descending effects. Moreover, it is not very sensitive to change in ranks since it assigns the minimum value -1 not only to the association of $\mathbf{q}$ with the inverse of the natural order $\mathbf{q}^*$, but also to other very different arrangements: H, J, K. The Bhat-Nayar coefficient r_{16} has a similar behavior. The Gordon index r_{15} detects the sign and the magnitude of the bilinear descending pattern L, but it also indicates a false absence of association in G, H, K. The average determinant achieves its extreme values for the bipolarity schemes (other than for $\mathbf{q}$ and $\mathbf{q}^*$). Moreover, r_{17} assumes opposite values for complementary configurations: (G,H), (I,J), (K,L) .

The main drawback of the rank correlations included in Table (2) is that most of them implicitly assume that the level of any one of the items is of equal importance with the level of any other item and hence we are crediting rankings with possessing more information than is intended.

4 Weighted rank correlation

The decision to weight or not to weight rank comparisons is a controversial issue. Those in favor of using "neutral" methods prefer not to weight comparisons; those opposed argue that giving more weight to agreement on certain comparisons and less weight to others increases flexibility. The use of the weights, as a matter of fact, avoids a direct assumption that there is a linear relation between two rankings and thus uncovers a potential nonlinear association, should it exist. In fact, any measure of rank correlation has an implicit weighting scheme. For instance, the Salvemini r_8 attributes zero weight to intermediate ranks. Also, the Spearman's r_1 gives greater weight to differences between items separated by more members of the ranking.

At least part of the problem is how to decide on a plausible set of weights. [29] showed that the numerous statistical methods for measuring association when the magnitude of intercategory distances cannot be ignored, group naturally in two classes: weighted rank correlation and correlation of scores. This section is devoted to the first type, whereas the second one will be treated in the next section.

The following formula is a weighted version of the Spearman coefficient r_1 that includes several special cases.

$$r_{1,w} = 1 - \frac{2\sum_{i=1}^{n} w_i\left(i - s_i\right)^2}{\max\limits_{{}_nP_n}\left\{\sum\limits_{i=1}^{n} w_i\left(i - s_i\right)^2\right\}} \tag{6}$$

An alternative generalization can be stated in the following terms

$$r'_{1,w} = \frac{\sum_{i=1}^{n}\left(w_i - \bar{w}\right)s_i}{\sum_{i=1}^{n}\left(w_i - \bar{w}\right)i} \qquad \text{with} \quad \bar{w} = n^{-1}\sum_{i=1}^{n} w_i \tag{7}$$

The Spearman coefficient r_1 is obtained for $w_i = i$ in (6) and in (7).

In[29] two weighted versions of the Kendall coefficient have been developed

$$\text{additive } r_{4,w,a} = \frac{\sum_{i<j}^{n}\left(w_i + w_j\right)h\left(s_i < s_j\right)}{\sum_{i=1}^{n}\left(n - i\right)w_i} - 1 \tag{8}$$

$$\text{multiplicative } r_{4,w,m} = \frac{2\sum_{i<j}^{n} w_i w_j h\left(s_i < s_j\right)}{\sum_{i<j}^{n} w_i w_j} - 1 \tag{9}$$

The usual Kendall coefficient r_4 is obtained from (9) for $w_i = 0.5, i = 1, 2, \cdots, n$.

Table 4: weighted rank correlations

Name	Formula
Weighted Spearman	
Mean rate	$r_{19} = 1 - \dfrac{2\sum_{i=1}^{n}\dfrac{(i-s_i)}{s_i}}{(n+1)L_1 - 2n}\ ;\quad L_1 = \sum_{i=1}^{n} i^{-1}$
Salama-Quade 82a	$r_{20} = 1 - \dfrac{2\sum_{i=1}^{n}(i-s_i)^2\left(i^{-1}+s_i^{-1}\right)}{(n+1)\sum_{i=1}^{n}\dfrac{[2i-(n+1)]^2}{is_i}}$
Salama-Quade 82b	$r_{21} = 1 - \dfrac{\sum_{i=1}^{n}\dfrac{(i-s_i)^2}{is_i}}{(n+1)L_1 - 2n}$
Salama-Quade 92	$r_{22} = 1 - \dfrac{6}{n(n-1)}\sum_{i=1}^{n}\dfrac{(i-s_i)^2}{i+s_i}$
Mango	$r_{23} = 1 - \dfrac{3\left[n^2(n+1)^2 - 4\sum_{i=1}^{n} i^2 s_i\right]}{n(n-1)(n+1)^2}$
Blest	$r_{24} = 1 - \dfrac{\left[12\sum_{i=1}^{n}(n+1-i)^2 s_i - n(n+2)(n+1)^2\right]}{n(n-1)(n+1)^2}$
Symmetrized Blest	$r_{25} = 1 - \dfrac{6\sum_{i=1}^{n}(i-s_i)^2[2(n+1)-(i+s_i)]}{n^4+n^3-n^2-n}$
Weighted Kendall	
Quade-Salama	$r_{26} = \dfrac{2\sum_{i<j}^{n}\left(i^{-1}+s_j^{-1}\right)h(s_i<s_j)}{\sum_{i<j}\left(i^{-1}+s_j^{-1}\right)} - 1$
Shieh/a	$r_{27} = \dfrac{2\sum_{i<j}^{n}(i*j)^2\, sgn(s_j-s_i)}{n(n^5/9+2n^4/15-5n^3/36-n^2/6+n/36+1/30)}$
Shieh/b	$r_{28} = \dfrac{2\sum_{i<j}^{n}[(n+1-i)(n+1-j)]^2\, sgn(s_j-s_i)}{n(n^5/9+2n^4/15-5n^3/36-n^2/6+n/36+1/30)}$

Table (4) shows some special cases of formulae (6)-(9) that have already been considered in the literature.

Coefficient r_{19} has its premise in the mean rate of change between the identity permutation $\mathbf{q}$ and the actual permutation s. Index r_{20}, suggested by [30], gives special attention to high-ranked items $(1, 2, \cdots,)$. Coefficients r_{21} and r_{22} were proposed by [29] as variants of the standard Spearman's r_1 The values of $r_{19} \cdots r_{23}$, however, are not antisymmetric under reversal.

The index r_{23}, Mango (1997), is a special case of (7) with $w_i = i^2$ and thus places emphasis on the relative importance of low ranks $(, \cdots, n-2, n-1, n)$. From another point of view, r_{23}, can be interpreted in terms of the sum of the ${}_nC_2$

second order minors extracted from the $(n \times 2)$ matrix having the actual ranking $\mathbf{s}$ as first column and $\mathbf{q}$ as second column. The index r_{24} proposed by [3] derives from (7) with $w_i = (n+1-i)^2$ and it can be interpreted as the differences between the accumulated ranks of the two orderings $\mathbf{q}$ and $\mathbf{s}$. The weighting scheme of r_{24} favors high ranks $(1, 2, \cdots,)$. Furthermore, $(r_{23} + r_{24}) = 2r_1$. In [6, 7] is advocated the use of r_{25} which is based on the weights $w_i = [2(n+1) - (i + s_i)]$; such a scheme represents not only the importance of the sorted values but also the importance of the current ranking. [15] observe that r_{25} is a version of the Blest index constrained to be symmetric under inversion (see also, [8]).

In their important survey, [29] formulated the new version r_{26} of Kendall's r_4 which involves the additive weights (8). On the other hand, [34] analyzed (9) with $w_i = h(i \leq \lfloor (n+1)p \rfloor)$ where $p = m/n$ and $m = \lfloor (n+1)p \rfloor$. The value of m must be determined on a case-by-case basis. For this reason, it appears to be unsuitable for a general use and we preferred using the weighting schemes applied to the Blest and Mango indices. The weighted rank correlation coefficients included in Table (4) have been computed for the rankings of Table (1).

Table 5: values of weighted rank correlations

	C	D	E	F	G	H	I	J	K	L
r_{19}	0.77	-0.23	0.91	-0.97	0.27	-0.27	0.96	-0.96	-0.23	0.23
r_{20}	0.95	-0.09	0.98	-0.97	0.62	-1.23	0.99	-0.25	-0.13	0.57
r_{21}	0.77	-0.23	0.91	-0.97	0.27	-0.60	0.96	-0.69	-0.29	0.23
r_{22}	0.37	-0.05	0.80	-0.93	0.47	-0.77	0.87	-0.86	-0.62	0.33
r_{23}	0.02	0.41	0.80	-0.93	0.83	-0.57	0.70	-0.90	-0.47	0.53
r_{24}	0.41	0.02	0.80	-0.93	0.57	-0.70	0.90	-0.80	-0.50	0.47
r_{25}	0.41	0.02	0.80	-0.93	0.57	-0.83	0.90	-0.70	-0.53	0.47
r_{26}	0.39	-0.46	0.72	-0.93	0.00	-0.73	0.83	-0.63	-0.31	-0.14
r_{27}	0.68	-0.89	0.72	-0.88	-0.52	-0.95	0.98	0.33	0.38	-0.55
r_{28}	-0.89	0.68	0.72	-0.88	0.95	0.52	-0.33	-0.98	0.55	-0.38

Even in this case, the results leave something to be desired. The mean rate r_{19} depicts well the bipolarity conditions, the U-shaped/HL and the inverted U/LH pattern, but it fails to identify all the other structures. The index r_{20} is relatively large for C, E, and I where the high-ranked items are in the first positions. A moderate degree of anticorrelation is attributed to F but for the other permutations there is no tendency for the ranks to run with or against each other. Coefficients r_{21} and r_{22} are in line with r_{20}, but tend to assume a wider variety of values. In addition, they assign a large negative value to G and H characterized by high-ranked items in the last positions. From another standpoint, r_{20}, r_{21}, r_{22} do not stress the inherent concordance for several nonlinearities or have the wrong sign or are confused (*e.g.* for the ceiling effect). The symmetrized Blest index r_{25} yields values of the same type as the Blest index r_{24} but provides a better description of the H pattern, that is, the U-shaped/HL configuration.

Since $r_{24}(\mathbf{q}, \mathbf{s}) = -r_{23}(\mathbf{q}, \mathbf{s}^*)$, the index of Mango and the index of Blest act as complementary statistics. In fact, the quadratic and the bilinear patterns are consistently reflected by the two coefficients. The signs of r_{23} and r_{24} are concordant; nevertheless, these two indices misstate the actual amount of agreement due to a ceiling or to a floor effect. On the other hand, a high value in both r_{23} and r_{24} constitutes a clear symptom that the configuration is ruled by antagonistic forces (e.g. the patterns E or F).

The weighted Kendall index r_{26} takes into account the bipolarities E, F and, at least partially, the quadratic interactions H and I, but the relationship contained in other arrangements is almost completely ignored (*e.g.* G, K, L). The coefficient r_{27} omits the strength of the linkage in J and K, but all the other values are well above 0.5. The coefficient r_{28} captures almost all the patterns with the exception of I and L. In addition, r_{27} and r_{28} give the same sign (but a different magnitude) to the pairs (C,D), (G, H), (I, J) and (K, L).

5 Correlation of scores

Scoring methods have been developed specifically for the analysis of ordered categorical data. A common procedure to measure agreement between two observers consists of first assigning arbitrary equal-interval scores to the ordinal levels, unless the particular case requires otherwise, and then applying classical statistical methods based on these scores (see, among the others, [28]).

For fixed n, consider the set of sample pairs $\{(x_i, y_i), i = 1, 2, \cdots, n\}$ from an absolutely continuous bivariate distribution function $H(X, Y)$, $F(X)$ and $G(Y)$. Many measures of dependence for the pair (X, Y) are of the form

$$T_n = \sum_{i=1}^{n} a_{q_i:n} b_{s_i:n} \tag{10}$$

where (q_i, s_i) is the pair of ranks associated with the i-th observation (x_i, y_i) from bivariate distribution H. Constants $\{a_{i:n}\}$, $\{b_{i:n}\}$, $i = 1, 2, \cdots, n$ are two sets of real numbers depending on the ranks and satisfying the constraints

$$\sum_{i=1}^{n} a_{q_i:n} = \sum_{i=1}^{n} b_{s_i:n} \tag{11}$$

$$a_i \leq a_{i+1}, \quad b_i \leq b_{i+1} \quad i = 1, 2, \cdots, n \tag{12}$$

In this section we are interested in exploring (10) when $\{a_{i:n}\}$ and $\{b_{i:n}\}$ are the expected value of order statistics $E(x_{i:n}) =_F m_{i:n}$ and $E(y_{i:n}) =_G m_{i:n}$.

In particular, we consider constants in the form

$$a_{q_i:n} = \frac{{}_F m_{i:n} - \mu_F}{\sqrt{\sum_{i=1}^{n} \left({}_F m_{i:n} - \mu_F\right)^2}}; \qquad b_{s_i:n} = \frac{{}_G m_{i:n} - \mu_G}{\sqrt{\sum_{i=1}^{n} \left({}_G m_{i:n} - \mu_G\right)^2}} \tag{13}$$

with $\mu_F = E(X), \mu_G = E(Y)$. The degree of concordance/discordance between two rankings is determined by calculating Pearson's product moment coefficient of correlation with $\{{}_F m_{i:n}\}$ and the $\{{}_G m_{i:n}\}$ in place of the ranks

$$r_n(F, G) = \frac{\sum_{i=1}^{n} \left({}_F m_{i:n} - \mu_F\right)\left({}_G m_{i:n} - \mu_G\right)}{\sqrt{\sum_{i=1}^{n} \left({}_F m_{i:n} - \mu_F\right)^2 \sum_{i=1}^{n} \left({}_G m_{i:n} - \mu_G\right)^2}} \tag{14}$$

The statistic $r_n(F, G)$, often attributed to Savage and van der Waerden, has a maximum value of 1 achieved when the model F and G are a linear transform of each other. The minimum possible value is attained if the rankings are exactly inverted, but it is not necessarily -1 because it depends on F and G. For intermediate values, $r_n(F, G)$ provides a measures of the dependence between the two rankings. In general, in carrying out the test, we reject the hypothesis of independence if the absolute value of (14) appears to be too large.

The models F and G generate the scores and may be chosen to conform to one's judgment about the general characteristics of the measurement. [20] proposed a rank correlation coefficient which emphasizes the concordance for the top-ranked items $(1, 2, 3, \cdots)$

$$ {}_F m_{i:n} = {}_G m_{i:n} = -\sum_{j=i}^{n} j^{-1} = M_i \rightarrow r_{29} = \frac{\sum_{i=1}^{n} M_i M_{s_i} - n}{n + M_1} \tag{15} $$

which is generated by reflected exponential distributions $F(x) = G(x) = e^x, x < 0$. Conversely, by using a positive exponential distribution $F(X) = G(x) = 1 - e^{-x}, x > 0$ we obtain

$$ {}_F m_{i:n} = {}_G m_{i:n} = -\sum_{j=n+1-i}^{n} j^{-1} = L_i \rightarrow r_{30} = \frac{\sum_{i=1}^{n} L_i L_{s_i} - n}{n + L_n} \tag{16} $$

which can be interpreted as a bottom-up rank correlation because it is especially sensitive to the concordance for low-ranked items $(\cdots, n-2, n-1, n)$. The coefficients r_{29} and r_{30} are not antisymmetric under reversal and have a mean value different from zero.

Crathorne (1925), Fieller *et al.* (1957), and many other used the expected values of the standard normal order statistics (or approximations of them) to define a measure of rank correlation corresponding to the Pearson's correlation coefficient, that is, the Fisher-Yates coefficient.

$$ {}_F m_{i:n} = {}_G m_{i:n} = L_i \rightarrow r_{31} = \frac{\sum_{i=1}^{n} N_i N_{s_i}}{\sum_{i=1}^{n} N_i^2} \tag{17} $$

Where N_i are the expected value of the i-th standard normal order statistic. Since F and G are symmetric about $x = 0$ and $y = 0$, respectively, then ${}_F m_{i:n} +_F m_{n-i+1:n} = 0$ and ${}_G m_{i:n} +_G m_{n-i+1:n} = 0$ implying that a similar score is attached to ordered position at equal depths from the extremes for each distribution. Furthermore, the absolute value of the scores increases as we go from the mediocre item to extreme items so that (17) is equally sensitive to agreement in both extremes but not in the center.

Table (6) reports the value of r_{29} and r_{30}. The scores (15) and (16) have been rescaled so that the extreme values of (14) will lie in the interval from -1 to 1. In addition, we have computed r_{31} for the approximation to the normal scores obtained by using the Hazen plotting positions

$$_{F}m_{i:n} = \ _{G}m_{i:n} = \Phi^{-1}\left(\frac{s_i - 0.5}{n}\right) \tag{18}$$

where $\Phi^{-1}(.)$ is the inverse cumulative normal distribution.

Table 6: Values of rank correlation of scores

	C	D	E	F	G	H	I	J	K	L
r_{29}	0.76	-0.21	0.91	-0.71	0.19	-0.52	0.96	-0.59	-0.36	0.15
r_{30}	-0.21	0.76	0.91	-0.71	0.94	-0.52	0.32	-0.59	-0.36	0.25
r_{31}	0.26	0.26	0.88	-0.96	0.63	-0.63	0.72	-0.72	-0.35	0.35

The values of r_{29} convey the information that there is agreement for configurations dominated by the concordance between low-ranked items (*e.g.* I). Coefficient r_{30} emphasizes the dependence in configuration characterized by the concordance between top-ranked items (*e.g.* G). A high positive value for both r_{29} and r_{30} is a signal of a bipolarity ascending pattern whereas a large negative value for both r_{29} and r_{30} may indicate either a bipolarity descending pattern or a quadratic link. Rank correlations based on the normal distribution r_{31} depict sufficiently well bipolarity conditions and quadratic relationships, but perform ineffectively for the other comparisons (C, D, K, L). We can add that, on the basis of our experiments, the results of r_{31} do not change very much if the plotting positions in (18) are substituted with other expressions.

6 Choice of a rank correlation

When the value of any typing method for $r\left(\mathbf{q},\mathbf{s}\right)$ is assessed, the two main characteristics that need to be considered are the robustness and the sensitivity. The former determines the degree of rank order inconsistency that can be withstood by the method before mismatches begin to occur. The sensitivity of $r\left(\mathbf{q},\mathbf{s}\right)$ is an estimate of its ability to differentiate between rankings. Robustness and sensitivity are antithetical requirements because more robust indices give greater stability against random change of the ranks whereas more sensitive coefficients offer a richer source of information on association patterns. Therefore, in order to choose a "good" index of association, some balancing of conflicting objectives will be required. A reasonable solution can be obtained by considering that ranking is an intrinsically robust process; thus, the choice of a coefficient should privilege its discriminatory power.

It is plain that a given value of a rank correlation coefficient does not in general define a unique permutation, except perhaps the maximum value of the coefficient. Nevertheless, many conventional rank correlations have a "resistance-to-change" that appears to be of little value for the purposes of rank comparisons. Moreover, a few of them are antisymmetric under reversal and only $r_2,\cdots,r_5,r_9,r_{16}$ are symmetric under inversion.

The weighted coefficients seem more flexible and can discriminate more easily between permutations than conventional rank correlations. It remains unclear, however, how to effectively choose among the various indices of this type. As a preliminary observation, we note that the sensitivity possessed by $r_{19},\cdots,r_{28}$, also in consideration that they fail to verify condition (2), seems inadequate to evaluate the majority of the situations described in Table (1). Finally, the version r_{26} of the Kendall coefficient has a negative bias that precludes its usage and application. As a consequence, the indices $r_1,\cdots,r_{22}$ and r_{26} are not considered suitable statistics to use when the capacity of an index to respond to changes in a permutation pattern is of concern.

Correlations of scores have received attention in a wide range of research disciplines because their definition gives the researcher the freedom to choose a suitable system of scores. We studied strengths and weaknesses of some correlations of scores defined as product-moment correlation between the expected value of the order statistics from two given distributions. Our analysis would suggest that this approach is less satisfactory than weighed rank correlation in reflecting certain patterns of agreement/disagreement between rankings. It must be noted, however, that the two approaches: rank correlation and correlation of scores, are not necessarily different. The Spearman and the Gini index, in fact, can be obtained from (14) using the order statistics from the uniform distribution (see [40]). The Blest index can be well approximated (see [13]) by a reflected power-function distribution $F(x) = 1 - \sqrt{-x}$ for $-1 < x < 0$, and $G(x) = x$ for $0 < x < 1$. On the other hand, the Mango index can be approximated by a power-function distribution $F(x) = \sqrt{x}$ and $G(x) = x$ for $0 < x < 1$. In this sense, the results achieved with correlations of scores do not appear an effective improvement over weighted rank correlations. Moreover, a specification of reliable models is required and any such choice implies a further variant of the index that may be discouraging for a non-expert user. Consequently, even the indices r_{29}, r_{30}, r_{31} are not considered further here. In summary, we have restricted our attention to $r_{23}, r_{24}, r_{25}, r_{27}, r_{28}$ which are the most promising indices discussed in the previous sections.

Let us suppose that the values of $r(\mathbf{q}, \mathbf{s})$ are rounded after the m-th decimal place

$$\frac{\lfloor r(\mathbf{q}, \mathbf{s})\, 10^m + 0.5 \rfloor}{10^m} \tag{19}$$

where $\lfloor . \rfloor$ denotes the integer part of the argument. The discriminatory power of $r(\mathbf{q}, \mathbf{s})$ can be quantified by the fraction of values assumed by (19) in relation to the maximum potential number of values.

$$\psi = \frac{\nu}{\min\left\{{}_nP_n, 2(10^m) + 1\right\}} \tag{20}$$

where ν is the number of distinct values that (19) takes on over ${}_nP_n$, measured with m decimal place accuracy. Thus $\psi = 1$ would indicate that $r(\mathbf{q}, \mathbf{s})$ has the minimum number of repeated values at the given level of approximation. Conversely, $\psi \approxeq 0$ would indicate that virtually all members of ${}_nP_n$ are considered of an identical type from the point of view of $r(\mathbf{q}, \mathbf{s})$. A value of ψ around 0.50 would mean that if one ranking is chosen at random then there would be a 50% probability that the next ranking chosen at random would be indistinguishable from the first.

A summary of (20) for the selected indices is given in Table (7) for $n = 9, \cdots, 12$. In particular, column 3-6 show the mean, the standard deviation, the standardized third moment γ_1 and the standardized coefficient of kurtosis γ_2. The last column reports the ratio [20] where the values have been rounded after the 4th decimal place ($m = 4$) to keep computations at a feasible level.

Table 7: summary statistics for some weighted rank correlations

n	Coefficient	μ	σ	γ_1	γ_2	ψ
9	r_{23}	0.0001	0.132	0.00	2.45	5.95
	r_{24}	0.0001	0.132	0.00	2.45	5.95
	r_{25}	0.0000	0.128	0.02	2.47	5.68
	r_{27}	0.0001	0.138	0.00	2.43	93.43
	r_{28}	0.0001	0.138	0.00	2.43	93.43
10	r_{23}	0.0001	0.117	0.00	2.51	9.02
	r_{24}	0.0001	0.117	0.00	2.51	9.02
	r_{25}	0.0000	0.114	0.02	2.52	8.74
	r_{27}	0.0001	0.120	0.00	2.49	98.41
	r_{28}	0.0001	0.120	0.00	2.49	98.41
11	r_{23}	0.0001	0.106	0.00	2.55	13.15
	r_{24}	0.0000	0.106	0.00	2.55	13.15
	r_{25}	0.0000	0.103	0.02	2.56	12.83
	r_{27}	0.0000	0.106	0.00	2.54	99.63
	r_{28}	0.0000	0.106	0.00	2.54	99.63
12	r_{23}	0.0001	0.096	0.00	2.59	18.54
	r_{24}	0.0000	0.096	0.00	2.59	18.54
	r_{25}	0.0000	0.093	0.02	2.60	18.16
	r_{27}	0.0001	0.094	0.00	2.58	99.92
	r_{28}	0.0000	0.094	0.00	2.58	99.92

The results suggest that all the rank correlations included in Table (7) are slightly positively biased (although the bias diminishes as the number of ranks increases) and their variance decreases with n. For large n, the distributions is nearly normal with zero mean. It may be also observed that the sensitivity of all the indices increases as n increases, but r_{27} and r_{28} can discriminate most easily between individual permutations. Thus, these indices should be preferred to measure the monotone association between two set of rankings and the choice between them depends on the weight that has to be assigned to each level of the configuration of ranks.

7 Conclusion

There are many methods of rank correlation, from simple ones such as the Blomqvist's coefficient to relatively complicated definitions invoking one or two system of weights and/or special rank transformations. Any of these methods describes a different aspect of the association between two permutations. The discussion in the previous sections has shown that important factors such as the context in which we do association analysis, the properties of the items to be ranked, the purpose of the study, may influence the choice of a particular weighting scheme for a measure of ordinal association. The flexibility of the formula and the high resolution over the set of all permutations are primary factors for a general coefficient.

In this paper we have looked at many rank correlations which emphasize or de-emphasize certain part of the scale by considering special characteristics of the ranks or by attaching to each comparison a weight that reflects the judgment of the evaluator about how much a rank matters. It is unlikely that any single coefficient could cope with or even detect the profusion of nonlinear relationships between rankings. Nonetheless, in response to the special needs arising from the peculiar situations discussed in our paper, a reasonably general answer could be given by a weighted Kendall's r_4 initially proposed by Shieh (1998) and slightly modified in the present paper.

The subject of dependence between permutations enjoys much current interest, but it seem that only conventional measures of rank correlation have been generally employed. We have only covered a subset of all rank correlations and there are many potential areas for future research *e.g.* the link between correlation of scores and copulas or the subject of partial rankings. However, the question of how to weight and integrate impacts on the different ranks is not trivial. After all, if one needs to know the proper set of weights before one can choose the proper measure of rank correlation, the strategy of avoiding bias seems circular.

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