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Corrigendum to "Resource-Monotonicity for House Allocation Problems"

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Abstract

Ehlers and Klaus (2003) study so-called house allocation problems and claim to characterize all rules satisfying *efficiency*, *independence of irrelevant objects*, and *resource-monotonicity* on two preference domains (Ehlers and Klaus, 2003, Theorem 1). They explicitly prove Theorem 1 for preference domain $\mathcal{R}_0$ which requires that the null object is always the worst object and mention that the corresponding proofs for the larger domain $\mathcal{R}$ of unrestricted preferences "are completely analogous."

Quesada (2009) in a recent working paper claims to have found a counterexample that shows that Theorem 1 is not correct on the unrestricted domain $\mathcal{R}$. In Lemma 1, we prove that Quesada's (2009) example in not a counterexample to Ehlers and Klaus (2003, Theorem 1). However, in Example 1 and Lemma 2, we demonstrate how to adjust Quesada's (2009) original idea to indeed establish a counterexample to Ehlers and Klaus (2003, Theorem 1) on the general domain $\mathcal{R}$.

Quesada (2009) also proposes a way of correcting the result on the general domain $\mathcal{R}$ by strengthening *independence of irrelevant objects* in two ways: in addition to requiring that the chosen allocation should depend only on preferences over the set of available objects (which always includes the null object), he adds two situations in which the allocation should also be invariant when preferences over the null object change. We here demonstrate that it is sufficient to require only one of Quesada's (2009) additional independence requirements to reestablish the result of Theorem 1 on the general domain $\mathcal{R}$.

Finally, while Quesada (2009) essentially replicates the original proofs of Ehlers and Klaus (2003) using his stronger independence condition, here we offer a short proof that uses the established result of Theorem 1 for the restricted domain $\mathcal{R}_0$.

JEL Classification: D63, D70

Keywords: corrigendum, indivisible objects, resource-monotonicity.

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1 House Allocation with Variable Resources

This section equals Section 2 in Ehlers and Klaus (2003). It is included in this working paper version for completeness (so that the reader does not necessarily have to look up Ehlers and Klaus, 2003).

Let N denote a finite set of agents, $|N| \geq 2$. Let K denote a set of potential real objects. Not receiving any real object is called "receiving the null object." Let 0 represent the *null object*. Each agent $i \in N$ is equipped with a preference relation R_i over all objects $K \cup \{0\}$. Given $x, y \in K \cup \{0\}$, $x \; R_i \; y$ means that agent i weakly prefers x to y, and $x \; P_i \; y$ means that agent i strictly prefers x to y. We assume that R_i is strict, *i.e.*, R_i is a linear order over $K \cup \{0\}$. Let $\mathcal{R}$ denote the class of all linear orders over $K \cup \{0\}$, and $\mathcal{R}^N$ the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $K' \subseteq K \cup \{0\}$, let $R_i|_{K'}$ denote the restriction of R_i to K' and $R|_{K'} = (R_i|_{K'})_{i \in N}$. Let $\mathcal{R}_0 \subsetneq \mathcal{R}$ denote the class of preference relations where the null object is the worst object. That is, if $R_i \in \mathcal{R}_0$, then all real objects are "goods": for all $x \in K$, $x \; P_i \; 0$.

An *allocation* is a list $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in K \cup \{0\}$, and none of the real objects in K is assigned to more than one agent. Note that 0, the null object, can be assigned to any number of agents and that not all real objects have to be assigned. Let $\mathcal{A}$ denote the set of all allocations. Let $\mathcal{H}$ denote the set of all non-empty subsets H of K. A *(house allocation) problem* consists of a preference profile $R \in \mathcal{R}^N$ and a set of real objects $H \in \mathcal{H}$. Note that the associated set of *available objects* $H \cup \{0\}$ includes the null object which is available in any economy. An *(allocation) rule* is a function $\varphi : \mathcal{R}^N \times \mathcal{H} \to \mathcal{A}$ such that for all problems $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, $\varphi(R, H) \in \mathcal{A}$ is *feasible*, *i.e.*, for all $i \in N$, $\varphi_i(R, H) \in H \cup \{0\}$. By feasibility, each agent receives an available object. Given $i \in N$, we call $\varphi_i(R, H)$ the *allotment* of agent i at $\varphi(R, H)$.

A natural requirement for a rule is that the chosen allocation depends only on preferences over the set of available objects.

Independence of Irrelevant Objects: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ and all $R' \in \mathcal{R}^N$ such that $R|_{H \cup \{0\}} = R'|_{H \cup \{0\}}$, $\varphi(R, H) = \varphi(R', H)$.

Next, a rule chooses only (Pareto) efficient allocations.

Efficiency: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, there is no feasible allocation $a \in \mathcal{A}$ such that for all $i \in N$, $a_i \; R_i \; \varphi_i(R, H)$, with strict preference holding for some $j \in N$.

When the set of objects varies, another natural requirement is *resource-monotonicity*. It describes the effect of a change in the available resource on the welfare of the agents. A rule satisfies *resource-monotonicity*, if after such a change, either all agents (weakly) lose or all (weakly) gain.

It is easy to see that in combination with *efficiency*, *resource-monotonicity* means that if for some fixed preference profile and some fixed set of objects, additional objects are available, then – this being good news – all agents (weakly) gain. Since we study *resource-monotonicity* together with *efficiency* we use the latter notion to formalize *resource-monotonicity*.

Resource-Monotonicity: For all $R \in \mathcal{R}^N$ and all $H, H' \in \mathcal{H}$, if $H \subseteq H'$, then for all $i \in N$, $\varphi_i(R, H') \, R_i \, \varphi_i(R, H)$.

2 Mixed Dictator-Pairwise-Exchange Rules

This section equals the first part of Section 3 in Ehlers and Klaus (2003) (until the statement of Theorem 1). It is included in this working paper version for completeness (so that the reader does not necessarily have to look up Ehlers and Klaus, 2003).

Our aim is to describe the class of rules that are *efficient*, *independent of irrelevant objects*, and *resource-monotonic*. We show that each such rule allocates the available objects in a sequence of steps as follows: At the first step there is either an agent who receives for all problems his most preferred object from the set of available objects—we call such an agent a dictator—or there are exactly two agents who partition the set of available real objects among themselves and for all problems their allotments result from a pairwise exchange using this partition as endowments. At the second step there is again either a dictator or a pairwise exchange (restricted to the remaining available objects); and so on. Here, we call such a rule a *mixed dictator-pairwise-exchange rule* (Ehlers, 2002). In Ehlers, Klaus, and Pápai (2002) we discuss essentially the same class of rules under the name "restricted endowment inheritance rules." After formally defining mixed dictator-pairwise-exchange rules, we briefly discuss another interpretation of these rules as so-called efficient priority rules.

For the formal description we use "(endowment) inheritance tables" (Pápai, 2000). For each real object $x \in K$, a one-to-one function $\pi_x : \{1, \ldots, |N|\} \to N$ specifies the inheritance of object x. Here agent $\pi_x(1)$ is initially endowed with x. If x is still available after $\pi_x(1)$ received an object, then agent $\pi_x(2)$ inherits x; and so on. Let Π^N denote the set of all one-to-one functions from $\{1, \ldots, |N|\}$ to N. An *inheritance table* is a profile $\pi = (\pi_x)_{x \in K}$ specifying the inheritance of each real object. We call an inheritance table π a *mixed dictator-pairwise-exchange inheritance table with respect to* $S = (S_1, S_2, \ldots, S_m)$ if

(i) $(S_1, S_2, \ldots, S_m)$ is a partition of N into singletons and pairs, *i.e.*, for all $t \in \{1, \ldots, m\}$, $2 \geq |S_t| \geq 1$,

(ii) row 1 and row $|S_1|$ of the inheritance table contain exactly S_1, *i.e.*, $\{\pi_x(1) \mid x \in K\} = S_1$ and $\{\pi_x(|S_1|) \mid x \in K\} = S_1$, and

(iii) row $(1 + \sum_{l=1}^{t-1} |S_l|)$ and row $(\sum_{l=1}^{t} |S_l|)$ contain exactly S_t, *i.e.*, for all $t \in \{2, \ldots, m\}$, $\{\pi_x(1 + \sum_{l=1}^{t-1} |S_l|) \mid x \in K\} = S_t$ and $\{\pi_x(\sum_{l=1}^{t} |S_l|) \mid x \in K\} = S_t$.

Given $i \in N$ and $H \in \mathcal{H}$, let $\text{top}(R_i, H)$ denote agent i's most preferred object under R_i in $H \cup \{0\}$.

Mixed Dictator-Pairwise-Exchange Rules, $\varphi^{(\pi,S)}$: Given a mixed dictator-pairwise-exchange inheritance table π with respect to $S = (S_1, S_2, \ldots, S_m)$, for all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ the allocation $\varphi^{(\pi,S)}(R, H)$ is inductively determined as follows:

Step 1:

(a) If $S_1 = \{i\}$, then $\varphi_i^{(\pi,S)}(R,H) = \text{top}(R_i,H)$.
(b) Let $S_1 = \{i,j\}$ $(i \neq j)$. If $\text{top}(R_i,H) = \text{top}(R_j,H) \equiv h_1 \in H$ and $\pi_{h_1}(1) = i$, then $\varphi_i^{(\pi,S)}(R,H) = h_1$ and $\varphi_j^{(\pi,S)}(R,H) = \text{top}(R_j, H\backslash\{h_1\})$. Otherwise, $\varphi_i^{(\pi,S)}(R,H) = \text{top}(R_i,H)$ and $\varphi_j^{(\pi,S)}(R,H) = \text{top}(R_j,H)$.

Step t: Let $H_{t-1} = \bigcup_{i \in \left(\cup_{l=1}^{t-1} S_l\right)} \{\varphi_i^{(\pi,S)}(R,H)\}$ denote the set of objects that are assigned up to Step t.

(a) If $S_t = \{i\}$, then $\varphi_i^{(\pi,S)}(R,H) = \text{top}(R_i, H\backslash H_{t-1})$.
(b) Let $S_t = \{i,j\}$ $(i \neq j)$. If $\text{top}(R_i, H\backslash H_{t-1}) = \text{top}(R_j, H\backslash H_{t-1}) \equiv h_t \in H\backslash H_{t-1}$ and $\pi_{h_t}(1 + \sum_{l=1}^{t-1} |S_l|) = i$, then $\varphi_i^{(\pi,S)}(R,H) = h_t$ and $\varphi_j^{(\pi,S)}(R,H) = \text{top}(R_j, H\backslash(H_{t-1} \cup \{h_t\}))$. Otherwise, $\varphi_i^{(\pi,S)}(R,H) = \text{top}(R_i, H\backslash H_{t-1})$ and $\varphi_j^{(\pi,S)}(R,H) = \text{top}(R_j, H\backslash H_{t-1})$.

Mixed dictator-pairwise-exchange rules are a subclass of endowment inheritance rules discussed in Ehlers, Klaus, and Pápai (2002).[1]

As already mentioned, instead of interpreting a mixed dictator-pairwise-exchange rule as an endowment inheritance rule, one can equivalently interpret it as a priority rule. Take the underlying inheritance table and use it as follows for each real object x: $\pi_x^{-1}(i) < \pi_x^{-1}(j)$ means "agent i has higher priority for object x than agent j." A rule violates the priority of agent i for object x if there is a preference profile under which i envies another agent j who obtains x even though i has a higher priority for x than j. A rule is a *priority rule* if it never violates the specified priorities. In an earlier version of his article, Ergin (2002) shows that efficient priority rules can be described through the so-called serial-bidictatorship algorithm, which for house allocation problems turns out to be equivalent to the algorithm underlying the corresponding mixed dictator-pairwise-exchange rules. For a further discussion of efficient priority rules we refer to Ehlers and Klaus (2002) and Ergin (2002).[2]

Our main result also applies to the domain $\mathcal{R}_0^N$ where all real objects are "goods".

Theorem 1. ***Ehlers and Klaus (2003, Theorem 1)***
Let $|K| > |N|$. On the domain $\mathcal{R}^N$ ($\mathcal{R}_0^N$), mixed dictator-pairwise-exchange rules are the only rules satisfying efficiency, independence of irrelevant objects, and resource-monotonicity.

[1] Endowment inheritance rules are based on *Gale's top trading cycle algorithm.* We omit their somewhat tedious definition and refer the interested reader to Pápai (2000). It is interesting to note that on the domain $\mathcal{R}_0^N$ some inheritance tables that do not satisfy the conditions of a mixed dictator-pairwise-exchange inheritance table may still generate an endowment inheritance rule that equals a mixed dictator-pairwise-exchange rule.

[2] For a more detailed discussion and comparison of endowment inheritance and priority rules see Kesten (2006).

3 Problems Concerning Ehlers and Klaus's (2003) Theorem 1 on Domain $\mathcal{R}$

3.1 Counterexamples to Theorem 1

We would like to note that Quesada (2009) uses very different notation and terminology compared to Ehlers and Klaus (2003). Since this correction will likely be read as an addition or complement to Ehlers and Klaus (2003), we return to the notation and terminology as used in Ehlers and Klaus (2003) and adjust Quesada's (2009) notation and terminology accordingly (except when quoting from Quesada, 2009).

In a recent working paper Quesada (2009) claims to have found a counterexample that shows that Ehlers and Klaus's (2003) Theorem 1, Theorem 1 for short, is not correct on the unrestricted domain $\mathcal{R}$. We quote from Quesada (2009, p. 2):

> "This characterization does not appear to be correct for the domain $\mathcal{R}$. To see this, consider the allocation rule f defined as follows: with $N = \{1, 2, 3\}$ and $x \in A$, the hierarchy of dictators 1–2–3 determines the allocation except if 2 prefers the null object to x and x is what both 1 and 3 prefer most, in which case 3 receives x and the hierarchy 1–2 determines the rest of the allocation. Whereas f satisfies independence, efficiency and monotonicity, it does not have a hierarchy of diarchies."[3]

First, in Lemma 1, we show that the rule f that Antonio Quesada describes above is not a counterexample to Theorem 1 on the general domain $\mathcal{R}$. However, Antonio Quesada did correctly suspect that there are problems with Theorem 1 on the general domain $\mathcal{R}$; in Example 1 we demonstrate how to adjust Quesada's original idea to indeed establish a counterexample to Theorem 1 on the general domain $\mathcal{R}$. In Lemma 2 we prove that our correction of Quesada's (2009) example is the "desired" counterexample that Quesada might have had in mind.

Lemma 1. *Quesada's rule violates resource-monotonicity*
Rule f as introduced by Quesada (2009, p. 2) violates resource-monotonicity.

Proof. Let $N = \{1, 2, 3\}$ and $x \in K$. Quesada's (2009) rule f is defined as follows: for all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$,

(a) if agent 2 prefers the null object to x and x is what agents 1 and 3 prefer most, the serial dictatorship corresponding to the order $(3, 1, 2)$ is used; and
(b) otherwise the serial dictatorship corresponding to the order $(1, 2, 3)$ is used.

Consider preferences R such that (a) applies and more specifically $x\, P_1\, 0 \ldots$, $0\, P_2\, x \ldots$, and $x\, P_3\, 0 \ldots$, i.e., object x and the null object are the two top ranked alternatives for all agents. Then, $f(R, \{x\}) = (0, 0, x)$ and agent 3 receives object x.

[3] With "hierarchy of diarchies" Antonio Quesada refers to our mixed dictator-pairwise-exchange rules.

Next, consider an object $y \in K \setminus \{x\}$ (note that such an object exists since the number of objects is assumed to be larger than the number of agents) and consider preferences R' such that (b) applies and $x\, P'_1\, 0\, P'_1\, y \ldots$, $y\, P'_2\, 0\, P'_2\, x \ldots$, and $y\, P'_3\, x\, P'_3\, 0 \ldots$, i.e., objects x, y, and the null object are the three top ranked alternatives for all agents. Note that now agent 3 now does not rank object x as his most preferred object and therefore the serial dictatorship corresponding to the order $(1,2,3)$ is used. Thus, $f(R', \{x,y\}) = (x, y, 0)$. Next, observe that $R|_{\{x,0\}} = R'|_{\{x,0\}}$. Hence, by *independence of irrelevant objects*, $f(R,\{x\}) = f(R',\{x\}) = (0,0,x)$. Now note that when adding object y to the problem $(R', \{x\})$, agent 3 switches from receiving object x to receiving the null object. Formally, $x = f_3(R',\{x\})\, P_3\, f_3(R', \{x,y\}) = 0$; a violation of *resource-monotonicity*. □

Example 1. Let $N = \{1,2,3\}$, $x \in K$ and $|K| \geq 2$. Rule φ is defined as follows: for all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$,

(a) if $x \in H$, $x\, P_2\, 0$, and x is what agents 1 and 3 prefer most in H, the serial dictatorship corresponding to the order $(3,1,2)$ is used; and
(b) otherwise the serial dictatorship corresponding to the order $(1,3,2)$ is used.

Note that φ is not a mixed dictator-pairwise-exchange rule because for any problem (R,H) such that $x \in H$ is what agents 1 and 3 prefer most in H, we have $\varphi_3(R,H) = x$ if $x\, P_2\, 0$ and $\varphi_1(R,H) = x$ if $0\, P_2\, x$.

Lemma 2. ***A counterexample to Theorem 1 on Domain*** $\mathcal{R}$
Rule φ as defined in Example 1 satisfies efficiency, independence of irrelevant objects, *and* resource-monotonicity.

Proof. It is obvious that rule φ as defined in Example 1 satisfies *efficiency* and *independence of irrelevant objects*. In order to check *resource-monotonicity*, let $R \in \mathcal{R}^N$ and $H, H' \in \mathcal{H}$ be such that $H \subseteq H'$. If (R,H) and (R,H') are both of type (a) (or both of type (b)), then *resource-monotonicity* is satisfied by φ because serial dictatorship rules are *resource-monotonic*.

We are left with the case where (R,H) and (R,H') are not of the same "type" (a) or (b).

First, suppose that (R,H) is of type (a) and (R,H') is of type (b). Then, by definition of φ, $\varphi_3(R,H) = x$. Now, if $\varphi_3(R,H') = x$, then *resource-monotonicity* is satisfied because the order in the serial dictatorship between 1 and 2 is unchanged in (a) and (b). Otherwise $\varphi_3(R,H') \neq x$ and because (R,H') is of type (b), at $\varphi(R,H')$ agent 1 or agent 3 receives an object in $H' \setminus H$ and the other agent either receives x or also an object in $H' \setminus H$. Now, *resource-monotonicity* is satisfied because agents 1 and 3 are weakly better off and agent 2 can choose from a larger set of objects under (R,H') than under (R,H).

Second, suppose that (R,H) is of type (b) and (R,H') is of type (a). Then, by definition of φ, $\varphi_3(R,H') = x$. Since (R,H) is of type (b) and (R,H) not of type (a), we must have $x \in H' \setminus H$. Then, $\varphi_3(R,H')\, R_3\, \varphi_3(R,H)$ and both agents 1 and 2 can choose from a larger set of objects at (R,H') than at (R,H). Since the order in the serial dictatorship between 1 and 2 is unchanged in (a) and (b), *resource-monotonicity* is satisfied. □

3.2 Corrections of Theorem 1

Quesada (2009, p. 3) proposes a correction of Theorem 1 on Domain $\mathcal{R}$ that involves strengthening *independence of irrelevant objects*. This new independence axiom, referred to as SIN, is described as follows:

> "This note shows that the characterization can be restored by strengthening independence of irrelevant objects. The new independence axiom suggested (see SIN in Section 3) associates the same allocation to two preference profiles P and Q when: (i) P restricted to the set of objects $B \subseteq A$ to be assigned coincides with Q restricted to B; (ii) for agents receiving the null object 0 under P, the preference for 0 does not diminish when passing from P to Q; and (iii) for agents receiving some object under P, that object is still more preferred in Q than 0. If (ii) and (iii) hold, the changes in the preference profile caused by moving the null object up or down the preference rankings may be considered irrelevant. By (ii), no agent receiving 0 has increased the preference of some object over 0, so it is reasonable for those agents to still receive 0. And, by (iii), no agent receiving some object has increased the preference of the null object over the received object, so it is reasonable for those agents to receive the same object as before."

The definition of SIN adjusted to the notation of Ehlers and Klaus (2003) is given next.

For all $i \in N$, all $R_i \in \mathcal{R}$, all $H \in K$, and all $x \in H \cup \{0\}$, the *lower contour set of* $R_i|_{H\cup\{0\}}$ *at the null object* is $L(R_i|_{H\cup\{0\}}, 0) \equiv \{x \in H \cup \{0\} : 0 \; R_i|_{H\cup\{0\}} \; x\}$.

Quesada's SIN Property: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ and all $R' \in \mathcal{R}^N$ such that $R|_H = R'|_H$, if for all $i \in N$,

(i) $\varphi_i(R, H) = 0$ implies $L(R_i|_{H\cup\{0\}}, 0) \subseteq L(R'_i|_{H\cup\{0\}}, 0)$ and
(ii) $\varphi_i(R, H) \neq 0$ implies $\varphi_i(R, H) \; P'_i \; 0$,

then $\varphi(R, H) = \varphi(R', H)$.

After defining SIN Quesada (2009, p. 5) states:

> "SIN strengthens the independence of irrelevant objects condition in Ehlers and Klaus (2003, p. 547), which asserts that $Q|_{B^+} = P|_{B^+}$ implies $f(Q, B) = f(P, B)$.[4] SIN declares the preference for 0 to be irrelevant if: (i) agents receiving 0 consider 0 at least as preferred as before; and (ii) agents not receiving 0 consider the object received as preferred to 0."

First, we would like to remark that Quesada's claim that SIN strengthens the independence of irrelevant objects condition in Ehlers and Klaus (2003, p. 547) is only true for *efficient* rules.[5] Since *independence of irrelevant objects* is used together with *efficiency*, this is not

[4] In our notation: $R|_{H\cup\{0\}} = R'|_{H\cup\{0\}}$ implies $\varphi(R, H) = \varphi(R', H)$.

[5] Consider the following situation: $N = \{1\}$, $K = \{x, y, z\}$, $z \; P_1 \; 0 \; P_1 \; x \; P_1 \; y$ and $0 \; P'_1 \; x \; P'_1 \; y \; P'_1 \; z$. Note that $R|_{\{x,y,0\}} = R'|_{\{x,y,0\}}$. A rule φ that would assign $\varphi(R, \{x, y, 0\}) = x$ and $\varphi(R', \{x, y, 0\}) = y$ would violate *independence of irrelevant objects*, but since only objects that are worse than 0 are assigned, SIN is not violated.

really a problem. Furthermore, it turns out that requirement (i) in the definition of SIN is not necessary for the correction of Theorem 1 on Domain $\mathcal{R}$. We formulate *independence of irrelevant objects* with the additional SIN property (ii) as a separate property.

Strong Independence of Irrelevant Objects: For all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$ and all $R' \in \mathcal{R}^N$ such that

(i) $R|_{H\cup\{0\}} = R'|_{H\cup\{0\}}$ or

(ii) $R|_H = R'|_H$ and for all $i \in N$, $\varphi_i(R,H) \neq 0$ implies $\varphi_i(R,H)\, P'_i\, 0$,

$\varphi(R,H) = \varphi(R',H)$.[6]

We are now ready to present a correction of Theorem 1 on Domain $\mathcal{R}$.

Theorem 2. *Let $|K| > |N|$. On the domain $\mathcal{R}^N$, mixed dictator-pairwise-exchange rules are the only rules satisfying efficiency, strong independence of irrelevant objects, and resource-monotonicity.*

Our proof of this correction differs in two ways from the one that Quesada (2009) presents. First, we use a "weaker strengthening" of *independence of irrelevant objects* (under *efficiency*, Quesada's SIN property implies our *strong independence of irrelevant objects* property, which implies Ehlers and Klaus's (2003) *independence of irrelevant objects* property). Second, while Quesada (2009) essentially replicates the original proofs of Ehlers and Klaus (2003, Theorem 1) using SIN, here we offer a comparably short proof that uses the established result of Ehlers and Klaus (2003, Theorem 1) for the restricted domain $\mathcal{R}_0$.

Before proving Theorem 2, we establish two useful implications of *efficiency* and *resource-monotonicity*.

The following lemma states that *efficiency* and *resource-monotonicity* imply that taking out an unassigned object does not change the assigned allocation.

Lemma 3. *Let φ be an efficient and resource-monotonic rule and $(R,H) \in \mathcal{R}^N \times \mathcal{H}$. If for all $i \in N$, $\varphi_i(R,H) \neq x \in H$, then $\varphi(R,H) = \varphi(R, H \setminus \{x\})$.*

Proof. Suppose by contradiction that $(R,H) \in \mathcal{R}^N \times \mathcal{H}$ is such that for all $i \in N$, $\varphi_i(R,H) \neq x \in H$ and $\varphi(R,H) \neq \varphi(R, H \setminus \{x\})$. Note that object x is neither assigned at (R,H) nor at $(R, H\setminus\{x\})$, i.e., $\varphi(R,H)$ is feasible for $(R, H \setminus \{x\})$. Hence, by *efficiency*, there exists an agent $j \in N$ such that $\varphi_j(R, H\setminus\{x\})\, P_j\, \varphi_j(R,H)$; contradicting *resource-monotonicity*. □

The following lemma states that *efficiency* and *resource-monotonicity* imply that adding an object that none of the agents who are assigned objects would prefer will not change the assigned allocation for these agents. For $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, we denote the set of agents who are assigned objects in H at $\varphi(R,H)$ by $N^+(\varphi, R, H) \equiv \{i \in N : \varphi(R,H) \in H\}$.

Lemma 4. *Let φ be an efficient and resource-monotonic rule and $(R,H) \in \mathcal{R}^N \times \mathcal{H}$. Let $x \in K \setminus H$ and $R' \in \mathcal{R}^N$ such that $R'|_{H\cup\{0\}} = R|_{H\cup\{0\}}$ and for all $i \in N^+(\varphi,R,H)$, $\varphi_i(R,H)\, P'_i\, x$. Then, for all $i \in N^+(\varphi,R,H)$, $\varphi_i(R,H) = \varphi_i(R', H\cup\{x\})$.*

[6] Note that part (ii) is a stronger version of the property "Independence of Truncations" (Ehlers, 2008): $R|_H = R'|_H$ and for all $i \in N$ such that $\varphi_i(R,H) \neq 0$ and all $x \in H$, $x\, P'_i\, 0$ implies $x\, P_i\, 0$.

Proof. Let $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, $x \in K \setminus H$, and $R' \in \mathcal{R}^N$ such that $R'|_{H\cup\{0\}} = R|_{H\cup\{0\}}$ and for all $i \in N^+(\varphi, R, H)$, $\varphi_i(R, H)\ P'_i\ x$. By *resource-monotonicity*, for all $i \in N^+(\varphi, R, H)$, $\varphi_i(R', H \cup \{x\})\ R_i\ \varphi_i(R, H)$. Hence, object x at $(R', H \cup \{x\})$ is not assigned to any agent in $N^+(\varphi, R, H)$ (it might be assigned to any of the agents who did not receive an object). Since $\varphi(R, H)$ is *efficiency* and only agents in $N^+(\varphi, R, H)$ received any objects in H, the only way to satisfy *resource-monotonicity* is to not change the assigned allocation for agents in $N^+(\varphi, R, H)$, i.e., for all $i \in N^+(\varphi, R, H)$, $\varphi_i(R, H) = \varphi_i(R', H \cup \{x\})$. □

3.3 Proof of Theorem 2

Throughout the proof, when referring to Ehlers and Klaus (2003, Theorem 1), we refer to Ehlers and Klaus (2003, Theorem 1) on the domain $\mathcal{R}_0$.

It is easy to verify that mixed dictator-pairwise-exchange rules satisfy *efficiency*, *strong independence of irrelevant objects*, and *resource-monotonicity* since no more than two agents "trade" at any step. In proving the converse, let $|K| > |N|$ and let φ be a rule satisfying *efficiency*, *strong independence of irrelevant objects*, and *resource-monotonicity*.

By Ehlers and Klaus (2003, Theorem 1), φ equals a mixed dictator-pairwise-exchange rules on the domain $\mathcal{R}_0$. Hence, on the domain $\mathcal{R}_0$, $\varphi = \varphi^{(\pi,S)}$. Suppose, by contradiction, that on the general domain $\mathcal{R}$, $\varphi \neq \varphi^{(\pi,S)}$. Then, there exists $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ such that $\varphi(R, H) \neq \varphi^{(\pi,S)}(R, H)$. Note that by Ehlers and Klaus (2003, Theorem 1), $R \notin \mathcal{R}_0^N$.

Step 1. Let $j \in N^+(\varphi, R, H)$, i.e., $\varphi_j(R, H) \neq 0$. Hence, by *efficiency*, $\varphi_j(R, H)\ P_j\ 0$. Let R'_j be obtained from R_j by making the null object the worst object, i.e., $R'_j \in \mathcal{R}_0$ and $R|_H = R'|_H$. Let $R' = (R'_j, R_{-j})$.[7] By construction of R', (ii) in the definition of *strong independence of irrelevant objects* applies to $R, R' \in \mathcal{R}^N$ and $\varphi(R, H) = \varphi(R', H)$. By the definition of $\varphi^{(\pi,S)}$ it also follows (for $\varphi_j^{(\pi,S)}(R, H) \neq 0$ as well as for $\varphi_j^{(\pi,S)}(R, H) = 0$) that $\varphi^{(\pi,S)}(R, H) = \varphi^{(\pi,S)}(R', H)$. Hence, $\varphi(R', H) \neq \varphi^{(\pi,S)}(R', H)$. We repeat this type of preference transformation for all agents $i \in N^+(\varphi, R', H)$ and denote the resulting profile by $\hat{R}$. Hence, $(\hat{R}, H) \in \mathcal{R}^N \times \mathcal{H}$ is such that $\varphi(\hat{R}, H) \neq \varphi^{(\pi,S)}(\hat{R}, H)$ and for all $i \in N^+(\varphi, \hat{R}, H)$, $\hat{R}_i \in \mathcal{R}_0$. Note that by Ehlers and Klaus (2003, Theorem 1), $\hat{R} \notin \mathcal{R}_0^N$.

Step 2. By *efficiency*, $\varphi(\hat{R}, H) \neq \varphi^{(\pi,S)}(\hat{R}, H)$ implies that there exists an agent j such that $\varphi_j(\hat{R}, H)\ \hat{P}_j\ \varphi_j^{(\pi,S)}(\hat{R}, H)$. Since, $\varphi_j^{(\pi,S)}(\hat{R}, H)\ \hat{R}_j\ 0$, this implies $\varphi_j(\hat{R}, H)\ \hat{P}_j\ 0$. Thus, by construction of $\hat{R}$, $j \in N^+(\varphi, \hat{R}, H)$ and $\hat{R}_j \in \mathcal{R}_0$.

Assume, without loss of generality, that the set of agents who are assigned the null object at $\varphi(\hat{R}, H)$ equals $N^0(\varphi, \hat{R}, H) \equiv N \setminus N^+(\varphi, \hat{R}, H) = \{1, \dots, k\}$ ($k \geq 1$ follows from $\hat{R} \notin \mathcal{R}_0^N$).

Next, we show that it is without loss of generality to assume that there exist $a_1, \dots, a_k \in K \setminus H$. If $|K \setminus H| < k$, then remove all objects from H that are not assigned at $\varphi(\hat{R}, H)$. Denote the remaining set of objects by $\hat{H}$ (note that $\hat{H} \subsetneq H$) and the new problem by $(\hat{R}, \hat{H})$. By Lemma 3, $\varphi(\hat{R}, H) = \varphi(\hat{R}, \hat{H})$. By $\hat{H} \subsetneq H$ and *resource-monotonicity*, for

[7] We use the notation $R_{-j} = R_{N\setminus\{j\}}$.

all $i \in N$, $\varphi_i^{(\pi,S)}(\hat{R}, H)\ \hat{R}_i\ \varphi_i^{(\pi,S)}(\hat{R}, \hat{H})$. Recall that there exists an agent j such that $\varphi_j(\hat{R}, H)\, \hat{P}_j\, \varphi_j^{(\pi,S)}(\hat{R}, H)$. Hence, $\varphi_j(\hat{R}, \hat{H}) = \varphi_j(\hat{R}, H)\, \hat{P}_j\, \varphi_j^{(\pi,S)}(\hat{R}, H)\, \hat{R}_j\, \varphi_j^{(\pi,S)}(\hat{R}, \hat{H})$. Thus, $\varphi_j(\hat{R}, \hat{H}) \hat{P}_j \varphi_j^{(\pi,S)}(\hat{R}, \hat{H})$ and $\varphi(\hat{R}, \hat{H}) \neq \varphi^{(\pi,S)}(\hat{R}, \hat{H})$. Since we have assumed that $|K| > |N|$, $a_1, \ldots, a_k \in K \setminus \hat{H}$ exist.

Hence, without loss of generality, we now consider $(\hat{R}, H)$ and $j \in N^+(\varphi, \hat{R}, H)$ with $\varphi_j(\hat{R}, H)\ \hat{P}_j\ \varphi_j^{(\pi,S)}(\hat{R}, H)$, $N^0(\varphi, \hat{R}, H) = \{1, \ldots, k\}$, and $a_1, \ldots, a_k \in K \setminus H$.

First, we focus on agent 1. Recall that $\varphi_1(\hat{R}, H) = 0$. Let $R^1 \in \mathcal{R}^N$ be such that $R^1|_{H\cup\{0\}} = \hat{R}|_{H\cup\{0\}}$ and preferences over a_1 are as follows:

(i) agent 1 ranks a_1 just above the null object, i.e., for all $x \in H$, $x\, \hat{P}_1\, 0$ implies $x\, P_1^1\, a_1\, P_1^1\, 0$,

(ii) if $i \in N^0(\varphi, \hat{R}, H) \setminus \{1\}$, then object a_1 is the worst object, i.e., for all $x \in H \cup \{0\}$, $x\ P_i^1\ a_1$, and

(iii) if $i \in N^+(\varphi, \hat{R}, H)$, then object a_1 is the worst object before the null object, i.e., for all $x \in H$, $x\ P_i^1\ a_1\ P_i^1\ 0$ (hence, $R_i^1 \in \mathcal{R}_0$ for agents in $N^+(\varphi, \hat{R}, H)$).

Let $H^1 \equiv H \cup \{a_1\}$ and consider (R^1, H^1). Agent 1 is the only agent who would prefer object a_1 to the object he receives at $\varphi(\hat{R}, H)$ (the null object). Hence, by *efficiency* and *resource-monotonicity*, $\varphi_1(R^1, H^1) = a_1$. By Lemma 4, for all $i \in N^+(\varphi, \hat{R}, H)$, $\varphi_i(R^1, H^1) = \varphi_i(\hat{R}, H)$. In particular, since $j \in N^+(\varphi, \hat{R}, H)$, $\varphi_j(R^1, H^1) = \varphi_j(\hat{R}, H)$.

Note that by the definition of $\varphi^{(\pi,S)}$, for all $i \in N \setminus \{1\}$, object a_1 is the worst object or it is the second worst object with the null object ranked immediately after it as worst object. Hence, for all $i \in N^+(\varphi^{(\pi,S)}, \hat{R}, H) \setminus \{1\}$, $\varphi_i^{(\pi,S)}(R^1, H^1)\ P_i^1\ a_1$. Furthermore, if $1 \in N^+(\varphi^{(\pi,S)}, \hat{R}, H)$, then by the definition of R^1 (i), $\varphi_1^{(\pi,S)}(R^1, H^1)\ P_1^1\ a_1$. Hence, by Lemma 4, for all $i \in N^+(\varphi^{(\pi,S)}, \hat{R}, H)$, $\varphi_i^{(\pi,S)}(R^1, H^1) = \varphi_i^{(\pi,S)}(\hat{R}, H)$.

If $j \in N^+(\varphi^{(\pi,S)}, \hat{R}, H)$, then $\varphi_j^{(\pi,S)}(R^1, H^1) = \varphi_j^{(\pi,S)}(\hat{R}, H)$. Hence, $\varphi_j(R^1, H^1) = \varphi_j(\hat{R}, H)\ \hat{P}_j\ \varphi_j^{(\pi,S)}(\hat{R}, H) = \varphi_j^{(\pi,S)}(R^1, H^1)$. Thus, $\varphi_j(R^1, H^1)\ P_j^1\ \varphi_j^{(\pi,S)}(R^1, H^1)$.

If $j \notin N^+(\varphi^{(\pi,S)}, \hat{R}, H)$, then $\varphi_j^{(\pi,S)}(R^1, H^1) \in \{a_1, 0\}$ (since above we have shown that for all $i \in N^+(\varphi^{(\pi,S)}, \hat{R}, H)$, $\varphi_i^{(\pi,S)}(R^1, H^1) = \varphi_i^{(\pi,S)}(\hat{R}, H)$). By the definition of R^1 (iii), $\varphi_j(\hat{R}, H)\ P_j^1\ a_1\ P_j^1\ 0$, which implies $\varphi_j(\hat{R}, H)\ P_j^1\ \varphi_j^{(\pi,S)}(R^1, H^1)$. Hence, $\varphi_j(R^1, H^1) = \varphi_j(\hat{R}, H)\ P_j^1\ \varphi_j^{(\pi,S)}(R^1, H^1)$. Thus, $\varphi_j(R^1, H^1)\ P_j^1\ \varphi_j^{(\pi,S)}(R^1, H^1)$.

To summarize, we have constructed a problem (R^1, H^1) with $j \in N^+(\varphi, R^1, H^1)$ such that $\varphi_j(R^1, H^1)\, P_j^1\, \varphi_j^{(\pi,S)}(R^1, H^1)$, and $|N^0(\varphi, R^1, H^1)| = |N^0(\varphi, \hat{R}, H)| - 1$. If $|N^0(\varphi, R^1, H^1)| > 0$, we repeat this construction $k-1$ times (adding objects $a_2, \ldots, a_k$ in a similar manner) to obtain a problem (R^k, H^k) with $j \in N^+(\varphi, R^k, H^k)$ such that $\varphi_j(R^k, H^k)\ P_j^k\ \varphi_j^{(\pi,S)}(R^k, H^k)$, and $|N^0(\varphi, R^k, H^)| = 0$.

Step 3. Note that by construction of problem (R^k, H^k), $N = N^+(\varphi, R^k, H^k)$, $\varphi(R^k, H^k) \neq \varphi^{(\pi,S)}(R^k, H^k)$, and $R^k \notin \mathcal{R}_0^N$. Now, we repeat Step 1 for all agents in N who do not yet rank the null object as their worst object. Using the same arguments as in Step 1, we then obtain $(\hat{R}^k, H^k) \in \mathcal{R}_0^N \times \mathcal{H}$ such that $\varphi(\hat{R}^k, H^k) \neq \varphi^{(\pi,S)}(\hat{R}^k, H^k)$. However, $\hat{R}^k \in \mathcal{R}_0$ and $\varphi(\hat{R}^k, H^k) \neq \varphi^{(\pi,S)}(\hat{R}^k, H^k)$ contradict Ehlers and Klaus (2003, Theorem 1). □

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