# Numerical Simulation of Nonoptimal Dynamic Equilibrium Models*

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**Abstract**

In this paper we present a recursive method for the computation of dynamic competitive equilibria in models with heterogeneous agents and market frictions. This method is based upon a convergent operator over an expanded set of state variables. The fixed point of this operator defines the set of all Markovian equilibria. We study approximation properties of the operator as well as the convergence of the moments of simulated sample paths. We apply our numerical algorithm to two growth models, an overlapping generations economy with money, and an asset pricing model with financial frictions.

KEYWORDS: Heterogeneous agents, taxes, externalities, financial frictions, competitive equilibrium, computation, simulation.

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# 1 Introduction

In this paper we present a recursive method for the computation of sequential competitive equilibria for dynamic economic models in which the welfare theorems may fail to hold because of the presence of incomplete agents' participation, taxes, externalities, incomplete financial markets, and other financial frictions. These models have become central to analyze the effects of various macroeconomic policies, the evolution of wealth and income distribution, and the variability of asset prices. However, computation of their equilibrium solutions may be a formidable task. Indeed, dynamic programming arguments may fail to apply, and a continuous Markov equilibrium may not exist. Therefore, existing numerical techniques cannot be readily extended to non-optimal economies.

We shall address the following issues for the computation and simulation of dynamic equilibrium solutions: *(i) Existence: Lack of Markov equilibria.* Even though the model may have a recursive structure, a Markovian equilibrium may not exist – or no Markov equilibrium may be continuous – over a natural space of state variables. We prove existence of a Markov equilibrium over an expanded state space. *(ii) Computation: Non-convergence of the algorithm.* Backward iteration over a candidate equilibrium function may not reach a Markovian equilibrium solution. Contraction arguments underlying dynamic programming methods usually break down for non-optimal economies. We prove convergence of our algorithm to a fixed-point solution that can generate all sequential competitive equilibria. *(iii) Approximation: Accuracy properties of the computed solution.* Approximation errors may cumulate over time. Consequently, as we refine the approximation we need to ensure that discretized versions of the algorithm approach an exact solution. Again, contraction arguments cannot be invoked to guarantee good approximation properties of the algorithm. We establish convergence of the computed solution to the set of competitive equilibria. *(iv) Simulation: Convergence of the moments from sample paths.* Standard laws of large numbers require certain regularity conditions – such as continuity of the law of motion – that would be rather imposing for the equilibria of these economies. We present a discretized method in which the moments from sample paths approach the set of moments of the invariant distributions of the model.

In dynamic competitive-markets economies with frictions the existence of Markovian equilibria has been well established under certain monotonicity properties on the equilibrium dynamics [e.g., see Bizer and Judd (1989), Coleman (1991), and Datta, Mirman and Reffett (2002)]. But existence of Markov equilibria remains largely unexplored in many other models in which these monotonicity conditions may not be satisfied. Regular examples of non-existence of Markovian equilibria have been found in one-sector growth models with taxes and externalities [Santos (2002)], in exchange economies with incomplete financial markets [Krebs (2004) and Kubler and Schmedders (2002)], and in overlapping generations (OLG) economies [Kubler and Polemarchakis (2004)]. For the canonical one-sector growth model with taxes and externalities, monotonicity conditions follow from fairly mild restrictions on the primitives, but monotone dynamics are much harder to obtain in multi-sector models with heterogeneous agents and incomplete financial markets. Duffie *et al.* (1994) dispense with such monotonicity requirements by expanding the state space with endogenous variables such as asset prices and individual consumptions. By a suitable randomization of the equilibrium correspondence [Blume (1982)] they then prove the existence of an ergodic invariant distribution for a wide class of discrete-time infinite-horizon models with exogenous short-sale constraints on asset holdings. Building on these methods, Kubler and Schmedders (2003) prove the existence of a Markovian equilibrium for a class of financial economies with collateral requirements.

We extend this existence result to various types of economies. Our state space includes agents' shadow values of investment. This choice of the state space seems suitable for computation. The set of all Markov equilibria can be characterized as the fixed-point solution of a convergent iterative procedure. (A key factor of convergence is that our operator is acting over candidate equilibrium sets on a compact domain.) Then, we develop a computable version of the theoretical algorithm. This numerical algorithm is shown to approximate the original fixed-point solution. Moreover, the moments derived from simulated paths of the computed solution converge to a set of moments of the invariant distributions of the model. We apply our methods to two growth economies, a stochastic OLG economy with money, and an asset pricing model with incomplete financial markets and heterogeneous agents. We illustrate the applicability of our algorithm by comparing our numerical solution with those generated from some other standard methods. These other methods may display low accuracy properties, fail to converge to the equlibrium solutions, or capture only one of the possible existing equilibria.

The computation of competitive equilibria for non-optimal economies has been of considerable interest in macroeconomics and finance [e.g., Castaneda, Diaz-Gimenez and Rios-Rull (2003), Krusell and Smith (1998), Heaton and Lucas (1996), Marcet and Singleton (1999), and Rios-Rull (1999)], but most of this literature does not deal with the problem of existence of a Markovian equilibrium. Kubler and Schmedders (2003) refine the analysis of Duffie *et al.* (1994) and develop a reliable computational algorithm over an expanded state space. But in the implementation of this algorithm they iterate over continuous equilibrium functions, and such iteration process does not guarantee convergence to a fixed-point solution. Also, their state space includes additional variables which seem hard to compute, and so their algorithm may not be computationally efficient.

The idea of enlarging the state space with the shadow values of investment was first suggested by Kydland and Prescott (1980) in their seminal study of time inconsistency. Abreu, Pierce and Stacchetti (APS, 1990) use a similar approach for the computation of sequential perfect equilibria in which they expand the state space with continuation utility values. The analyses of Kydland and Prescott and APS have been extended in several directions involving strategic decisions [e.g., Atkenson (1991), Chang (1998), Judd, Yeltekin and Conklin (2003), Marcet and Marimon (1998) and Phelan and Stacchetti (2001)], but none of these papers are concerned with the computation of sequential equilibria for competitive-market economies with heterogeneous agents. To the best of our knowledge, the only related paper is Miao (2006) who sets forth a recursive solution method for the model of Krusell and Smith (1998). However, as in the original APS approach Miao's state space includes expected continuation utilities over the set of sequential competitive equilibria, and this choice of the state space does not seem operative for the computation of equilibrium solutions in the present framework.

Finally, for nonoptimal economies convergence properties of numerical algorithms and convergence of the simulated moments remain largely unexplored. As already discussed, Duffie *et al.* (1994) show existence of an ergodic distribution (which validates a law of large numbers for these economies). This result is not practical for computational purposes as it is usually hard to locate the ergodic set. In the absence of continuity of the equilibrium law of motion, other ways to validate laws of large numbers for these economies would be to resort to monotonicty assumptions on the equilibrium dynamics [Santos (2008)] or to artificial expansions of the noise space [Blume (1979)]. These latter approaches seem less attractive for these economies.

We proceed as follows. In Section 2 we present our general framework and lay out our theoretical algorithm. Section 3 studies the numerical implementation of our algorithm and its convergence properties. Sections 4-6 explore the existence and computation of recursive equilibria for various families of models. We conclude in Section 7.

# 2 General Theory

In this section, we first set out a general analytic framework that encompasses various competitive equilibrium models. We then present our numerical approach and main results on existence and global convergence to the Markovian equilibrium correspondence.

## 2.1 The Analytical Framework

Time is discrete, $t = 0, 1, 2, \cdots$ . The state of the economy includes a state vector of endogenous variables $x$ and vector of exogenous shocks $z$. Vector $x$ belongs to a compact domain $X$ and contains all predetermined variables, such as agents' holdings of physical capital, human capital, and financial assets. The exogenous state vector follows a Markov chain $(z_t)_{t\geq 0}$ over a finite set $Z$. This Markovian process is described by positive transition probabilities $\pi(z'|z)$ for all $z, z' \in Z$. The initial state, $z_0 \in Z$, is known to all agents in the economy. Then $z^t = (z_1, z_2, ..., z_t) \in Z^t$ is a history of shocks, often called a date-event or node. Let $y$ denote the vector of all other endogenous variables. These variables could be equilibrium prices or choice variables such as consumption and investment.

In all our models the dynamics of the state vector $x$ is conformed by a system of non-linear equations:

$$\varphi(x_{t+1}, x_t, y_t, z_t) = 0. \tag{2.1}$$

Function $\varphi$ incorporates technological constraints as well as individual budget constraints. For some models, such as the growth models analyzed in Section 4, function $\varphi$ is known and we can explicitly solve for $x_{t+1}$ as a function of $(x_t, y_t, z_t)$. In other applications such as in various models with adjustment costs, vector $x_{t+1}$ may not admit an analytical representation.

Let $m$ denote a vector of shadow values of the marginal investment return for all assets and all agents. This vector lies in a compact space $M$, and it will be a function of existing variables such as prices, rates of interest, and marginal utilities and productivities:

$$m_t = h(x_t, y_t, z_t). \tag{2.2}$$

Let us assume that a sequential competitive equilibrium exists and can be represented by a sequence

$\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$ satisfying (2.1), (2.2) and the additional sytem of equations

$$\Phi\left(x_t, y_t, z_t, E_t\left[m_{t+1}\right]\right) = 0, \tag{2.3}$$

where $E[m]$ is the expectations operator. Function $\Phi$ may describe individual optimality conditions (such as Euler equations), market-clearing conditions, various types of budget restrictions, and resource constraints. We assume that equations (2.1)-(2.3) fully characterize any sequential competitive equilibrium, and that $\varphi, h$, and $\Phi$ are continuous functions.

## 2.2 Recursive Equilibrium Theory

In order to compute the set of equilibria for the model economy we define the equilibrium correspondence $V^*(x, z)$ containing all the equilibrium vectors $m$ for any given state $(x, z)$. From this correspondence $V^*$ we can generate recursively the set of sequential competitive equilibria as $V^*$ is the fixed point of an operator $B : V \longmapsto B(V)$ that links state variables to future equilibrium states. Operator $B$ embodies all equilibrium conditions such as agents' optimization and market-clearing conditions from any initial node $z$ to all immediate successor states $z_+$. This operator is analogous to the expectations correspondence defined in Duffie *et al*. (1994), albeit it is defined over a smaller set of endogenous variables.

More precisely, let $B(V)(x, z)$ be the set of all values $m = h(x, y, z)$ satisfying the following temporary equilibrium conditions: For given $x, z$ there exist $y$ and $m_+(z_+) \in V(x_+, z_+)$ with $z_+ \in Z$ such that

$$\Phi(x, y, z, \sum_{z_+ \in Z} \pi(z_+|z)\, m_+(z_+)) = 0,$$

and

$$\varphi(x_+, x, y, z) = 0. \tag{2.4}$$

Note that operator $B$ is well defined as a sequential competitive equilibrium is assumed to exist. Also, $B$ is monotone in the sense that if $V \subset V'$ then $B(V) \subset B(V')$.[1] Moreover, if $V$ has a closed

[1] For correspondences $V, V'$ we say that $V \subset V'$ if $V(x, z) \subset V'(x, z)$ for all $(x, z)$.

graph then $B(V)$ also has a closed graph since the above functions $\varphi, h, \Phi$ are all assumed to be continuous. Indeed, in all our models below operator $B$ satisfies the following

**Assumption 2.1** *Operator $B$ preserves compactnes in the sense that if $V$ is compact valued, then $B(V)$ is also compact valued.*

Using this assumption we can show existence of a fixed-point solution $V^*$ which is globally convergent for every initial guess $V_0 \supset V^*$. Convergence, should be understood as pointwise convergence[2] in the Hausdorff metric [e.g., see Hildenbrand (1974)]. If $V^*$ is a continuous correspondence then convergence will be uniform over all points $(x, z)$.

**Theorem 2.1** *(convergence) Let $V_0$ be a compact-valued correspondence such that $V_0 \supset V^*$. Let $V_n = B(V_{n-1}), n \geq 1$. Then, $V_n \to V^*$ as $n \to \infty$. Moreover, $V^*$ is the largest fixed point of the operator $B$, i.e., if $V = B(V)$, then $V \subset V^*$*

Theorem 2.1 provides the theoretical foundations of our algorithm. The iterative process starts as follows: For all $(x, z)$, pick a sufficiently large compact set $V_0(x, z) \supset V^*(x, z)$. Then apply operator $B$ to $V_0$ and iterate until a desirable level of convergence is attained. This is possible since $\lim_{n\to\infty} V_n$ equals the equilibrium correspondence $V^*$. An important advantage of our approach is that if there are multiple equilibria, we can find all of them. Finally, under assumption 2.1 by the measurable selection theorem [Hildenbrand (1974)] it follows that from operator $B$ we can select a measurable policy function $y = g^y(x, z, m)$, and a transition function $m_+(z_+) = g^m(x, z, m; z_+)$, for all $z_+ \in Z$. These functions give a Markovian characterization of a dynamic equilibrium in the enlarged state space.

Note that the equilibrium shadow value correspondence $V^*$ may not be single-valued; hence, there could be multiple equilibrium selections in which none of them is continuous. Moroever, there may not be an equilibrium function $y = g(x, z)$, and hence a simple recursive equilibrium may not exist.[3] Kubler and Schmedders (2002) construct an example economy with multiple equilibria. They show that the model does not admit a recursive solution $g(x, z)$:

[2] Later, we will establish uniform convergence of the simulated moments even though the equilibrium correspondence $V^*$ is only upper semicontinuous.

[3] Of course, if the competitive equilibrium is always unique then there is a continuous function $y = g(x, z)$.

$$\Phi(x, g(x,z), z, \sum_{z_+ \in Z} \pi(z_+|z)\, h(f(x, g(x,z), z), g(f(x, g(x,z)), z_+)) = 0. \qquad (2.5)$$

where $x_+ = f(x, y, z)$. Kubler and Schmedders (2003) propose a computation procedure to recover such Markov equilibria numerically by a related expansion of the state space. But their computational algorithm relies on the assumption that the policy correspondence is a continuous function. Their algorithm may fail if there are multiple equilibria or if the policy function is not continuous. Our approach overcomes this problem as we illustrate by the various examples in the coming sections.

# 3 Numerical Implementation

Numerical implementation of our theoretical results requires the construction of a computable algorithm that approximates the fixed point of operator $B$. In this section we develop and study properties of such an algorithm.

We first partition the state space into a finite set of simplices $\{X^j\}$ with non-empty interior and maximum diameter $h$. Over this partition we define a family of step correspondences which take constant values over each $X^j$. To obtain a computer representation of a step correspondence we resort to an outer approximation in which each set-value is defined by $N$ elements. Using these two simplifications we get a discretized version of operator $B$, which we denote by $B^{h,N}$. By a suitable selection of an initial condition $V_0$, the sequence $\{V_{n+1}^{h,N}\}$ defined recursively as $V_{n+1}^{h,N} = B^{h,N} V_n^{h,N}$ converges to a limit point $V^{*,h,N}$ containing the equilibrium correspondence $V^*$. Moreover, the sequence of fixed points $\{V^{*,h,N}\}$ approaches the equilibrium correspondence $V^*$ as the accuracy of the discretizations goes to the limit. It should be understood that convergence is uniform in economies where the equilibrium corespondence is continuous. At a later stage, we address the issue of convergence of the moments obtained from simulations of our numerical approximations. This problem has been hardly addressed in the literature, and again it has to cope with the fact that the equilibrium correspondence may not be continuous.

### 3.1 The Numerical Algorithm

Let $\{X^j\}$ be a finite family of simplices with non-empty interior such that $\cup_j X^j = X$ and $int(X^j) \cap int(X^i)$ is empty for every pair $X^i, X^j$. Define the *mesh size* $h$ of this discretization as

$$h = \max_j diam\left\{X^j\right\}.$$

Consider a correspondence $V : X \times Z \to 2^M$ that takes values in space $M$. Then, its step approximation $V^h$ over the partion $\{X^j\}$ takes constant set-values $V^h(x, z)$ on each simplex $X^j$ and is conformed by the union of sets $V(x, z)$ for $x \in X^j$ for given $z$. That is, for each $z$

$$V^h(x, z) = \cup_{x \in X^j} V(x, z). \tag{3.1}$$

Accordingly, we can define operator $B^h$ that takes a correspondence $V$ into the step correspondence $[B(V)]^h$. By similar arguments as above, we can prove that $B^h$ has a fixed point solution $V^{*h}$. Moreover, we shall soon clarify the sense in which the correspondence $V^{*h}$ constitutes an approximation to $V^*$.

As already mentioned, to obtain a computer representation of the step correspondence we also perform a discretization on the image space. We say that $\mathcal{C}^N(V(x, z)) \supseteq V(x, z)$ is an $N$-element outer approximation of $V(x, z)$ if $\mathcal{C}^N(V(x, z))$ can be generated by $N$ elements. In what follows we assume that this approximation satisfies a strong uniform convergence property.[4] Namely, for any $\varepsilon > 0$ there is $0 < N^* < \infty$ such that $d[\mathcal{C}^N(V(x, z)), V(x, z)] \leq \varepsilon$ for all $N > N^*$, and all $V(x, z)$. For instance, this later property can be satisfied if the outer approximation is generated by convex combinations of $N$ points as $M$ is a compact set.

We are now ready to put forward a computable version of operator $B$. That is, we can define a new operator $B^{h,N}$ that sends a correspondence $V$ to the step correspondence $[B(V)]^h$ and then each set-value is adjusted with the $N$-element outer approximation so as to get $\mathcal{C}^N\left([B(V)]^h\right)$. Sections 4 to 6 illustrate examples of this type of operators, and their application in different dynamic models. Let us first show that our discretized operator has good convergence properties: The fixed point of this operator $V^{*,h,N}$ contains the equilibrium correspondence $V^*$ and it approaches $V^*$ as we refine the discretizations. The proof of this result extends the convergence arguments of Beer (1980) to a

[4] Again, convergence should be understood in the Hausdorff metric $d$ (see *opt. cit.*).

dynamic setting.

**Theorem 3.1** *Suppose that* $V_0 \supseteq V^*$ *is an upper-semicontinuous correspondence. Consider the recursive sequence defined by* $V_{n+1}^{h,N} = B^{h,N} V_n^{h,N}$ *for given* $h$ *and* $N$ *and with initial condition* $V_0$*. Then: (i)* $V_n^{h,N} \supseteq V^*$ *for all* $n$*; (ii)* $V_n^{h,N} \to V^{*,h,N}$ *uniformly as* $n \to \infty$*; and (iii)* $V^{*,h,N} \to V^*$ *as* $h \to 0$ *and* $N \to \infty$*.*

The output of our numerical algorithm is summarized by the equilibrium correspondence $V_n^{h,N}$ from operator $B^{h,N}$. By Theorem 3.1, we have that $graph[\mathcal{C}^N \left([B(V_n^{h,N})]^h\right)]$ can be made arbitrarily close to $graph[B(V^*)]$ for appropriate choices of $n$, $h$, and $N$. As $graph[\mathcal{C}^N \left([B(V_n^{h,N})]^h\right)]$ is compact, by the measurable selection theorem [Hildenbrand (1974)] we can choose an approximate equilibrium selection $y = g_n^{y,h,N}(x, z, m)$, and a transition function $m_+ (z_+) = g_n^{m,h,N}(x, z, m; z_+)$. From these approximate equilibrium functions we can generate simulated paths $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$.

### 3.2 Convergence of the Simulated Moments

To assess model's predictions, analysts usually calculate moments of the simulated paths $\left(x_t(z^t), y_t(z^t)\right)_{t=0}^{\infty}$ from a numerical approximation. The idea is that the simulated moments should approach steady-state moments of the true model. Under continuity of the policy function, Santos and Peralta-Alva (2005) establish various convergence properties of the simulated moments. They also provide examples of non-existence of stochastic steady-state solutions for non-continuous functions, and lack of convergence of emprirical distributions to some invariant distribution of the model. Hence, it is not clear how economies with distortions should be simulated, since for these economies the continuity of the policy function does not usually follow from standard economic assumptions.

We now outline a reliable simulation procedure that circumvents the lack of continuity of the equilibrium law of motion. We append two further steps to the standard model simulation. First, we discretize the image space of the approximate equilibrium selection so that this function can take on a finite number of points. Then, the simulated moments are generated by a finite Markov chain that has an invariant distribution, and every empirical distribution from the simulated paths converges almost surely to some ergodic invariant distribution of the Markov chain. Second, following Blume (1982) and Duffie *et al*. (1994) we randomize over continuation values of operator $B$. We construct a new operator $B^{cv}$ that is a convex-valued correspondence in the space of probability measures. It follows then that there is an invariant distribution $\mu^* \in B^{cv}(\mu^*)$. Moreover, as we refine the approximations the simulated moments from our numerical approximations are shown to converge to the moments of some invariant distribution $\mu^*$.

$(i)$ *Discretization of the equilibrium law of motion*: In order to make the analysis more transparent, let $S = X \times M$. Let $\chi_n^{h,N} : S \times Z \to S \times Z$ be a selection from $graph[\mathcal{C}^N \left([B(V_n^{h,N})]^h\right)]$. Note that function $\chi_n^{h,N}$ is simply defined from the above functions $y = g_n^{y,h,N}(x,z,m)$, and $m_+(z_+) = g_n^{m,h,N}(x,z,m;z_+)$ and the law of motion for state variable $x$ as given by equation (1). Then, $\chi_n^{h,N}$ gives rise to a time-homogeneous Markov process $(s,z) \to s_+(z_+)$ for $s = (x,m)$ and all $z_+ \in Z$. Now, let $A_\gamma$ be a set with a finite number of points in $S$ such that $d(A_\gamma, S) < \gamma$ so that each point in $S$ is within a $\gamma$-ball of some point in $A$. Let $\chi_n^{h,N,A_\gamma}(s,z) = \arg min_{s_+ \in A_\gamma} d(s_+, \chi_n^{h,N}(s,z))$. If there are various solution points $s_+$ we just pick arbitrarily one solution $s_+$. Hence, the new discretized function $\chi_n^{h,N,A_\gamma}$ takes values in the finite set $A_\gamma \times Z$, and gives rise to a Markov chain that has an invariant distribution $\nu_n^{*,h,N,A_\gamma}$. Further, the moments of a simulated path $\left(s_t, z^t\right)_{t=0}^{\infty}$ converge almost surely to those of some ergodic invariant distribution $\nu_n^{*,h,N,A_\gamma}$ [e.g., see Stokey, Lucas and Prescott (1989), Ch. 11].

$(ii)$ *Randomization over continuation equilibrium sequences*: We can view operator $B : V^* \to V^*$ as a correspondence in the space of probability measures $\mu$ on $S \times Z$. That is, $\nu \in B(\mu)$ if there is a selection $\chi$ of $B$ such that $\nu = \chi \cdot \mu$, where $\chi \cdot \mu$ denotes the action of function $\chi$ on probability measure $\mu$ [e.g., see Stokey, Lucas and Prescott (1989)]. Following Blume (1982) and Duffie *et al.* (1994) we convexify the image of $B$. Thus, if $\nu$ and $\nu'$ are two probability measures that belong to the range of $B$ we assume that every convex combination $\lambda\nu + (1-\lambda)\nu'$ also belongs to the range of $B$. We let $B^{cv}$ denote the convexification[5] of operator $B$ over the space of probability measures $\mu$ on $S \times Z$. The new operator $B^{cv}$ is a convex-valued, upper semicontinuous correspondence. Since $S \times Z$ is assumed to be compact, the set of probability measures $\mu$ on $S \times Z$ is also compact in the weak topology of measures. Therefore, operator $B^{cv}$ has a fixed point solution; i.e., there exists an invariant probability, $\mu^* \in B^{cv}(\mu^*)$.

$(iii)$ *Convergence of the simulated moments to population moments of the model*: For given

[5] Duffie et *al*. (1994) argue that such convexification amounts to a weak form of sunspot equilibria since the randomization proceeds over equilibrium distributions rather than over an external parameter or extraneous sunspot variable.

function $\chi_n^{h,N,A_\gamma}$ and a randomly selected sequence $\left(z^t\right)_{t=0}^{\infty}$, we generate an approximate equilibrium path $(s_t)_{t=0}^{\infty}$. Let $f : S \times Z \to R_+$ be a function of interest. Then, $\frac{1}{T}\sum_{t=0}^{T} f(s_t, z_t)$ represents a simulated moment or some other statistic. Since $\chi_n^{h,N,A_\gamma}$ defines a Markov chain, it follows that $(s_t, z_t)_{t=0}^{\infty}$ must enter an ergodic set in finite time. Therefore, $\frac{1}{T}\sum_{t=0}^{T} f(s_t, z_t)$ must converge almost surely to $\int f(s,z)d\nu_n^{*,h,N,A_\gamma}$ as $T \to \infty$for some ergodic invariant distribution $\nu_n^{*,h,N,A_\gamma}$. We now link convergence of invariant distributions $\nu_n^{*,h,N,A_\gamma}$of numerical approximations to some invariant distribution of the original model $\mu^*$so that the simulated statistics converge almost surely to those of some invariant distribution $\mu^*$.

**Theorem 3.2** *Let* $\left(\nu_n^{*,h,N,A_\gamma}\right)$ *be a sequence of invariant distributions corresponding to functions* $\left(\chi_n^{h,N,A_\gamma}\right)$. *Then, every limit point of* $\left(\nu_n^{*,h,N,A_\gamma}\right)$ *converges weakly to some invariant distribution* $\mu^* \in B^{cv}(\mu^*)$.

To summarize our work in this section, convergence of the simulated moments involves discrete approximations over the following margins:

1. *Discretization of the domain*: $h$ mesh size of the family of simplices $\{X^j\}$.
2. *Discretization of the image*: $N$ number of elements in every outer approximation.
3. *Finite iterations*: $n$ number of iterations over operator $B^{h,N}$.
4. *Finite Markov chain*: $\gamma$ maximum distance of every point in $S$ to some point in set $A_\gamma$.
5. *Finite simulations*: $T$ lenght of a simulated path $(s_t, z_t)_{t\geq 0}$.
6. *Convexification of equilibrium distributions*: $B^{cv}$ regularized operator in the space of distributions with a convex image.

Thus, for every $\epsilon > 0$ we can make the aforementioned parameters sufficiently close to their respective limits so that for a given path $(s_t, z_t)_{t=0}^{\infty}$ generated under function $\chi_n^{h,N,A_\gamma}$, there are invariant distributions $\mu^*, \mu'^*$ of $B^{cv}$ such that $\int f(s,z)d\mu^* + \epsilon \leq \frac{1}{T}\sum_{t=0}^{T} f(s_t, z_t) \leq \int f(s,z)d\mu'^* - \epsilon$ almost surely. Therefore, for a sufficiently fine approximation the moments from simulated paths are close to the set of moments of the invariant distributions of the model. Of course, if $B^{cv}$ has a unique invariant distribution $\mu^*$ then $\mu'^* = \mu^*$ and the above expression reads as $\int f(s,z)d\mu^* + \epsilon \leq \frac{1}{T}\sum_{t=0}^{T} f(s_t, z_t) \leq \int f(s,z)d\mu^* - \epsilon$.

# 4 Non-Optimal Growth Models

In this section we present a standard stochastic growth model with taxes, heterogeneous agents, and incomplete markets. This framework comprises several macroeconomic models that are often simulated by numerical methods. We illustrate the applicability of our algorithm with two simple specifications of the model, and contrast its performance against standard numerical methods. In the first application, we study a representative-agent deterministic economy with capital income taxes. We show that a continuous Markov equilibrium may not exist; moreover, standard computation methods would usually fail to converge or yield inaccurate solutions. In the second application, we consider a stochastic economy with heterogeneous agents. For our simple parameterization, the competitive equilibrium is unique, and hence there is a continuous Markovian solution. We compare the solution of our accurate algorithm against a simplified algorithm that uses an approximate aggregation strategy. We show that this latter algorithm yields a rather poor approximation of the equilibrium correspondence and simulated statistics are strongly biased. Therefore, the first numerical experiment alerts us of the dangers of using continuous approximations when the true solution may not be continuous, and the second numerical experiment alerts us of the dangers of using heuristic simplifications as they may introduce large errors in the equilibrium law of motion.

## 4.1 Economic Environment

The economy is populated by a finite number of agents, $I$, who live forever. The vector of shocks $z$ affects the overall productivity level, as well as individual income and preferences. Capital is the only asset in this economy, and hence financial markets are technically incomplete.

Each agent $i$ has preferences represented by the intertemporal objective

$$E\left[\sum_{t=0}^{\infty}\left(\beta^i\right)^t u^i\left(c_t^i, z_t\right)\right], \tag{4.1}$$

where $\beta \in (0,1)$ is the discount factor, and $c_t$ is consumption of the aggregate good at a given node $z^t$. Function $u(\cdot, z_t)$ is increasing, strictly concave, and twice continuously differentiable.

Stochastic consumption plans $\left(c_t^i\right)_{t\geq 0}$ are financed from after-tax capital returns, wages, profits, and commodity endowments. These values are expressed in terms of the single good, which is taken as the numeraire commodity of the system at each date-event. For a given rental rate $r_t$ and wage

$w_t$ household $i$ offers $k_t^i \geq 0$ units of capital to the production sector, and supplies inelastically $l_t^i(z_t) \geq 0$ units of labor. For simplicity, we abstract from leisure considerations.

Each household $i$ is subject to the following sequence of budget constraints

$$\begin{aligned} k_{t+1}^i\left(z^t\right) + c_t^i\left(z^t\right) &= (1-\delta)\, k_t^i\left(z^{t-1}\right) + (1-\tau_k(K)) r_t\left(z^t\right) k_t^i\left(z^{t-1}\right) + \\ &\quad + w_t\left(z^t\right) l_t^i\left(z^t\right) + e_t^i\left(z^t\right) + T_t^i\left(z^t\right) + \pi^i(z^t) \\ k_{t+1}^i\left(z^t\right) &\geq 0, \text{ for all state histories } z^t = (z_0, ..., z_t), \text{ and } k_0^i \text{ given.} \end{aligned} \tag{4.2}$$

Capital income is taxed according to function $\tau_k$, which depends on the aggregate capital stock, $K_t$. This tax function is assumed to be positive, continuous, and bounded away from 1. Tax revenues are rebated back to consumers as lump-sum transfers $T_t^i$. Finally, $\pi_t^i$ denotes profits.

The production sector is made up of a continuum of identical units that have access to a constant returns to scale technology in individual factors. Hence, without loss of generality we shall focus on the problem of a representative firm. After observing the current shock vector $z$ the firm hires $K$ units of capital and $L$ units of labor. The total quantity produced of the single aggregate good is given by a production function $A_t F\left(K_t, L_t\right)$, where $A_t$ is the firm's total factor productivity and $F\left(K_t, L_t\right)$ is the direct contribution of the firm's inputs to the production of the aggregate good. Hence, at each date event $z^t$, the representative firm seeks to maximize one-period profits by an optimal choice of factors $(K, L)$,

$$\pi_t = max A_t F\left(K_t, L_t\right) - r_t K_t - w_t L_t. \tag{4.3}$$

We shall maintain the following standard conditions on function $F$:

**Assumption 4.1** $F : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ *is increasing, concave, continuous and linearly homogeneous. This function is continuously differentiable at each interior point* $(K, L)$*; moreover,* $\lim_{K\to\infty} D_1 F\left(K, L\right) = 0$ *for each given* $L > 0$.

## 4.2 Sequential and Recursive Competitive Equilibrium

The present model contemplates several deviations from a frictionless world and so a competitive equilibrium cannot usually be recast as the solution to an optimal planning program. The model includes individual uninsurable shocks to preferences and labor, capital income taxes, and an aggregate shock to production. Households can hold capital to transfer wealth, but they may be unable to smooth out consumption since there is only one single asset and capital holdings must be non-negative.

**Definition 4.1** *A sequential competitive equilibrium (SCE) is a tax function* $\tau_k(K)$, *and a collection of vectors* $\left(\left\{c_t^i(z^t), k_{t+1}^i(z^t)\right\}_i, K_t(z^t), L_t(z^t), w_t(z^t), r_t(z^t)\right)_{t\geq 0}$ *that satisfy*

*(i) Constrained utility maximization: For each household* $i$, *the sequence* $\left(c_t^i, k_{t+1}^i\right)_{t\geq 0}$ *maximizes the objective (4.1) subject to the sequence of budget constraints (4.2).*

*(ii) Profit maximization: For each* $z^t$, *vector* $(K_t(z^t), L_t(z^t))$ *maximizes profits (4.3).*

*(iii) Market clearing: For each* $z^t$ *and its predecessor node* $z^{t-1}$,

$$K_t\left(z^t\right) + \sum_{i=1}^{I} c_t^i\left(z^t\right) = A_t F\left(K_t\left(z^t\right), L_t\left(z^t\right)\right) + (1-\delta) K_t\left(z^t\right) + \sum_{i=1}^{I} e_t^i\left(z^t\right),$$

$$\sum_{i=1}^{I} k_t^i\left(z^{t-1}\right) = K_t\left(z^t\right) \text{ and } \sum_{i=1}^{I} l^i\left(z_t\right) = L_t\left(z^t\right).$$

Note that the equilibrium quantities $\left(K_t(z^t), L_t(z^t)\right)_{t\geq 0}$ may be inferred from households' aggregate supply of these factors. Hence, we may refer to a SCE as simply a sequence of vectors $\left(\left\{c_t^i(z^t), k_{t+1}^i(z^t)\right\}_i, r_t(z^t), w_t(z^t)\right)_{t\geq 0}$. There does not seem to be a general proof of existence of competitive equilibria for infinite-horizon economies with distortions. We are aware of a related contribution by Jones and Manuelli (1999), but their analysis is not directly applicable to models with incomplete markets or externalities. Hence, in the Appendix we outline a proof of the following result.

**Proposition 4.2** *A SCE exists.*

For computational purposes we need to bound the equilibrium values of the key variables of the model. In the Appendix below we show that there are positive constants $K^{max}$ and $K^{min}$ such that for every equilibrium sequence of physical capital vectors $\left(k_{t+1}^i(z^t)\right)_{t\geq 0}$ if $K^{max} \geq \sum_{i=1}^{I} k_0^i(z^0) \geq K^{min}$ then $K^{max} \geq \sum_{i=1}^{I} k_{t+1}^i(z^t) \geq K^{min}$ for all $z^t$. Moreover, $K^{min} > 0$ if $\lim_{K\to 0} D_1 F(K, L) = \infty$ for some positive $L$. Hence, in what follows the domain of aggregate capital will be restricted to the interval $[K^{min}, K^{max}]$, and it should be understood that $K^{min} = 0$ only if $\lim_{K\to 0} D_1 F(K, L)$ is bounded for all given $L > 0$. This implies that every equilibrium sequence of factor prices $\left(r_t(z^t), w_t(z^t)\right)_{t\geq 0}$ is bounded.

We also need to bound the equilibrium shadow values of investment. To accomplish this task, we define an auxiliary value function of an individual sequential optimization problem. For a given sequence of factor prices and aggregate capital $(\mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0})) = \left(r_t(z^t), w_t(z^t), K_t\left(z^t\right)\right)_{t\geq 0}$, let

$$J^i(k_0^i, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0})) = \max E \sum_{t=0}^{\infty} \beta^t u^i(c_t(z^t), z_t)$$

$$s.t.$$

$$k_{t+1}^i\left(z^t\right) + c_t^i\left(z^t\right) = (1-\delta)\, k_t^i\left(z^{t-1}\right) +$$

$$(1 - \tau_k(K_t\left(z^t\right)))r_t\left(z^t\right) k_t^i\left(z^{t-1}\right) +$$

$$+ w_t\left(z^t\right) l_t^i + e_t^i + T_t^i\left(z^t\right) + \pi_t^i,$$

$$k_{t+1}^i\left(z^t\right) \geq 0, k_0^i \text{ given.}$$

For every bounded sequence $(\mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0})) = \left(r_t(z^t), w_t(z^t), K_t\left(z^t\right)\right)_{t\geq 0}$, the value function $J^i(k_0^i, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0}))$ is well defined, and continuous. Moreover, mapping $J^i(\cdot, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0}))$ is increasing, concave, and differentiable with respect to the initial condition $k_0^i$. Further, the partial derivative $D_1 J^i(k_0^i, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0}))$ varies continuously with $(k_0^i, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0}))$ [cf. Rincon-Zapatero and Santos (2009)]. The next result readily follows from these regularity properties of the value function.

**Proposition 4.3** *For all SCE* $\left(\left\{c_t^i(z^t), k_{t+1}^i(z^t)\right\}_i, r_t(z^t), w_t(z^t)\right)_{t\geq 0}$ *with* $K^{max} \geq \sum_{i=1}^I k_0^i(z^0) \geq K^{min}$, *there is a constant* $\gamma$ *such that* $0 \leq D_1 J^i(k_0^i, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0})) \leq \gamma$ *for all* $z^t$.

Observe that these bounds apply to the following types of utility functions: $(i)$ Both function $u(\cdot, z)$ and its derivative are bounded, $(ii)$ function $u(\cdot, z)$ is bounded, and its derivative function is unbounded, and $(iii)$ both function $u(\cdot, z)$ and its derivative are unbounded. Phelan and Stacchetti (2001) deal with case $(i)$ and Krebs (2004) and Kubler and Schmedders (2003) consider utility functions of type $(iii)$. We provide a uniform method of proof that covers all the three cases, as well as production functions with bounded and unbounded derivatives. As a matter of fact, Proposition 4.3 fills an important gap in the literature, since no general results are available on upper and lower bounds for factor prices and marginal utilities for production economies with heterogeneous consumers and market frictions.

For any initial distribution of capital $k_0$ and a given shock $z_0$, we define the Markov equilibrium correspondence $V^* : \mathbf{K} \times \mathbf{Z} \to \mathbf{R}^I_+$ as

$$V^*(k_0, z_0) = \left\{ \begin{array}{c} (\cdots, D_1 J^i(k^i_0, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0})), \cdots) : \\ \left(\left\{c^i_t(z^t), k^i_{t+1}(z^t)\right\}_i, r_t, w_t\right)_{t\geq 0} \text{ is a SCE} \end{array} \right\}, \tag{4.4}$$

where $\mathbf{K} = \{k : K^{max} \geq \sum_{i=1}^I k^i \geq K^{min}\}$. Hence, the set $V^*(k_0, z_0)$ contains all current equilibrium shadow values of investment $m_0 = (\cdots, m^i_0, \cdots)$, for every household $i$.

**Corollary 4.4** *Correspondence $V^*$ is nonempty, compact-valued, and upper semicontinuous.*

This corollary is a straightforward consequence of Propositions 4.2 and 4.3. Note that by the envelope theorem we must have $D_1 J^i(k^i_0, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0})) \geq (1-\delta+(1-\tau_k)r_0(z_0))D_1 u^i(c^i_0, z_0)$, with equality when $c^0_i > 0$. Moreover, Proposition 4.3 implies $0 \leq D_1 J^i(k^i_0, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}), \mathbf{K}(\mathbf{z_0})) \leq \gamma$, and so $c^i_0 = 0$ is only possible if the derivative of the utility function $u^i$ is bounded at $c^i_0 = 0$.

The second key element of our analysis is operator $B$ which is defined as follows. For any given correspondence $V : K \times Z \to R^I_+$ let $B(V)(k, z)$ be the set of values $m = (\cdots, m^i, \cdots)$, with $0 \leq m^i \leq \gamma$ for all $i$, such that there is some vector $(c, k_+, r, w, m_+, \lambda, \zeta) \in R^I_+ \times R^I_+ \times R_+ \times R_+ \times \left(R^I_+\right)^N \times R^I_+ \times R^I_+$, with $m_+(z_+) \in V(k_+, z_+)$ for all $z_+ \in Z$ that satisfies all individual and aggregate *temporary equilibrium conditions.*

## 4.3 Numerical example 1: A model with capital income taxes

Let us first consider a deterministic version of the above model with a representative agent and capital income taxes. To further simplify our analysis, assume that capital is the only production factor with full depreciation $\delta = 1$, and the utility function is logarithmic. Let

$$f(k) = k^{1/3}, \ \beta = 0.95. \tag{4.5}$$

Assume that there is a piecewise linear, tax schedule given by

$$\tau(K) = \begin{cases} 0.10 & \text{if } K \leq 0.160002 \\ 0.05 - 10(K - 0.165002) & \text{if } 0.160002 \leq K \leq 0.170002 \\ 0 & \text{if } K \geq 0.170002. \end{cases} \tag{4.6}$$

Then, a continuous Markov equilibrium fails to exist [cf. Santos (2002, Prop. 3.4)]. However, it follows from the foregoing analysis that a recursive equilibrium in an adequately expanded state space does exist.

*Implementation of our algorithm*

Following the notation of our general theoretical framework, we can write:

$$\varphi(k_{t+1}, c_t) = f(k_t) - c_t - k_{t+1}, \text{ and} \tag{4.7}$$

$$m_t = h(k_t, c_t) = \frac{r_t\,(1 - \tau_k(K_t))}{c_t} = \frac{\frac{1}{3}k_t^{-2/3}(1 - \tau(k_t))}{c_t}. \tag{4.8}$$

Similarly, aggregate feasibility and the intertemporal optimality conditions for the household can be summarized by the Euler equation

$$\Phi(k_t, c_t, m_{t+1}) = \frac{1}{c_t} - \beta m_{t+1}. \tag{4.9}$$

The, let $B(V)(k_t)$ be the set of values $m_t$ such that there is $(c_t, k_{t+1})$ and $m_{t+1} \in V(k_{t+1})$ satisfying the temporary equilibrium conditions (4.7- 4.9).

For the numerical implementation of our algorithm we exploit the low dimensionality of the state space and compactness of the equilibrium correspondence. Specifically, notice that for each given $k_t$ the shadow values of investment, $m(k_t)$, lie in a compact interval $[\underline{m}(k_t), \overline{m}(k_t)]$. Hence, our numerical algorithm starts by approximating the upper and lower bound functions $\underline{m}(k_t)$ and $\overline{m}(k_t)$ using step functions. Notice, however, that these functions may be discontinuous. Hence, the standard strategy of approximating these functions only at the vertex points of the triangulation may not work. In our case it is necessary to obtain bounds for all values within each of the simplices. Some technical details are relegated to Appendix B. Here, we just illustrate some properties of our numerical approximation.

Figure 4.1 presents our initial guess (left panel), $V_0^{h,N}$, and the correspondence defined by the area (right panel) between the upper and lower approximated functions $\underline{m}(k_t)$ and $\overline{m}(k_t)$. A useful feature of this example is that the backward shooting algorithm can be used to obtain highly accurate solutions. (Of course, for stochastic versions the shooting method no longer works.) The dots in the Figure below represent an approximate solution derived via backward shooting.

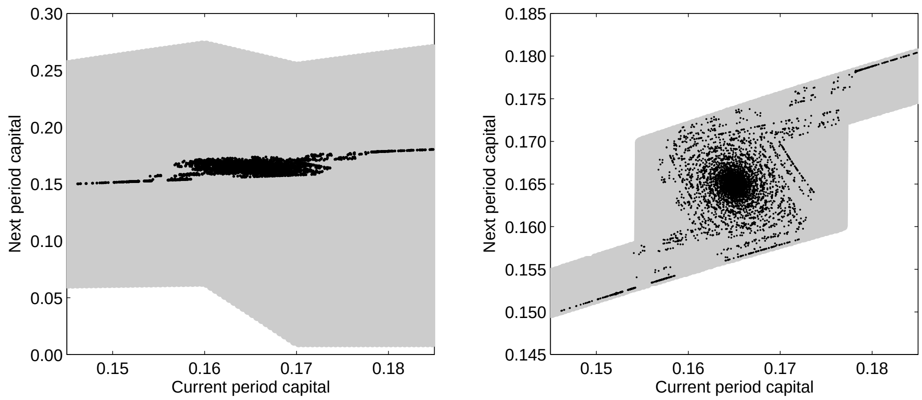


Figure 4.1: Initial (left grey area) and limiting correspondence (right grey area) vs solution obtained via the backward shooting method (black dots).

Since the limiting correspondence is not single valued near the middle steady state, our method is signaling the possibility of a multiple valued equilibrium correspondence. The resulting policy correspondence is illustrated in Figure 4.2 below together with the solution obtained via the shooting method.In this specific example both our method and the shooting algorithm yield highly accurate

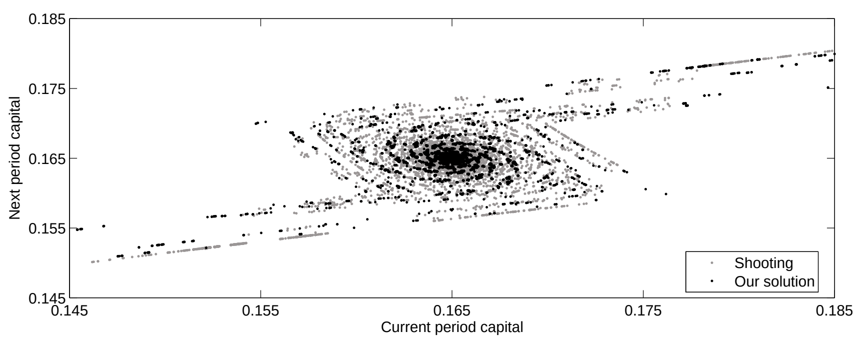


Figure 4.2: Equilibrium correspondence computed with our method vs the backwards shooting method.

solutions. However, we remark that shooting cannot be used for stochastic models whereas our algorithm will be use below in several examples with uncertainty.

*Comparing with other computational algorithms*

A standard practice in quantitative analysis is to assume that a continuous policy function exists. Hence, let

$$k_1 = g(k, \xi),$$

where $g$ is a continuous approximation defined by a finite vector of parameters $\xi$. We obtain an estimate for $\xi$ by forming an Euler equation system over as many points as the dimensionality of the parameter space

$$u'(k^i, g(k^i, \xi)) = \beta u'(g(k^i, \xi), g(g(k^i, \xi), \xi)) \cdot \left[ f'(g(k^i, \xi))(1 - \tau(g(k^i, \xi))) + (1 - \delta) \right].$$

The choice of the grid points, $k^i$, for the Euler equation may be dependent on the functional approximations for the policy function (e.g. Chebyshev polynomials could be evaluated at the Chebyshev nodes). Here, we assume that $g(k, \xi)$ belongs to the class of piecewise linear functions. First, we should note that this approximation failed to converge in several instances. In particular, we found that vector $\xi$ could oscillate with no discernible pattern across different iterations. As expected, the area of the domain where the lack of convergence occurred was close to the middle steady state. Figure 4.3 below displays some representative functions from different iterations of the algorithm. Second, in some other cases the distance between candidate policy functions was relatively small, but this does not mean that these policies are close to the true solution. Of course, for points near the middle steady state solution a continuous policy function will arbitrarily redirect the convergence of initial conditions to one of the remaining two competitive steady-states solutions.

In summary, the equilibrium correspondence of this model cannot be represented by a continuous law of motion. Traditional computational methods based on iterations of continuous functions may either fail to converge or yield inaccurate solutions that highly distort the dynamics of competitive equilibria.

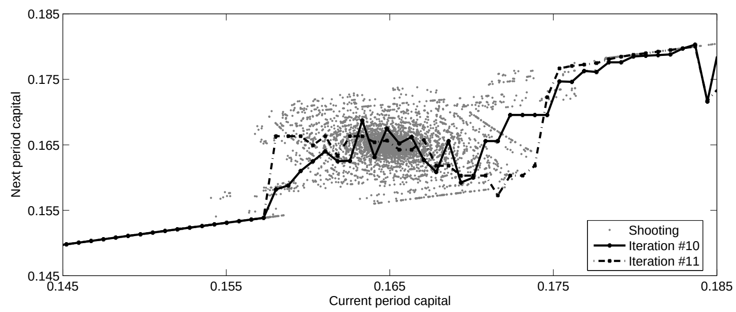


Figure 4.3: Different iterations of a standard solution method vs Shooting solution

### 4.4 Numerical example 2: A model with two agents and no taxes

We now consider a specification of the model with two agents who face idiosyncratic and aggregate uncertainty. There are no taxes. Both agents have the same utility function, $u^i(c^i) = \frac{(c^i)^{1-\sigma}}{1-\sigma}$, with discount factor, $\beta^1 = \beta^2 = 0.95$. The capital share is $\alpha = 0.34$ and depreciation rate $\delta = 0.06$. Total factor productivity is a random variable with two possible values: $A_g = 1.0807$ and $A_b = 1.0593$. Each agent has a random endowment of labor, $l^i$, which can take two possible values, $l_b = 0$ and $l_g = 1$. These idiosyncratic shocks do not affect the aggregate labor supply; that is, $l_t^1 + l_t^2 = 1$ at all date-events. Productivity and labor endowments are assumed to be jointly driven by a Markov process with transition matrix

$$\Pi = \begin{pmatrix} 0.5 & 0.4 & 0.06 & 0.04 \\ 0.6 & 0.3 & 0.06 & 0.04 \\ 0.45 & 0.35 & 0.15 & 0.05 \\ 0.5 & 0.3 & 0.15 & 0.05 \end{pmatrix}.$$

Here, entry $\Pi_{p,q}$ is the probability of moving to state $p$ from the current state $q$.[6]

*Implementation of our algorithm*

Mapping this model into the notation of our general theoretical framework is simple. The vector of endogenous predetermined variables is given by the capital holdings of each agent, $x_t = (k_t^1, k_t^2)$, while the vector of current endogenous variables contains the consumption and investment choices of each agent, $y_t = (c_t^1, c_t^2, i_t^1, i_t^2)$. Equilibrium interest rates and wages can be explicitly written in terms of the aggregate capital $K_t$ and unit labor supply. Hence, capital and the shadow multipliers

[6]Notice there are four possible states $(A_g, l_g, l_b), (A_g, l_b, l_g), (A_b, l_g, l_b)$, and $(A_b, l_b, l_g)$.

for investment are determined by the following equations

$$\varphi(x_{t+1}, x_t, y_t, z_t) \;=\; (i_t^1 + (1-\delta)k_t^1 - k_{t+1}^1, i_t^2 + (1-\delta)k_t^2 - k_{t+1}^2), \tag{4.10}$$

$$\left(m_t^1, m_t^2\right) \;=\; h(x_t, y_t, z_t) = ((r_t + 1 - \delta)\left(c_t^1\right)^{-\sigma}, (r_t + 1 - \delta)\left(c_t^2\right)^{-\sigma}). \tag{4.11}$$

with $r_t = \theta A_t K_t^{\theta - 1}, w_t = (1-\theta) A_t K_t^{\theta}$. Finally, intertemporal optimality and all individual and aggregate constraints are collected in a function $\Phi$ defined as $\Phi(x_t, y_t, z_t, E_t[m_{t+1}]) =$

$$\begin{gathered} ((r_t + 1 - \delta)\, k_t^i + w_t l_t^i - c_t^i - i_t^i, \\ \left(c_t^j\right)^{-\sigma} - \beta E_t m_{t+1}^j(z_{t+1}) + \lambda_t^j, \text{ for } j = 1, 2, \\ \sum_i (c_t^i + i_t^i) - A_t K_t^{\theta}, \end{gathered} \tag{4.12}$$

where $\lambda_t^i$ is the multiplier associated to the constraint $k_{t+1}^i \geq 0$.

Our algorithm operates as follows. Let $V$ be any given correspondence, then $BV(x,z)$ is the set of all values $\left(m_t^1, m_t^2\right)$ for which one can find values $c_t^1, c_t^2, i_t^1, i_t^2, k_{t+1}^1, k_{t+1}^2$, and $\left(m_{t+1}^1, m_{t+1}^2\right) \in V(k_{t+1}^1, k_{t+1}^2, z_{t+1})$ at all successors $z_{t+1}$ that satisfy (4.10-4.12). Appendix B explains further details of the operation of this algorithm that considers multiple agents.

*Comparing with other computational algorithms*

A commonly employed method to solve this type of models is the "approximate aggregation" procedure pioneered by Krusell and Smith (1998). A key insight of this method is that in equilibrium aggregate variables may be well approximated as functions of simple statistics. In particular, the stochastic process driving aggregate capital is assumed to be characterized by a finite vector of moments. Individual decisions are computed on the basis of such expectations for aggregate variables. And a fixed point is reached if the simulated moments from the individual decision rules match those of the law of motion for aggregate capital.

In our baseline model, the algorithm is applied in the following way. Start with a guess on a parameterized functional form for the first moment of aggregate capital. Then, use value function to compute the problem of the representative household

$$\nu(k^i; K, z) = \max\{U(c) + \beta E[\nu(k'^i; K', z')|z, \varepsilon]\} \tag{4.13}$$

$$s.t.\ \ c + k^{i\prime} = r(K,z)k^i + w(K,z)\varepsilon^i + (1-\delta)k^i$$

$$k^{i\prime} \geq B$$

$$\log K_{t+1} = a(z)\log K_t + b(z) \tag{4.14}$$

The algorithm estimates coefficients $(a(z_g), b(z_g), a(z_b), b(z_b))$ and individual policy functions in the following fashion: $(i)$ Start with initial parameter estimates; $(ii)$ Solve the dynamic programming problem of each agent (4.13); $(iii)$ Construct aggregate capital time series by aggregating the resulting individual time series simulations; $(iv)$ Perform a regression over the stationary region to obtain new estimates for such coefficients. This process stops when there is no variation in the coefficient estimates and the $R^2$ and standard error of the aforementioned regression are sufficiently good.

An obvious advantage of "Approximate Aggregation" is computational cost. Indeed, the algorithm can accommodate an arbitrary number of agents and idiosyncratic shocks. Surprisingly, relatively little is known about the accuracy properties of the solutions and of the simulated moments for this type of algorithms. In Table 4.4 below, we compare some quantitative properties of the "Approximate Aggregation" method described above to those of our algorithm. $EE_i$ refers to the Euler equation residuals, and $Mean(k_i)$ is the average of simulated capital values for each agent $i = 1, 2$.

| Method | $Mean(\|EE_1\|)$ | $Mean(\|EE_2\|)$ | $Mean(k_1)$ | $Mean(k_2)$ |
|---|---|---|---|---|
| Approx. Aggregation | $1.57 \times 10^{-2}$ | $2.71 \times 10^{-2}$ | 2.8196 | 4.5210 |
| Our Algorithm | $5.14 \times 10^{-4}$ | $7.58 \times 10^{-4}$ | 3.0898 | 3.8623 |

Table 4.4: Euler equation residuals and simulated moments of alternative solution methods.

Even though in this case the model always has a unique competitive equilibrium which may be generated by a continuous policy function, we can see that our method yields higher accuracy of approximation as measured by Euler equation residuals. Further, our non-linear equilibrium approximation results in substantially different simulated statistics for individual wealth from those of the approximate aggregation method. Indeed, approximation errors for these simple moments are of the order of 10 percent.

## 5 A Stochastic OLG Economy

Overlapping generations (OLG) models have become central in the analysis of several macro issues such as the funding of social security, the optimal profile of savings and investment over the life cycle, the effects of various fiscal and monetary policies, and the evolution of future interest rates and asset prices under current demographic trends.[7]

As already stressed, there are no known convergent procedures for the computation of sequential competitive equilibria in OLG models even for frictionless economies with complete financial markets. We now illustrate that our approach delivers a reliable, computable algorithm for the solution of competitive equilibria in a general class of OLG models.

### 5.1 Economic Environment

The economy is conformed by a sequence of overlapping generations that live for two periods. The primitive characteristics of the economy are defined by a stationary Markov chain. At every time period $t = 0, 1, 2, \cdots$ a new generation is born. Each generation is made up of $I$ agents, who are present in the economy for two periods. More specifically, for a household of type $i$ born at time $t$ preferences are defined over consumption bundles of the goods available at times $t$ and $t+1$, and the agent can only trade goods and assets in these two periods. The economy starts with an initial generation who is only present in the initial period $t = 0$. This generation is endowed with the aggregate supply of assets $\theta_0$. At each node $z^t$, there exist spot markets for the consumption good and $J$ securities. These securities are specified by the current vector of prices, $q_t(z^t) = (\cdots, q_t^j(z^t), \cdots)$, and the vectors of future dividends $d_r(z^r) = (\cdots, d_r^j(z^r), \cdots)$ promised to deliver at future information sets $z^r|z^t$ for $r > t$. We assume that the vector of security prices $q_t(z^t)$ is non-negative – a condition implied by free disposal of securities – and the vector of dividends $d_t(z^t)$ is positive in all components and depends only on the current realization of the vector of shocks $z_t$; hence, $(d_t(z_t))_{t\geq 0}$ is a time invariant Markov chain.

For simplicity, we assume that every utility function $U^i$ is separable over consumption of different dates. For an agent $i$ born in period $t$, let $c_{y,t}^i\left(z^t\right)$ denote the consumption of the aggregate good

[7] For instance, see Champ and Freeman (2002), Conesa and Krueger (1999), Geanakoplos, Magill and Quinzii (2003), Gourinchas and Parker (2002), Imrohoroglu, Imrohoroglu, and Joines (1995), Storelesletten, Telmer and Yaron (2004), and Ventura (1999).

in period $t$ over the history of shocks $z^t$, and let $c^i_{o,t+1}\left(z^{t+1}|z^t\right)$ denote the consumption in period $t+1$ for every successor node $z^{t+1}|z^t$ of $z^t$. Then the intertemporal objective $U^i$ is defined as

$$U^i\left(c^i_y, c^i_0; z^t, z^{t+1}\right) = u^i\left(c^i_{y,t}\left(z^t\right), z_t\right) + \beta \sum_{z^{t+1}\in\mathbf{Z}} v^i\left(c^i_{o,t+1}\left(z^{t+1}\right), z_{t+1}\right)\pi\left(z^{t+1}|z^t\right) \tag{5.1}$$

The one-period utilities $u^i$ and $v^i$ satisfy the following conditions:

**Assumption 5.1** *For each $z \in \mathbf{Z}$ the one-period utility functions $v^i(\cdot, z), u^i(\cdot, z) : \mathbf{R}_+ \to \mathbf{R}\cup\{-\infty\}$ are increasing, strictly concave, and continuous. These functions are also continuously differentiable at every interior point $c > 0$.*

Each agent $i$ born at $t = 1, 2, \cdots$ is endowed with a vector of goods $e^i_t = (e^i_{y,t}, e^i_{o,t+1})$ and trades an asset portfolio $\theta^i$ to attain desirable amounts of consumption. The endowment process $(e^i_t(z^t)) = (e^i_{y,t}(z^t), e^i_{o,t+1}(z^{t+1}|z^t))$ follows a time invariant Markov chain; hence $e^i_{y,t}(z^t) = e^i_y(z_t)$, and $e^i_{o,t+1}(z^{t+1}|z^t) = e^i_o(z_{t+1})$ for every agent $i$ and every $t$. Given prices $\left(q_t\left(z^t\right)\right)_{t\geq 0}$, a consumption-savings plan $\left(c^i_{y,t}\left(z^t\right), c^i_{o,t+1}\left(z^{t+1}\right), \theta^i_t\left(z^t\right)\right)$ must obey the following two-period budget constraints:

$$\theta^i_{t+1}\left(z^t\right)\cdot q_t\left(z^t\right) + c^i_{y,t}\left(z^t\right) \leq e^i_{y,t}\left(z_t\right),\ \text{for } \theta^i_{t+1}\left(z^t\right) \geq 0, \tag{5.2}$$

$$c^i_{o,t+1}\left(z^{t+1}\right) \leq \theta^i_{t+1}\left(z^t\right)\cdot\left(q_{t+1}\left(z^{t+1}\right) + d_t\left(z_{t+1}\right)\right) + e^i_{o,t+1}\left(z_{t+1}\right),\ \text{all } z^{t+1}|z^t. \tag{5.3}$$

For an initial stock of securities $\theta^i_0$ each agent $i$ at time $t = 0$ seeks to maximize the total quantity of consumption $c^i_{o,0}(z_0)$ for given endowments of the aggregate good $e^i_o$ and the vector of securities $\theta^i_0$. More precisely,

$$c^i_{o,0}\left(z_0\right) = \theta^i_0\cdot\left(q_0\left(z_0\right) + d_0\left(z_0\right)\right) + e^i_o\left(z_0\right). \tag{5.4}$$

## 5.2 Sequential and Recursive Competitive Equilibrium

In this economy the aggregate commodity endowment is bounded by a portfolio-trading plan [Santos and Woodford (1997)], and hence asset pricing bubbles cannot exist in a SCE.

**Definition 5.1** *A SCE is a collection of vectors* $\{(c^i_{y,t}(z^t), c^i_{o,t+1}(z^{t+1}|z^t), \theta^i_{t+1}(z^t))^I_{i=1}, q_t(z^t)\}_{t\geq 0}$ *such that*

*(i) Utility maximization: For every household $i$ and all $t$, vector* $(c^i_{y,t}(z^t),\ c^i_{o,t+1}(z^{t+1}|z^t),$ $\theta^i_{t+1}(z^t))$ *maximizes the objective (5.1) subject to (5.2)-(5.3). For every household $i$ of the starting generation,* $c^i_{o,0}(z_0)$ *satisfies (5.4).*

*(ii) Market clearing: For each* $z^t$,

$$\begin{aligned}\sum_{i=1}^{I}\left(c^i_{y,t}(z^t)+c^i_{o,t}(z^t)\right) &= \sum_{j=1}^{J} d^j_t(z_t)+\sum_{i=1}^{I}\left(e^i_{yt}(z_t)+e^i_{ot}(z_t)\right)\\ \sum_{i=1}^{I}\theta^{ji}_{t+1}(z^t) &= 1, j=1,\cdots,J.\end{aligned}$$

Note that to circumvent technical issues concerning existence of a SCE, we still maintain the short-sale constraint $\theta_t \geq 0$ for all $t$. Then, the existence of a SCE can be established by standard methods [e.g., Balasko and Shell (1980), and Schmachtenberg (1988)]. Moreover, by similar arguments used by these authors it is easy to show that every sequence of equilibrium asset prices $(q_t(z^t))_{t\geq 0}$ is bounded.

Then, we define the Markov equilibrium correspondence $V^*: \mathbf{\Theta}\times\mathbf{Z}\to\mathbf{R}^{JI}_{++}$ as

$$V^*(\theta_0,z_0)=\left\{\left(...\left(q^j_0(z_0)+d^j_0(z_0)\right)D_1v^i\left(c^i_0(z_0),z_0\right)...\right):(c_y,c_o,\theta,q)\ \text{ is a SCE}\right\}.$$

From the above results on existence of SCE for OLG economies we obtain

**Proposition 5.2** *Correspondence $V^*$ is nonempty, compact-valued, and upper semicontinuous.*

## 5.3 Numerical Example: A monetary model

We consider a simplified version of the OLG model with money of Benhabib and Day (1982) and Grandmont (1985). This simple model is useful because it can be solved with arbitrary accuracy. Hence, it is possible to compare the true solution of the model with other numerical approximations. Extensions to a stochastic environment are easy to handle by our algorithm, but may become problematic when using other algorithms.

Each individual receives an endowment $e_1$ of the perishable good when young and $e_2$ when old. There is a single asset, money, that pays zero dividends at each given period. The initial old agent is endowed with the existing money supply $M$. Let $P_t$ be the price level at time $t$. An agent born in period $t$ chooses consumption $c_{1t}$ when young, $c_{2t+1}$ when old, and money holdings $M_t$ to solve

$$\max u\left(c_{1t}\right)+\beta v\left(c_{2t+1}\right)$$

subject to

$$\begin{aligned} c_{1t}+\frac{M_t}{P_t} &= e_1, \\ c_{2t+1} &= e_2+\frac{M_t}{P_{t+1}}. \end{aligned}$$

A sequential competitive equilibrium for this economy is a sequence of prices $(P_t)_{t\geq 0}$, and sequences of consumption and money holdings $\{c_{1t}, c_{2t+1}, M_t\}_{t\geq 0}$ such that individual solves the budget-constrained utility maximization problem and markets clear:

$$c_{1t}+c_{2t}=e_1+e_2, \text{ and } M_t=M \text{ for all } t.$$

A sequential competitive equilibrium can be characterized by the following first-order condition:

$$\frac{1}{P_t}u'\left(e_1-\frac{M}{P_t}\right)=\frac{1}{P_{t+1}}\beta v'\left(e_2+\frac{M}{P_{t+1}}\right).$$

Let $b_t=M/P_t$ be real money balances at $t$. Then,

$$b_t u'\left(e_1-b_t\right)=b_{t+1}\beta v'\left(e_2+b_{t+1}\right).$$

Hence, all competitive equilibria can be generated by an offer curve in the $(b_t, b_{t+1})$ space.[8] A simple recursive equilibrium would be described by a function $g$ such that $b_{t+1}=g\left(b_t\right)$.

In the remainder of this section, we restrict our attention to the following parameterizations:

$$u\left(c\right)=c^{0.45},\ v\left(c\right)=-\frac{1}{7}c^{-7},\ \beta=0.8,$$

[8] We can also use the $(c_{1t}, c_{2t+1})$ space as in Cass, Okuno, and Zilcha (1979).

$M = 1$, $e_1 = 2$, and $e_2 = 2^{6/7} - 2^{1/7}$. For this simple example, the offer curve is backward bending. Hence, the equilibrium correspondence is multi-valued, and standard methods – based on the computation of a continuous equilibrium function $b_{t+1} = g(b_t)$ – may portray a partial view of the equilibrium dynamics.

The solution is illustrated in Figure 5.1.

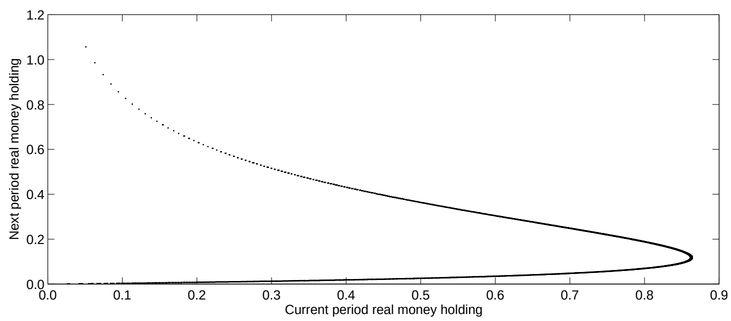


Figure 5.1: "Exact" offer curve for the OLG model.

*Implementation of our algorithm.*

Note that the implementation our numerical algorithm of Section 3 is fairly straightforward. In fact, since the shadow multipliers of investment lie on a compact subset of $R$, we can follow the same computational steps as in the growth model of the previous section. Then, upper and lower bound functions are selected to compute the fixed point that can generate all competitive equilibria. The results from this algorithm are reported in Figure 5.2 where the dark grey area represents the initial correspondence, the light grey area represents the fixed point of algorithm $B^{h,N}$, and the dotted line is the equilibrium correspondence constructed using the equilibrium selection algorithm of Section 3.

For this example, we find that the policy correspondence and time series from our method generate an Euler equation residual of order $10^{-6}$, so that the solution obtained with our algorithm is indistinguishable from the "exact" solution.

*Comparing with other computational algorithms*

A common practice in OLG models is to start the search with an equilibrium guess function of the form $b' = g(b)$. In several numerical experiments we obtained that either the upper part or the lower part of the offer curve. Which one one will obtain depends on the initial guess. This

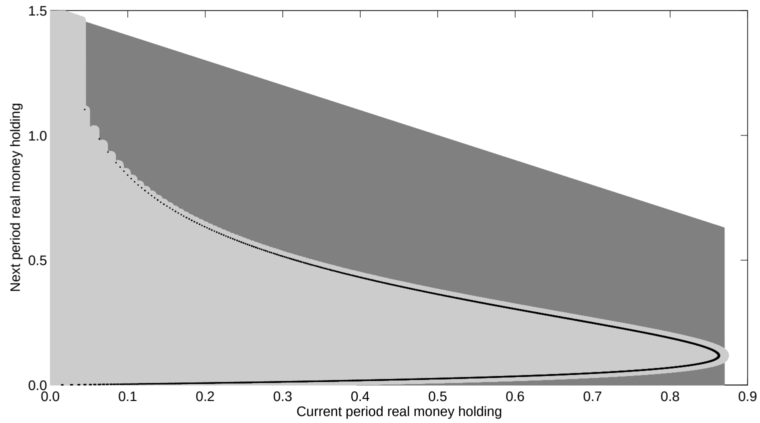


Figure 5.2: Initial guess, limiting correspondence, and approximated equilibrium policy correspondence from operator $B^{h,N}$.

strong dependence on initial conditions is a rather undesirable feature of this method. In particular, note that for initial conditions where the method yields the lower part of the actual equilibrium correspondence all competitive equilibria converge to autarchy. Indeed, zero real monetary holdings are the unique absorbing steady state associated with the lower part of the equilibrium correspondence. Hence, even in the deterministic version of the model, we need a global approximation of the equilibrium correspondence to analyze the various predictions of the model. As shown in Figure 5.3, in the approximate equilibrium correspondence there is cyclical equilibrium in which real money holdings oscillate between 0.85296237892 and 0.09517670718. It is also known that the model has a three-period cycle. But if we just iterate over the upper part of the offer curve we find that money holdings converge monotonically to $\frac{\bar{M}}{p} = 0.418142579084$, as illustrated by the dashed line of Figure 5.3. Indeed, the upper part of the equilibrium correspondence is monotonic, and can at most have cycles of period two, whereas the model generates lots of equilibrium cycles of various periodicities.

In conclusion, for OLG economies standard computational methods based on iteration of continuous functions may miss some important properties of the equilibrium dynamics.

# 6 Asset Pricing Models with Incomplete Markets

There is an important family of macroeconomic models that incorporate financial frictions in the form of sequentially incomplete markets, borrowing constraints, transactions costs, cash-in-advance

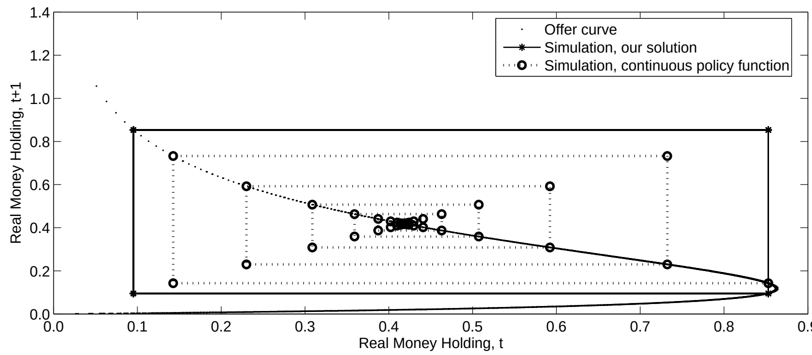


Figure 5.3: Time-series behavior of different numerical solutions.

constraints, and margin and collateral requirements. These models are commonly used to assess the effects of monetary policies, and the variability of macroeconomic aggregates such as asset prices, consumption, interest rates, and inflation.[9] Fairly general conditions rule out the existence of financial bubbles in these economies, and hence equilibrium asset prices are determined by the expected value of future dividends [Santos and Woodford (1997)]. However, in the presence of financial frictions the equivalence of competitive equilibria and optimal allocations breaks down, and standard computational methods are of limited application. The purpose of this section is to illustrate the applicability of our proposed algorithm in a model with collateral requirements taken from Kubler and Schmmeders (2003). Our choice of the state space simplifies the computations, and becomes instrumental to solve the model by a reliable iteration procedure.

## 6.1 Economic Environment

The economy is populated by a finite number of agents. At each date, agents can trade quantities of the unique aggregate good as well as a fixed set of assets that span the horizon of the economy. There are various financial frictions: Incomplete markets, collateral requirements, and short-sale constraints.

Each agent $i$ maximizes the intertemporal objective

$$E\left[\sum_{t=0}^{\infty}\left(\beta^i\right)^t u^i\left(c_t^i\right)\right], \tag{6.1}$$

[9] For instance, see Campbell (1999), Heaton and Lucas (1996), Huggett (1993), Krebs and Wilson (2004), Mankiw (1986), and Telmer (1993). For some monetary models see Bewley (1980), Lucas (1982), and Santos (2006).

subject to a sequence of budget constraints. We assume that $\beta^i \in (0,1)$, and $u^i$ is strictly increasing, strictly concave and continuously differentiable with derivative $(u'^i(0) = \infty$. At each node $z^t$, there exist spot markets for the consumption good and $J$ securities. These securities are specified by the current vector of prices, $q_t(z^t) = (\cdots, q_t^j(z^t), \cdots)$, and the vectors of dividends $d(z^r) = (\cdots, d^j(z^r), \cdots)$ promised to deliver at future information sets $z^r|z^t$ for $r > t$. The vector of security prices $q_t(z^t)$ is non-negative, and that the vector of dividends $d_t(z_t)$ is positive and depends only on the current realization of the vector of shocks $z_t$. Also, at each node $z^t$ the agent receives $e^i(z_t) > 0$ units of the consumption good.

There is also a market for one-period bonds available at all times. A bond is a promise to one unit of the consumption good at all successor nodes $z^{t+1}|z^t$. Bonds are are in zero net supply, and are specified by the price vector $p_t(z^t)$. Agents can default on bond payments, and hence they required to hold at least $k^j \geq 0$ units worth of each security $j$ as collateral. In case of default, the buyer of the bond will garnish the collateral wealth.

For a given a price process $\left(q_t\left(z^t\right), p_t\left(z^t\right)\right)_{t\geq 0}$, each agent $i$ chooses desired quantities of consumption, real securities and bond holdings $(c_t^i\left(z^t\right), \theta_{t+1}^i\left(z^t\right), \phi_{t+1}^i(z^t))_{t\geq 0}$ subject to the following sequence of budget constraints

$$c_t^i\left(z^t\right) - \phi_t^i(z^{t-1}) \min\left\{1, \sum_j k^j \frac{q_t^j(z^t)}{q_{t-1}^j(z^{t-1})}\right\} + \theta_{t+1}^i\left(z^t\right) \cdot q_t\left(z^t\right) = \tag{6.2}$$

$$e^i\left(z_t\right) + \theta_t^i\left(z^{t-1}\right) \cdot \left(q_t\left(z^t\right) + d\left(z_t\right)\right) - \phi_{t+1}^i(z^t) p_t(z^t),$$

$$-k^j \phi_{t+1}^i(z^t) \leq q_t^j(z^t) \theta_{t+1}^{ij}(z^t), \text{ for } j = 1..J, \tag{6.3}$$

$$0 \leq \theta_{t+1}^i\left(z^t\right), \text{ all } z^t, \ \theta_0^i \text{ given.} \tag{6.4}$$

Note that (6.4) imposes non-negative holdings of real securities, and (6.3) is meant to limit the amount of bond debt to a fraction of collateral wealth. The minimum in expression (6.2) above reflects that it is optimal to default on previous bond short-sales whenever the promised payment is larger than the cost of loosing the collateral.

## 6.2 Sequential and Recursive Competitive Equilibrium

**Definition 6.1** *A sequential competitive equilibrium (SCE) for this economy is a collection of vectors* $\left(c_t\left(z^t\right), \theta_{t+1}\left(z^t\right), \phi_{t+1}(z^t), p_t(z^t), q_t\left(z^t\right)\right)_{t\geq 0}$ *such that (i) for each agent i the plan* $(c_t^i\left(z^t\right),$ $\theta_{t+1}^i\left(z^t\right), \phi_{t+1}^i(z^t))_{t\geq 0}$ *maximizes the objective (6.1) subject to (6.2)-(6.4), and (ii) markets clear:*

$$\sum_i^I c_t^i\left(z^t\right) = \sum_j^J d^j\left(z_t\right) + \sum_i^I e_t^i\left(z^t\right), \tag{6.5}$$

$$\sum_i^I \theta_{t+1}^{ji}\left(z^t\right) = 1, \text{ for } j = 1, \cdots, J, \tag{6.6}$$

$$\sum_i^I \phi_{t+1}^i\left(z^t\right) = 0, \text{ at all } z^t. \tag{6.7}$$

For the recursive specification of equilibria the state space includes the space of exogenous shocks $Z$, the space of possible values for share prices, $Q$, the distribution of shares $\Theta = \left\{\theta \in R_+^{JI} : \sum_{i=1}^I \theta^{ji} = 1 \text{ for all } j\right\}$ and bond holdings $\Delta = \left\{\phi \in R_+^I : \sum_{i=1}^I \phi^i = 0\right\}$. The equilibrium shadow value correspondence $V^* : Q \times \Theta \times \Delta \times Z \rightarrow R_+^{JI}$ is then defined as

$$V^*\left(q_-, \theta_0, \phi_0, z_0\right) = \left\{\left(..., \left(q_0^j\left(z_0\right) + d^j\left(z_0\right)\right) U_1^i\left(c_0^i\left(z_0\right)\right), ...\right) : \left(c_t, \theta_{t+1}, \phi_{t+1}, q_t, p_t, \lambda_t, \gamma_t\right)_{t\geq 0} \text{ is a SCE}\right\}.$$

Observe that, for every $(q_-, \theta_0, \phi_0, z_0)$, the set $V^*\left(q_-, \theta_0, \phi_0, z_0\right)$ contains all equilibrium $JI$-vectors $m_0 = (\cdots, m_0^{ji}, \cdots)$ of shadow values of investing in each asset $j$ for every agent $i$. It follows that operator $B : V \longmapsto B(V)$ is defined as: For each $(q_-, \theta, \phi, z) \in Q \times \Theta \times \Delta \times Z$, the set $B(V)(q_-, \theta, \phi, z)$ contains all values $m = (\cdots, m_+^{ji}, \cdots)$ such that there is some vector $\left(c, \theta_+, \phi_+, q, q_+, p, \lambda, \gamma\right)$ satisfying all the equilibrium conditions with $m_+ = \left(..., m_+^{ji}\left(z_+\right), ...\right) \in V\left(q, \theta_+, \phi_+, z_+\right)$ for each $z_+ \in Z$.

Under similar regularity conditions Kubler and Schmedders (2003) show existence and compactness of the equilibrium set. Building on the previous literature we can then derive the following result.

**Proposition 6.2** *Correspondence* $V^*$ *is nonempty, compact-valued, and upper semicontinuous.*

We now illustrate an application of our algorithm for a model with two agents and two assets.

### 6.3 Numerical Example

There are two infinitely lived agents $i = 1, 2$, and a real security that generates a sequence of random dividends. Following Kubler and Schmedders (2003) we choose the auxiliary variable,

$$\omega = \frac{\theta q + \phi \min\left\{1, k\frac{q_+}{q}\right\}}{q},$$

Then, the set of predetermined variables is reduced to $y = \left(\omega, d_t, \left\{e_t^h\right\}_{h=1}^2\right)$. Further, the budget constraints also simplify to

$$c_t^1 = e_t^1 + \omega_t q_t + \theta_t (d_t - q_t) - \phi_t p_t \tag{6.8}$$

$$c_t^2 = e_t^2 + (1 - \omega_t) q_t + (1 - \theta_t)(d_t - q_t) + \phi_t p_t \tag{6.9}$$

$$0 \leq \theta_t \leq 1.$$

With this simplification it is no longer necessary to keep track of last period or next period prices. This change of variable is actually not needed for our methods but it will speed up computations.

*Implementation of our algorithm*

Under the above change of variable, it becomes easier to consider the related shadow value

$$\hat{m}_t^i \equiv (q_t) u'^i(c_t^i). \tag{6.10}$$

From the above definition, and the individual constraints (6.8-6.9) we can solve for $\theta_t$ and $q_t$ as functions of $\hat{m}_t^1, \hat{m}_t^2, y_t, p_t, \phi_t$. Hence, given a correspondence $V$, we have that $\left(\hat{m}_t^1, \hat{m}_t^2\right)$ will belong to $BV$ if we can find bond holdings $\phi_t$ and prices $(p_t, q_{t+1})$, a wealth level, $\omega_{t+1}$, and continuation values for the shadow investment values, $\left(\hat{m}_{y_{t+1}}^1, \hat{m}_{y_{t+1}}^2\right) \in V(y_{t+1})$ for all successor nodes, which satisfy the individual budget constraints as well as the intertemporal optimality conditions

$$(d_t - q_t) U_1^i \left(c_t^i\right) + \beta E_t m_{y_{t+1}}^i + q_t \lambda_{c,t}^i + \lambda_{ss}^i = 0 \tag{6.11}$$

$$-p_t U_1^i(c_t^i) + \beta E_t \left[\frac{k}{q_t} m_{y_{t+1}}^i | \Omega_A\right] + \beta E_t \left[\frac{m_{y_{t+1}}^i}{q_{t+1}} | \Omega_B\right] + k \lambda_{c,t}^i = 0 \tag{6.12}$$

where $\Omega_A = \left\{(b_t, q_t, q_{t+1}) : \min\left\{b_t, k\frac{q_{t+1}}{q_t}\right\} = k\frac{q_{t+1}}{q_t}\right\}$, $\Omega_B = \left\{(b_t, q_t, q_{t+1}) : \min\left\{b_t, k\frac{q_{t+1}}{q_t}\right\} = b_t\right\}$, and

$$\omega_{t+1} = \frac{\theta_t q_{t+1} + \phi_t \min\left\{b_t, k\frac{q_{t+1}}{q_t}\right\}}{q_{t+1}}. \tag{6.13}$$

*Comparing with other computational algorithms*

Kubler and Schmedders (2003) enlarge the state space with all exogenous and endogenous variables, and wealth. Recursive equilibrium is constructed from a correspondence that maps the enlarged state space into the set of all endogenous variables. As we have seen in our previous examples, the computational cost of approximating a set operator grows exponentially in the dimension of the domain and range of the operator. Hence, in the end these authors proceed with a computational algorithm that iterates over *functions* from the enlarged state space into the set of all endogenous variables. Unfortunately, iteration over functions does not guarantee of convergence to the equilibrium correspondence, and can only identify one one equilibrium at a time. In contrast, our proposed algorithm constructs recursive equilibria from an operator that maps the enlarged state space into the space of shadow multipliers of investment. This is a lower dimensional object that makes the algorithm more amenable to computation.

To illustrate the performance of our algorithm, assume both agents have identical utilities $u = \frac{c^{1-\sigma}}{1-\sigma}$, with a common coefficient of risk aversion of $\sigma = 2$ and $\beta_1 = \beta_2 = 0.95$. There are four possible values for the aggregate endowment, $\overline{e} \in (9.9, 10.5, 9.9, 10.5)$, with dividends $d = 0.3 \cdot \overline{e}$, and individual endowments

$$\begin{aligned} e^1 &\in (1.386, 2.205, 5.544, 5.145), \\ e^2 &= 0.7\,\overline{e} - e^1. \end{aligned}$$

Also, the transition matrix driving individual shocks

$$\Pi(z'|z) = \begin{bmatrix} 0.4 & 0.4 & 0.1 & 0.1 \\ 0.4 & 0.4 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{bmatrix}.$$

We simulate the economy using the decision rules obtained from our method, as well as from the algorithm based on continuous value function iteration. The resulting simulated statistics are summarized in Table 6.3 below. As before, $EE_i$ denotes the Euler equation residual over the computed solution and $c_i$ is consumption for $i = 1, 2$.

| | *mean* $c_1$ ($\sigma(c_1)$) | *mean* $c_2$ ($\sigma(c_2)$) | $mean(\|EE_1\|)$ | $mean(\|EE_2\|)$ |
|---|---|---|---|---|
| Continuous Markov equilibrium | 4.96 (0.78) | 5.26 (0.78) | $5.05 \times 10^{-6}$ | $3.27 \times 10^{-8}$ |
| Our algorithm | 4.96 (0.78) | 5.26 (0.78) | $2.41 \times 10^{-5}$ | $9.01 \times 10^{-6}$ |

Table 6.3: Simulated moments from alternative solution methods.

Note that for this benchmark calibration both methods deliver identical simulated moments. As a straightforward consequence of Theorem 3.1, we obtain that convergence to a continuous function can only occur when the equilibrium is always unique. Hence, we have shown uniqueness of equilibria for the present model. Of course, iterations over a continuous functions may fail to converge, and for models in which there is no continuous equilibrium selection this procedure may lead to poor approximations of the equilibrium dynamics.

# 7 Concluding Remarks

This paper provides a theoretical framework for the computation and simulation of dynamic competitive-markets economies in which the welfare theorems may fail to hold because of market frictions or the existence of an infinite number of generations. We have applied these methods to some macroeconomic models with heterogeneous agents, taxes, sequentially incomplete markets, borrowing limits, short-sales, and collateral requirements.

For optimal economies, sequential competitive equilibria are generated by a continuous policy function which is the fixed-point solution of a contractive operator. Continuity of the policy function allows for various methods of approximation and functional interpolation, and it is essential to validate laws of large numbers for the simulated paths. Moreover, differentiability and contractive properties are useful for the derivation of error bounds that can guide the computation process. But for OLG models and economies with distortions several papers [e.g. Krebs (2004), Kubler and Polemarchakis (2004), Kubler and Schmedders (2002), and Santos (2002)] have shown that a continuous Markov equilibrium may not exist. We establish a general result on the existence of a Markovian equilibrium solution in a suitably expanded space of state variables. We construct a numerical algorithm that has desirable approximation properties and guarantees convergence of the moments computed from simulated paths.

There are three main features of our algorithm that should be of interest for quantitative work in this area. First, the existence of a Markovian competitive equilibrium is obtained in an enlarged space of state variables. Our choice of the marginal utility values of assets returns is dictated by computational considerations. This is a minimal addition to the state space to restore existence of a Markovian equilibrium and with the property that the extra added variables enter linearly into the Euler equation. Second, the algorithm iterates in a space of candidate equilibrium sets – rather than in a space of functions. Iteration over candidate equilibrium sets guarantees convergence to the fixed-point solution even if Markov equilibria are not continuous. Moreover, we also establish some desirable approximation properties of the computed solutions. And third, the algorithm provides a reliable method for model simulation. We resort to a further discretization of the equilibrium law of motion so that it becomes a Markov chain. It should be stressed that the usual simulation over a continuum of values cannot be justified on theoretical grounds: The simulated moments may fail to converge to the set of moments of the invariant distributions of the model. Other ways to restore laws of large numbers for the simulated paths of these economies would be by imposing monotonicity assumptions on the equilibrium dynamics [Santos (2008)] or by expanding artificially the noise process [e.g., Blume (1979)]. These latter approaches seem to be of more limited economic interest.

Of course, our methods have to face some computational challenges. Iteration over sets is computationally much more costly than iteration over functions. Therefore, the expansion of the state space along with iteration over sets should certainly be manifested into an additional computational burden. Besides, our general convergence results lack error bounds. This lack of accuracy should be expected since our models cannot be restated as optimization programs, and miss some common concavity, differentiability and contractive properties. In terms of numerical implementation the innovative techniques for error estimation proposed by Judd, Yeltekin, and Conklin (2003) seem to be of limited application for our economies. These authors use outer and inner approximations over convex sets. It is not clear to us that an outer approximation over convex sets will converge to the convex hull of the equilibrium correspondence. Moreover, inner convex approximations may be hard to find. Still, these techniques may work well in some applications.

There are several directions in which our analysis can be extended. For example, in the preceding sections we considered exogenous short-sales constraints and exogenous borrowing limits. We could incorporate borrowing constraints that depend on future income [e.g., Miao and Santos (2005)]. These general borrowing schemes arise in financial models and in the modelling of the public sector so as to allow for various types of fiscal policy rules. In most quantitative studies of recursive equilibrium with fiscal policy, the government must balance the budget in each state of the world. This is a rather strong assumption. Another extension is to the area of policy games. As our algorithm includes all the shadow values of investment, it can deal with heterogeneity and market frictions. For example, we can generalize the model of Phelan and Stacchetti (2001) to include heterogeneous agents and various types of financial frictions.

# A Appendix: Proofs

In this Appendix we prove the key results formally stated in Sections 2 and 3. All remaining results follow from similar arguments.

**Proof of Theorem 2.1:** Let $V_0 \supset V_0^*$, let $V_n = B(V_{n-1}), n \geq 1$. Consider $V_N^U = \cup_{n=N}^{\infty} V_n$. Then $V_{N+1}^U = B(V_N^U)$ and $V_{N+1}^U \subset V_N^U$ for all $N \geq 1$. It follows that the sequence $\{V_N^U\}$ must converge to a non-empty set $V^U$. Further, $V^U = \cap_{N=n} V_N^U$, for all $n \geq 1$, and so $V^U = B(V^U)$. We next prove that $V^U = V^*$, Indeed, by the monotonicity of operator $B$ we get that $V^* \subset V^U$; also, $V^U \subset V^*$ since every fixed point conforms an equilibrium – given that the transversality conditions are trivially satisfied in this model. To complete the proof of the theorem, just note that $V^* \subset V_n \subset V_n^U$ for all $n \geq 1$. Since we have already established that $V_n^U \rightarrow V^*$, we get that $V_n \rightarrow V^*$.

**Proof of Theorem 3.1:** $(i)$ Obvious. Operator $B^{h,N}$ is monotone, $V_0 \supseteq V^*$ and $B^{h,N}(V^*) \supset V^*$.

$(ii)$ Proof follows similar arguments as in proof of Theoreom 2.1. Actually, it is possible that $V_n^{h,N} \subset V^{*,h,N}$.

$(iii)$ Note that operator $B^{h,N}$ varies continuously with $h$ and $N$. Hence, the set of fixed points of operators $B^{h,N}$ is an upper semicontinuous correspondence on parameter values $h$ and $N$. Since $V^* \subset V^{*,h,N}$, we get that $V^{*,h,N} \rightarrow V^*$ as $h \rightarrow 0$ and $N \rightarrow \infty$.

**Proof of Theorem 3.2:** The proof follows directly from Blume (1982), Theorems 2.1 and 3.1. The sequence of operators $\{B^{h,N,A_\gamma}\}$ converges to $B$, and the set of fixed points is an upper semicontinuous correspondence. Moreover, the convexified operator $B^{cv}$ has a fixed point $\mu^* \in B^{cv}(\mu^*)$.

**Proof of Proposition 4.2:** As in the original work of Bewley (1972), the existence of a SCE can be established by approximating the infinite-horizon economy by a sequence of finite economies. This is the strategy followed by Jones and Manuelli (1999), but their proof is incomplete and does not apply to sequential competitive economies. As is usual in this approximation argument the hardest part of the proof is to provide upper bounds for equilibrium allocations and prices over all the finite-horizon economies. We nevertheless skip this part since these bounds follow from the proof of Proposition 4.3 below.

Hence, following Jones and Manuelli (1999), we consider the following steps for the proof of a SCE: (i) Existence of an equilibrium for a finite horizon economy. This result is covered by the general proofs of existence of competitive equilibria for economies with taxes and externalities of Arrow and Hahn (1971), Mantel (1975), and Shafer and Sonneschein (1976). (ii) Uniform bounds for equilibrium allocations and prices of finite-horizon economies. As already pointed out, these bounds can be established by the method of proof of Proposition 4.3. (iii) Existence of SEC as a limit point of finite equilibria. The preceding steps (i) and (ii) guarantee that there is a collection of vectors $\left(c_t(z^t), k_{t+1}(z^t), K_t(z^t), L_t(z^t), \overline{K}_t(z^t), w_t(z^t), r_t(z^t)\right)$ that can be obtained as a limit of equilibria of finite economies. It is obvious that for such limiting solution the market clearing conditions must be satisfied at each $z^t$, and that one period-profits must be maximized. Moreover, for each agent $i$ the limiting allocation $\left(c_t^i(z^t), k_{t+1}^i(z^t)\right)$ must satisfy the sequence of budget constraints (4.2), and it is optimal since the discounted objective (4.1) is continuous in the product topology over the set of feasible consumption plans $\left(c_t^i\left(z^t\right)\right)_{t\geq 0}$ which are preferred to the endowment allocation $\left(e_t^i\left(z_t\right)\right)_{t\geq 0}$. This is because feasible consumption plans $\left(c_t^i\left(z^t\right)\right)_{t\geq 0}$ are bounded above (see below) and the endowment process $\left(e_t^i\left(z_t\right)\right)_{t\geq 0}$ is bounded below by a positive quantity.

**Proof of Proposition 4.3:** We first show that there are positive constants $K^{max}$ and $K^{min}$ such that for every equilibrium sequence of physical capital vectors $\left(k_{t+1}^i(z^t))\right)_{t\geq 0}$ if $K^{max} \geq \sum_{i=1}^I k_0^i(z^0) \geq K^{min}$ then $K^{max} \geq \sum_{i=1}^I k_t^i(z^{t+1}) \geq K^{min}$ for all $z^t$. The existence of $K^{max}$ follows directly from Assumption 4.1. In particular, by Assumption 4.1 the marginal productivity of capital converges to zero as $K$ goes to $\infty$ for every fixed $L > 0$. Also, it obvious that $K^{min} \geq 0$. We now claim that if $\lim_{K\to 0} D_1 F\left(K, L\right) = \infty$ for some given positive $L$, then $K^{min} > 0$. For if not, there is a sequence of equilibrium capitals $\left(k_{t+1}^i(z^t)\right)_{t\geq 0}$ such that $\sum_{i=1}^I k_t^i(z^{t+1})$ is arbitrarily close to 0 for some $z^{t+1}$. Under the system of budget constraints (4.2), it follows that there is an arbitrarily small number $\varepsilon \geq 0$ such that $c_t^i(z^t) \geq e_t^i(z^t) - \varepsilon$ for every $i$. Therefore, modulo an arbitrarily small number the derivative $D_1 u(c_t^i(z^t), z_t)$ is bounded by $D_1 u(e_t^i(z^t), z_t)$, and $D_1 F\left(K_t, L_t\right)$ is arbitrarily large. These latter two conditions together are not compatible with utility maximization, since the existence of $K^{max}$ implies that future consumption $c_t^i(z^r|z^t)$ for $r > t$ is uniformly bounded. Consequently, if $\lim_{K\to 0} D_1 F\left(K, L\right) = \infty$ for some positive $L$, then $K^{min} > 0$.

Since $L$ takes on a finite number of positive values, our bounds $K^{max}$ and $K^{min}$ imply that there are constants $r^{max}$ and $w^{max}$ such that for every equilibrium sequence of factor prices $\left(r_t^i\left(z^t\right), w_t^i\left(z^t\right)\right)_{t\geq 0}$ we have $0 \leq r_t(z^t) \leq r^{max}$ and $0 \leq w_t(z^t) \leq w^{max}$ for all $z^t$. Hence, the value function $J^i(k_0^i, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}))$ is well defined, and as already pointed out the derivative $D_1 J^i(\cdot, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}))$ is continuous in $(k_0^i, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}))$. Moreover, by a simple notational change it follows from (4.2) that function $J^i$ can be rewritten as $J^i(a_0^i(z_0), z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}))$, where $a_0^i = e_0^i(z_0) + r_0 k_0^i$. Then we can conclude that $0 \leq D_1 J^i(k_0^i, z_0, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0})) \leq \gamma$, since $e_0^i(z_0)$ is bounded below by a positive number, and as shown above all feasible vectors $(k_0^i, \mathbf{r_0}(\mathbf{z_0}), \mathbf{w_0}(\mathbf{z_0}))$ lie in a compact set.

# B Appendix: Computation

In this appendix we provide details relevant for the numerical implementation of our method for some of the models considered in sections 4-6.

## B.1 Algorithm for the model with capital income taxes

The givens are a family of simplices that partition the state space (in this simple example we employ $N$ closed intervals of uniform size), $\left\{K^j\right\}$, an initial step valued correspondence $V_0^h \supseteq V^*$, and a parameter that determines the accuracy of approximation of the image of the equilibrium correspondence, $\mu > 0$. Let $\underline{m}_0^{K^j}$ and $\overline{m}_0^{K^j}$ be step functions representing the lowest and highest values of the shadow price of investment according to correspondence $V_0^h$ over simplex $K^j$.

We exploit the low dimensionality of this simple example and compute tight upper and lower bound functions for the equilibrium corresponence. To do so, we consider a version of operator $B^{h,N}$ that generates monotone sequences of step functions that converge to a limit satisfying $V^*(k) \subseteq [\underline{m}^{*K^j}(k), \overline{m}^{*K^j}(k)]$ for all $k$. The algorithm works as follows. Initial upper and lower bounds for the equilibrium correspondence are taken as given. Consider then any element $K^j$ of the state space partition and the lower bound value $\underline{m}^{K^j}$. Then we test, conditional on the current shadow value of investment taking values between the postulated lower bound and up to $\mu$ more, whether there is any $k_0 \in K^j$ such that the one period temporal conditions can be satisfied. If the answer is negative, we can postulate a new candidate lower bound, $\underline{m}^{K^j} + \mu$, over the $K^j$ element. A symmetric operation is performed for the upper bound. The details are below.

1. For each $K^j$, we set $\underline{m}_{n+1}^{K^j} = \hat{B}^{h,N}\underline{m}_n^{K^j} = \underline{m}_n^{K^j}$ either if

$$\min_{k_0 \in K^j, m, m_+} \left| \frac{1}{k_0^{1/3} - k_1(m)} - \beta m_+ \right| = 0 \tag{B.1}$$

$$\begin{aligned} & s.t. \\ & m \in \left[\underline{m}_{V_n}^{K^j}, \underline{m}_{V_n}^{K^j} + \mu\right] \\ & m_+ \in \left[\underline{m}_{V_n}^{K^i}(k_1), \overline{m}_{V_n}^{K^i}(k_1)\right] \\ & \text{for } k_1(m) \in K^i \end{aligned} \tag{B.2}$$

or if $\underline{m}_n^{Kj} + \mu > \overline{m}_n^{Kj}$. If any of these two conditions does not hold, then $\underline{m}_{n+1}^{Kj} = \hat{B}^{h,N}\underline{m}_n^{Kj} = \underline{m}_n^{Kj} + \mu$. A symmetric procedure can be used to define $\overline{m}_{n+1}^{Kj} = \hat{B}^{h,N}\overline{m}_n^{Kj}$.

2. Repeat step 1 until the sequence of functions functions $\underline{m}_n^{Kj}$ ,$\overline{m}_n^{Kj}$ have converged (up to a desired accuracy level) to their limits $\underline{m}^{*Kj}$, $\overline{m}^{*K}$.

Notice that the equilibrium correspondence will be contained at all iterations of operator $\hat{B}^{h,N}$. Further, $\hat{B}^{h,N}$ is monotone decreasing by construction, and generates a convergent sequence of compact and convex valued correspondences.

## B.2 Numerical Algorithm for the asset pricing model with two agents and incomplete markets

We now present one implementation of our algorithm in a model with multiple agents. Observe that, in contrast to the previous example where for each value of the state the shadow values of investment would lie in a closed interval, here we have that for each value of the state the shadow values lie in a compact subset of $R^2$.

An important feature of the asset pricing example of subsection 6.1 is that the only variable that takes a continuum of values is the fraction of wealth held by agent 1, $\omega \in [0,1]$, which we partition in a collection of intervals of equal lenght, $\{\Theta^j\}$. We choose the following representation for correspondences. Let any correspondence $V$, its step approximation $V^h$, and $\mu > 0$ be given. Then, $\mathcal{C}(V^h)$ is constituted by the smallest collection of boxes of the form $\left[\hat{m}^1_{\Theta^j,n}, \hat{m}^1_{\Theta^j,n} + \mu\right] \times \left[\hat{m}^2_{\Theta^j,n}, \hat{m}^2_{\Theta^j,n} + \mu\right]$ that contain $V^h$ at each given $\Theta^j$. This is possible because our analysis focuses on compact valued correspondences.

The computable version of operator $B^{h,N}$ takes as given $\tau = 0$ and an initial correspondence $\mathcal{C}(V_0^h) \supseteq V^*$, and works as follows:

1. Let the current value for the exogenously given variables be given. For a representative element

$\left[\hat{m}^1_{\Theta^j,n},\hat{m}^1_{\Theta^j,n}+\mu\right]$ associated to $\mathcal{C}(V^h_0)$ solve

$$\min_{\hat{m}^1,\hat{m}^2,\hat{m}^1_{y'},\hat{m}^2_{y'}} \sum_i (\left\|(d-q)\,U^i_1\left(c^i\right)+\beta E\hat{m}^i_{y'}+q\lambda^i_c+\lambda^i_{ss}\right\|+ \tag{B.3}$$

$$\left\|-pU^i_1(c^i)+\beta E\left[\frac{k}{q}\hat{m}^i_{y'}|\Omega_A\right]+\beta E\left[\frac{\hat{m}^i_{y'}}{q}|\Omega_B\right]+k\lambda^i_c\right\|)$$

$s.t.$

$$\left(\hat{m}^1,\hat{m}^2\right)\in\left[\hat{m}^1_{\Theta^j,n},\hat{m}^1_{\Theta^j,n}+\mu\right]\times\cup_n\left[\hat{m}^2_{\Theta^j,n},\hat{m}^2_{\Theta^j,n}+\mu\right] \tag{B.4}$$

$$\left(\hat{m}^1_{y'},\hat{m}^2_{y'}\right)\in\mathcal{C}(V^h) \tag{B.5}$$

$$\omega'=\frac{\theta q'+\phi\min\left\{b,k\frac{q'}{q}\right\}}{q'} \tag{B.6}$$

$$\text{aggregate and individual constraints,} \tag{B.7}$$

wiht $\Omega_A=\left\{(b,q,q'):\min\left\{b,k\frac{q'}{q}\right\}=k\frac{q'}{q}\right\}$, $\Omega_B=\left\{(b,q,q'):\min\left\{b,k\frac{q'}{q}\right\}=b\right\}$.

2. If the minimum above is equal to zero then $\left[\hat{m}^1_{\Theta^j,n},\hat{m}^1_{\Theta^j,n}+\mu\right]$ also belongs to $\mathcal{C}(V^h_{\tau+1})=B^{h,N}\mathcal{C}(V^h_\tau)$, otherwise it is eliminated. A symmetric procedure is done to determine if a representative element $\left[\hat{m}^2_{\Theta^j,n},\hat{m}^2_{\Theta^j,n}+\mu\right]$ will be part of $\mathcal{C}(V^h_{\tau+1})=B^{h,N}\mathcal{C}(V^h_\tau)$. [10]

3. Perform steps 1 and 2 for all values of the exogenously given variables, elements of the state space partition, and elements in $\mathcal{C}(V^h)$.

4. Increase $\tau$ by one, and repeat steps 1-3 until convergence in the sequence $\left\{V^{h,N}_\tau\right\}$ is achieved.

[10] The only change required is that constraint (B.4) must be changed to $(\hat{m}_1,\hat{m}_2)\in\cup_n\left[m^1_{\Theta^j,n},m^1_{\Theta^j,n}+\mu\right]\times\left[m^2_{\Theta^j,n},m^2_{\Theta^j,n}+\mu\right]$.

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