Consequences of Modeling Habit Persistence[*]

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Abstract

In this paper, we study the stationary and non-stationary equilibria of a deterministic, pure exchange, two-period overlapping generations model with habit persistence. We show that preferences with multiplicative habits can lead to quite different equilibrium outcomes compared to subtractive ones. The two most commonly adopted habit specifications can differ in terms of homotheticity, gross substitutability, and uniqueness of equilibria. We illustrate these differences in terms of steady state equilibria, as well as local dynamics.

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1 Introduction

A recent strand of the economic literature has analyzed theoretical and empirical implications of endogenous preferences, those that depend on time, personal experience, or social conditions. Among the different forms of endogenous preferences the ones displaying habits have received particular attention. The literature on habit persistence specifies preferences either using subtractive (SH) or multiplicative habits (MH), according to the terminology introduced by Carroll (2000).[1] Although habits have helped shed some light on several economic phenomena, not much attention has been paid to the economic consequences of these two alternative formulations. For example, Abel (1990) and Constantinides (1990) reach similar conclusions using the two alternative formulations in the context of risk premium. This may seem to suggest, *prima facie*, that the qualitative properties of models with SH and MH are pretty similar. However, Wendner (2003), shows that the two most commonly used habit specifications may easily lead to opposite implications regarding household savings behavior. He shows that in response to an increase in the strength of habits, young households may increase savings in the case of SH, while they may lower savings in the case of MH. Furthermore, Carroll (2000) points out that under plausible parameter values, SH may give rise to a not well-defined utility in stochastic environments while this is not the case using MH.

The objective of this paper is to study the two alternative formulations adopted in the literature and to examine their differences in terms of multiplicity of steady states, dynamic indeterminacy, and local dynamics. Understanding how optimal savings decisions change according to different preference specifications is crucial since intertemporal consumption decisions are at the core of macroeconomic analysis. We explore the consequences of specifying different forms of habit persistence in a pure exchange, two-period lived overlapping generations model. Habit persistence in this setting has been previously explored. Lahiri and Puhakka (1998) consider SH in a pure exchange overlapping generations framework and show that increasing the strength of habits raises desired savings and might lead to endogenous cycles. In a similar framework, Bunzel (2006) argues that both SH and MH yield dynamic behaviors that are qualitatively very similar. In this study, we show that this finding do not always hold.

Recently, Chen and Ludvigson (2006) point out the lack of theoretical studies on the functional form of habit persistence. Using semi-parametric and structural econometric approaches, they "reverse engineer" the habit specification that best matches the cross-sections of asset returns on a relatively large portfolio. Their empirical findings suggest that the habit function should be non-linear in current and past consumption rather than linear. Hence, they seem to suggest that a MH specification is empirically preferred to a SH one. A theoretical investigation of the underlying properties of the two habit formulations is thus needed.

The main contribution of this paper is to show that modeling preferences with MH or SH yields theoretical predictions that are not necessarily equivalent. We find that modeling habit persistence with MH may not always yield concave or homothetic preferences over consumption. This modeling choice can give rise to multiple monetary steady states, and hence stationary sunspots equilibria may arise. In addition, the stability properties of an

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[1]Some examples of MH are Abel (1990) and Carroll, Overland, and Weil (2000) and for SH, Campbell and Cochrane (1999) and Constantinides (1990).

economy with MH change drastically as we vary the strength of habits. Adopting the hyperbolic absolute risk aversion (HARA) class of preferences, we find instances in which there is a unique steady state under both specifications. However, we also find cases in which there are multiple steady states under MH while there is a unique steady state under SH. Thus, the resulting qualitative properties of the economy under SH and that under MH can be quite different. If the choice of how to model savings behavior in the presence of habits is not innocuous, the predictions and policy prescriptions based on the two types of habit persistence could be quite different as well.

In the next section, we introduce the general model with MH and SH and study the corresponding concavity and homotheticity properties of these preferences. Then, we analyze the steady state equilibria and their associated dynamics. In section 3, we provide a general result on the difference in the dynamics under the Classical case. We present numerical examples that suggest, once again, non-negligible discrepancies across MH and SH in terms of local stability properties of the steady states. The last section summarizes our main findings and concludes. The proofs can be found in the Appendix.

2 The Model

This paper builds on Gale (1973) and Lahiri and Puhakka (1998) by considering a pure exchange overlapping generations model. Complex dynamics and endogenous cycles have been shown to emerge in overlapping generation models with production and/or a variety of different assumptions concerning market imperfections.[2] Here we abstract from technology to reinforce the point we are trying to make: with habits, differences in dynamics and in steady states properties need not to be driven by the production side of the economy, but rather by the modeling choice of preferences alone. Economic activity takes place over infinite discrete time. Each generation lives for two periods and has perfect foresight. Agents are endowed with w_1 units of the single good when young, and with w_2 units when old.

Utility is derived from consumption in both periods. However, due to the presence of habit formation, utility of a given level of consumption when old depends on consumption when young.[3] Formally:

$$V(c_1, c_2) = u(c_1) + \beta\, u\,(c_1, c_2; \gamma) \tag{1}$$

where $\gamma \in (0,1)$ denotes the strength of habits in the instantaneous utility function and β is the discount factor. We assume that the function $u(.)$ is well-behaved, i.e. strictly increasing, strictly concave, homogeneous, and twice continuously differentiable. If one considers SH, then preferences are typically given by $u\,(c_1, c_2; \gamma) = u(c_2 - \gamma c_1)$ as in Lahiri and Puhakka (1998).[4] If, instead, the instantaneous utility is specified with MH, we have $u\,(c_1, c_2; \gamma) = u\,(c_2/c_1^\gamma)$. Regardless of the specification considered, the importance of past consumption in determining the utility derived by the "effective" consumption is increasing with γ. Each agent maximizes utility subject to budget constraints; $c_1 = w_1 - s_t$ when young, and $c_2 = w_2 + R_t s_t$ when old. s_t and R_t denote savings when young and the gross nominal interest rate on savings at time t, respectively. As pointed out by Lahiri and Puhakka (1998), we need

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[2]See Boldrin and Woodford (1990) for a survey.

[3]Specifically, both the absolute level of consumption in the second period and the change in consumption between the two periods are important. The higher the consumption when young, the more consumption in the following period is required to derive a given level of utility.

[4]In this case, one needs to impose also $\gamma < \frac{c_1}{c_2}$.

to impose conditions on the parameters to ensure that the indifference curves are downward sloping. These conditions for the subtractive and multiplicative case are respectively:

$$\frac{u'(c_1)}{u'(c_2 - \gamma c_1)} - \gamma\beta > 0 \tag{2}$$

$$\frac{u'(c_1)}{c_1^{-(1+\gamma)} u'(c_2 c_1^{-\gamma})} - \gamma\beta > 0. \tag{3}$$

In what follows, we first study the concavity, homothetic properties, and the steady state equilibria that emerge in each case. We then explore the local dynamic properties of the equilibria under each habit specification.

2.1 Preferences

One of the most important differences across alternative habit specifications is in terms of their implications for the underlying preferences. Alonso-Carrera, Caballé, and Raurich (2005) show that when habits are introduced multiplicatively in a capital accumulation model, the consumers' objective function might fail to be concave. In this paper, we find a similar result for pure exchange overlapping generation models as stated in Proposition 1.

Proposition 1 (Concavity) *Consider a pure exchange two-period lived overlapping generations model. Then: (i) under SH persistence, the utility function is strictly concave in consumption, and (ii) under MH persistence, concavity is not always ensured.*
Proof. *See Appendix.* ∎

Proposition 1 implies that one needs to be cautious when solving for the consumption plan that maximizes consumers' utility under MH. The next proposition outlines another crucial implication in terms of the savings function resulting from the consumer's problem. The adoption of MH or SH can greatly affect the savings behavior, even if other fundamentals remain the same.

Proposition 2 (Homotheticity) *If the instantaneous utility function u(.) is homogeneous of degree n, then the SH specification yields homothetic preferences whereas the MH specification does not.*
Proof. *See Appendix.* ∎

It is well known that, in overlapping generations models, the saving function is linear in income/endowments if the lifetime utility function is homothetic (De la Croix and Michel, 2002). Thus, in general, we expect to observe substantial differences in the underlying dynamics between SH and MH specifications. This particular comparison is carried out in section 3.

2.2 Steady State Equilibria

In this section, we show that if one adopts SH, then there exists a unique monetary steady state while under MH multiple monetary equilibria cannot be ruled out. The optimal

savings function for a young agent in this model with MH is defined as follows:

$$s^* = \arg\max u(w_1 - s) + \beta u\left(\frac{Rs + w_2}{(w_1 - s)^\gamma}\right). \tag{4}$$

Thus, the equation that defines the optimal savings can be written as:

$$F(s) = -u'(w_1 - s) + \beta u'\left(\frac{Rs + w_2}{(w_1 - s)^\gamma}\right)\left(\frac{R(w_1 - s) + \gamma(Rs + w_2)}{(w_1 - s)^{\gamma+1}}\right) = 0. \tag{5}$$

The implicit function for optimal savings under SH, on the other hand, is given by the following equation:

$$G(s) = -u'(w_1 - s) + (R + \gamma)\beta u'((R + \gamma)s + w_2 - \gamma w_1) = 0. \tag{6}$$

To close the model and to analyze the steady states, we introduce an outside asset into the economy. Following Lahiri and Puhakka (1998), we assume that there is a government that borrows from and lends to the public. This approach is clearly equivalent to injecting valueless fiat money into the economy. Government liabilities in period t are denoted by b_t, and the real deficit by d_t. The government's budget constraint for period t is then given by:

$$d_t + R_{t-1}b_{t-1} = b_t. \tag{7}$$

Asset market equilibrium requires that $b_t = s_t$ for all t. Setting the government's deficit to zero, the law of motion for this economy is defined as follows:

$$s_{t+1} = s_t R_t \tag{8}$$

which also represents the offer curve for this economy. If $b_t > 0$, then the economy is in the Samuelson case according to the terminology first coined by Gale (1973). Instead, if $b_t < 0$, then the economy is in the Classical case.

Using a general instantaneous utility function does not always yield an explicit expression for optimal savings, nor for the gross interest elasticity of savings. Thus, it is not possible to explicitly analyze equation (8). However, we can characterize a crucial difference between SH and MH in a general setting where no explicit functional form assumptions regarding the utility are made. The following proposition states that the standard results on uniqueness of steady states do not necessarily apply to the case with MH. This result will then have crucial implications for the local dynamics of the economy.

Proposition 3 (Uniqueness) *Restrict the parameter space such that the consumer's problem is concave. Under SH, there always exists a unique monetary steady state. Under MH, on the other hand, multiplicity of monetary steady states cannot be ruled out.*
Proof. *See Appendix.* ∎

The economic intuition behind this result is related to the possible lack of gross substitutability under the MH specification. It is well known that when consumption in the first and the second periods are weak gross substitutes, then the steady state is unique.[5] This

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[5]The first to study the gross substitutability condition in a two-period overlapping generation model is Grandmont (1985).

condition can be verified by studying the sign of the intertemporal elasticity of substitution (IES). Under the subtractive specification, IES is always positive, whereas under the multiplicative case it can be negative. Formally, the IES between consumption when young and old can be written as:

$$\Psi(c_1, c_2) = \frac{\frac{1}{c_1 \frac{\partial V}{\partial c_1}} + \frac{1}{c_2 \frac{\partial V}{\partial c_2}}}{-\frac{\frac{\partial^2 V}{(\partial c_1)^2}}{\left(\frac{\partial V}{\partial c_1}\right)^2} + 2\frac{\frac{\partial^2 V}{\partial c_1 \partial c_2}}{\frac{\partial V}{\partial c_1}\frac{\partial V}{\partial c_2}} - \frac{\frac{\partial^2 V}{(\partial c_2)^2}}{\left(\frac{\partial V}{\partial c_2}\right)^2}}. \tag{9}$$

The crucial term in Ψ is $\frac{\partial^2 V}{\partial c_1 \partial c_2}$; if this is negative then gross substitutability could break down. One can verify that under SH:

$$\frac{\partial^2 V}{\partial c_1 \partial c_2} = -\gamma\beta u''(c_2 - c_1) > 0.$$

When we consider MH, instead, we obtain the following:

$$\frac{\partial^2 V}{\partial c_1 \partial c_2} = -\gamma\beta c_1^{-\gamma-1}\left[u'(c_2 c_1^{-\gamma}) + c_2 c_1^{-\gamma} u''(c_2 c_1^{-\gamma})\right].$$

The sign of this expression depends on the underlying parameter values of the model.

Under intertemporally non-separable preferences, the elasticity of substitution between current and future consumption is time variant. In particular, different functional forms for the period utility imply different degrees of IES. A subtractive specification implies that consumption in the first and the second periods are always gross substitutes, whereas this is not necessarily true when a multiplicative formulation is adopted. Due to lack of gross substitutability, it is not possible to rule out the existence of multiple monetary steady states under MH. In the next section, we derive a general result on the dynamics in the Classical case. We also present numerical examples that highlight the differences between MH and SH modeling choices.

3   Dynamics

One clear message so far is that we should expect differences in the local dynamics between the two specifications due to homotheticity or the lack thereof. Before we proceed any further, let us establish a general result for the Classical case which follows directly from a theorem by Gale (1973). The following proposition together with Proposition 2 suggest that alternative habit formulations can yield drastically different results in terms of dynamic indeterminacy in the Classical case.

Proposition 4 (Gale, 1973) *In the Classical case, if the utility function is either (i) separable, or (ii) homothetic then there exist a unique path approaching the steady state given an initial condition.*

Proof. See the proof of Theorem 5 by Gale (1973, p.25). ∎

Weak gross substitutability implies that, if it exists, there is at most one equilibrium price sequence which converges to the steady state. Proposition 4 and Lemma 1 suggest that this is not necessarily the case for MH.

Lemma 1 *In the Classical case, SH lead to determinate dynamic equilibrium whereas MH may display dynamic indeterminacy.*

Proof. Recall that both habit specifications imply non-separable preferences. In addition, Proposition 2 proved that preferences are non-homothetic under MH while they are homothetic under SH. These facts together with Proposition 4 above complete the proof. ∎

In general, the local dynamic properties are determined by the slope of the offer curve given by:

$$\frac{ds_{t+1}}{ds_t} = R_t \left(1 + \frac{1}{\epsilon} \right) \tag{10}$$

where ϵ is the gross interest rate elasticity of savings. Since we can not obtain a closed form solution for the offer curve neither under MH nor under SH, it is not possible to study its behavior analytically. For this reason, we provide numerical examples that illustrate how the dynamics can differ. In particular, we show that multiple monetary steady states are possible under MH.

3.1 Numerical Examples

For our numerical examples, let us consider the class of preferences with hyperbolic absolute risk aversion (HARA). This is a general class that nests, as special cases, the family of utility functions with a constant coefficient of relative risk aversion (CRRA), the one with a constant absolute risk aversion (CARA), and the quadratic utility. The HARA family is commonly used in the finance literature which deals with asset pricing and savings' behavior.[6] Carroll and Kimball (1996) show how CARA and CRRA specifications imply a savings behavior that is qualitatively very different from the one corresponding to a HARA formulation. In particular, they show that under HARA preferences and uncertainty, the consumption function is concave, whereas it is linear when CARA and CRRA specifications are adopted.[7] Therefore, the underlying choice of preferences is crucial.

The HARA utility function takes the following form:

$$u(c) = \frac{1}{1-\sigma} \left[\sigma \left(A + \frac{\alpha c}{\sigma} \right)^{1-\sigma} \right] \tag{11}$$

where A and σ are real numbers, and $\alpha > 0$. Incorporating MH in the HARA family yields the following von Neumann-Morgenstern utility function for our example:

$$V(c_1, c_2) = \frac{1}{1-\sigma} \left[\sigma \left(A + \frac{\alpha c_1}{\sigma} \right)^{1-\sigma} \right] + \beta \frac{1}{1-\sigma} \left[\sigma \left(A + \frac{\alpha}{\sigma} \frac{c_2}{c_1^\gamma} \right)^{1-\sigma} \right]. \tag{12}$$

Typically, to characterize the dynamic properties of this economy, we need to determine the slope of the offer curve at each of the steady states. The marginal rate of substitution at the

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[6]See Ingersoll (1987) for a survey of this literature.

[7]Carroll (2000, p.68) points out other reasons to avoid CRRA in habit formation models. He states that: "CRRA utility in combination with the subtractive formulation...has several theoretical problems, the gravest of which is that for microeconomically plausible parameterizations of consumption variation the equation [of consumption] accumulation can easily lead to a zero or negative argument to the [utility] function...generating infinite negative utility".

initial endowment point should be studied as well. However, the HARA utility function may not always be defined at the origin. Formally, the utility function is defined if $A + \frac{\alpha c}{\sigma} \geq 0$. By substituting the budget constraint for each period into the respective flow utilities we obtain an upper and a lower bound on savings:

$$A + \frac{\alpha(w_1 - s_t)}{\sigma} \geq 0 \qquad\qquad (B1)$$

and

$$A + \frac{\alpha}{\sigma} \frac{(w_2 + R_t s_t)}{(w_1 - s_t)^{\gamma}} \geq 0. \qquad\qquad (B2)$$

Therefore, we need to make sure that these additional restrictions are met when studying the examples.

For the benchmark case, we formulate a simple economy that is in line with general empirical observations. Throughout our numerical analysis we set $\beta = 0.55$ which corresponds to an annual discount rate of 0.97 when the generation considered lasts 20 years. Since there is a lack of evidence in terms of the empirically plausible range of the parameter measuring the strength of habits, we decided to adopt the following calibration strategy for our benchmark example. First, we use data from Gourinchas and Parker (2002) for the life cycle profiles of income and consumption. The endowment parameters of our economy are set to mimic their income data. Based on their data, we computed the ratio of average consumption (income) between the age 45-65 to the average consumption (income) between the age of 26-44. This procedure yields 0.933 as the ratio of average consumption when old to young, and 1.102 as the ratio of average income when old to young. Second, given that there is a non-concavity issue we are forced to use (6), which are the first order conditions under SH, for calibration purposes. We also employ (9) to calibrate the model and retrieve the corresponding values for γ and σ. We force the IES to be one, a value confirmed by most recent studies.[8] We assign the following parameter values to the utility function: $A = -1$, and $\alpha = 1$. The resulting calibrated values are $\gamma = 0.440$ and $\sigma = 0.226$. Finally, using all these same parameter values we compute also the MH case. In Table 1, we report our benchmark example.

	SH	MH
$s^{s.s.}$	-0.089	-0.745
$c_1^{s.s.}$	1.089	1.745
$c_2^{s.s}$	1.012	0.357
ϵ	-6.457	-1.335
$(1 + \frac{1}{\epsilon})$	0.845	0.251
PM	-0.266	-0.068
$detH$	0.072	0.134
$MRSC$	0.55	0.804
$\Psi(c_1^{s.s.}, c_2^{s.s})$	1	0.958

Table 1: Benchmark example.

The tables throughout our paper provide information regarding steady state allocations

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[8]See Guvenen (2006).

for savings ($s^{s.s.}$), consumption when young ($c_1^{s.s.}$) and when old ($c_2^{s.s}$), the gross interest rate elasticity of savings (ϵ), and the slope of the offer curve evaluated at the steady state ($1 + \frac{1}{\epsilon}$). The tables also report the principal minor (PM) and the determinant of the associated Hessian at steady state ($detH$) as well. We do so in light of Proposition 1 that points out the potential lack of concavity under the MH case. Finally, we report the conditions established in equations (2) and (3): we have downward sloping indifference curves and meaningful economic equilibria when $MRSC$ is positive. In addition, we provide the steady state intertemporal elasticity of substitution for each example ($\Psi(c_1^{s.s.}, c_2^{s.s})$). Note that the qualitative properties of the offer curve and that of the steady states can be replicated under many other parameterizations. Clearly, both economies are in the Classical case where steady states savings are negative. In the benchmark case, the local dynamics of the monetary steady states are similar: they are both stable. However, we find that by varying slightly the strength of habits, γ, the dynamic properties of the steady state can greatly differ as illustrated by the following examples. As we increase the strength of habits, the number of steady state monetary equilibria and the corresponding local dynamics vary depending on the specific habit formulation adopted.

Example 1 *Holding all other parameters constant, if we set $\gamma = 0.45825$ we find two monetary steady states under MH and a unique monetary steady state under SH. The local dynamics associated with these two different sets of steady states are quite different.*

	SH	MH	
	Steady State	Steady State I	Steady State II
$s^{s.s.}$	-0.069	-0.744	0.771
$c_1^{s.s.}$	1.069	1.744	0.229
$c_2^{s.s}$	1.033	0.358	1.873
ϵ	-8.184	-1.330	0.007
$(1 + \frac{1}{\epsilon})$	0.878	0.248	142.677
PM	-0.275	-0.068	-221.299
$detH$	0.072	0.135	15.753
$MRSC$	0.550	0.796	0.337
$\Psi(c_1^{s.s.}, c_2^{s.s})$	0.999	0.929	0.012

Table 2: Multiplicity of equilibria under MH but not under SH.

It is interesting to note that, as we increase the strength of habits by only 4 percent relative to our benchmark example,two steady states with opposite stability properties emerge under MH, whereas the steady state for SH remains unique and stable. Steady state I for the MH case in example 1 is stable with converging monotone dynamics. Steady state II, on the other hand, is unstable. Sunspot equilibria can be constructed as a lottery between two deterministic steady states as in Cass and Shell (1983). Furthermore, since multiple steady states exist under MH, bifurcation phenomena are possible as we vary the strength of habits. By changing γ, we can move from a situation with a unique steady state to another where multiple steady states are possible. We may witness the economy switching from a situation with one steady state to a situation with two steady states or vice versa. Another interesting observation is that MH exhibit a Classical steady state and a Samuelsonian

steady state, suggesting that there are discontinuities in the offer curve probably due to the aforementioned lack of concavity.

As we can see from Table 2, multiple steady states are possible when MH are considered. This situation is non-generic since the only way to have multiple monetary steady states is for consumers to be indifferent between two (or more) levels of savings when $R = 1$ holds. In other words, we make sure that the maxima we obtain from the consumers' optimization problem yield exactly the same utility level. Recall from Proposition 3 that multiple maxima do never occur under SH.

Example 2 *Holding all other parameters constant, if $\gamma = 0.600$ we have a unique monetary steady state under both habit specifications. However, these steady states have very different local dynamics.*

	SH	MH
$s^{s.s.}$	0.075	-0.729
$c_1^{s.s.}$	0.924	1.729
$c_2^{s.s}$	1.177	0.373
ϵ	6.244	-1.264
$(1 + \frac{1}{\epsilon})$	1.160	0.209
PM	-0.350	-0.066
$detH$	0.069	0.137
$MRSC$	0.549	0.743
$\Psi(c_1^{s.s.}, c_2^{s.s})$	0.977	0.773

Table 3: Unique equilibrium under SH and MH.

The first thing to notice in this example is that for the same parameterization the two habit specifications yield very different predictions. As one can see from Table 3, the economy under SH is in the Samuelsonian case whereas the economy under MH is in the Classical case. Furthermore, under SH, we find that the monetary steady state is locally unstable because $(1 + \frac{1}{\epsilon})$ is greater than one and endogenous arising volatility is not possible ($\epsilon > 0$). Under MH, we observe monotonic dynamics converging to the steady state. If the choice of how to model savings behavior in the presence of habits is not innocuous, the predictions and policy prescriptions based on the two types of habit persistence could be quite different.

3.2 Endowment profiles or outer utility?

In a recent paper, Bunzel (2006) argues that, regardless of whether one adopts SH or MH, the dynamical properties of the pure exchange economy are very similar. She imposes an outer utility function that is CRRA and for tractability she considers only the case in which the second period endowments are zero. Here we show that Bunzel's irrelevance results are driven by the specific choice of the utility function and not by the particular endowment profile considered. If one adopts the HARA utility function, then the MH specification can indeed affect the qualitative properties of the dynamical system even when there is no second period endowment.

Example 3 *Setting $\gamma = 0.897155$, $w_1 = 3$, and $w_2 = 0$, we find two monetary steady states under MH and a unique monetary steady state under SH. Local dynamics associated with these two sets of steady states are quite different.*

	SH	MH	
	Steady State	Steady State I	Steady State II
$s^{s.s.}$	1.985	1.257	2.424
$c_1^{s.s.}$	1.015	1.742	0.576
$c_2^{s.s}$	1.985	1.257	2.424
ϵ	-0.200	-0.116	0.097
$(1 + \frac{1}{\epsilon})$	-3.984	-7.639	11.315
PM	-0.799	-0.059	-4.264
$detH$	0.158	0.016	0.177
$MRSC$	1.006	1.085	1.019
$\Psi(c_1^{s.s.}, c_2^{s.s})$	0.896	1.181	0.191

Table 4: Multiplicity of equilibria under MH but not under SH.

As we can see from Table 4, not all steady states are unstable under both habit specifications. The local dynamics around the steady state under SH display non-damped oscillations. This is also the case for the steady state I under MH but steady state II has diverging monotonic dynamics, instead.

The point we want to convey by presenting all the previous examples is to show how the stability properties of an economy can differ depending on whether the specification is SH or MH. Lahiri and Puhakka (1998) show that, under the subtractive specification, switches between a Samuelsonian and a Classical economy are possible as the strength of habits varies. We extend their findings to show that, as habits grow stronger, differences in savings behavior across MH and SH cases are possible. The response of savings to a variation in the gross interest rate depends crucially on the adopted specification. In general, with SH, the marginal utility of consumption when old is decreasing in c_1 at a decreasing rate. However, this might not be the case when we use the MH specification. In the latter case, marginal utility of effective consumption when old is decreasing in c_1 at an increasing or at a decreasing rate depending on parameter values. This implies that, agents might value consumption growth more under the multiplicative specification than under the subtractive specification.

4 Conclusion

The objective of this paper is to examine the differences between two alternative formulations used in the habit persistence literature. In particular, we explore the consequences of adopting multiplicative versus subtractive habits in terms of multiplicity of steady states, dynamic indeterminacy, and local dynamics. We find that in a pure exchange two-period lived overlapping generations model, adopting MH as opposed to SH can result in very different outcomes. First, we show that in general under MH, the optimization problem may not

be concave and that preferences are not homothetic. Second, we find that multiple monetary steady states cannot be ruled out under MH. Third, we provide numerical examples that illustrate how the local dynamic properties of the economy crucially depend on the type of habits considered as well as their strength.

Finally, it is worth to point out that we did not fully explore the differences in the global dynamics of the two alternative specifications. Since discontinuities in the offer curve are possible under MH due to the non-concavity of preferences, dynamic equilibrium may not exist for some initial conditions. However, the offer curve under SH is continuous. Our educated guess, therefore, is that SH and MH could yield very different time series patterns. This suggests that the results on the irrelevance of modeling habits by Bunzel (2006) may not hold under a more general specification of preferences even at the global level. We leave this for future research.

Our findings indicate that there are important differences on how savings behave across the two most commonly adopted habit specifications. The choice of how to model savings behavior in the presence of habits is not as innocuous as it might seem. Hence, the predictions and policy prescriptions based on the two types of habit persistence modeling choices could be quite different.

Appendix

Proof of Proposition 1

Proof. (i) The principal minor of the Hessian associated with the SH specification is given by $u''(c_1) + \gamma^2 \beta u''(c_2 - \gamma c_1)$. Note that this expression is always negative. The determinant is given by $\beta u''(c_1) u''(c_2 - \gamma c_1)$ which is always positive. Therefore, the Hessian is negative definite and concavity is ensured.

(ii) Repeating the process for the case of MH, we find that the first principal minor is negative if and only if:

$$u''(c_1) + \gamma\beta(\gamma+1)\frac{c_2}{c_1^{\gamma+2}}u'\left(\frac{c_2}{c_1^\gamma}\right) + \gamma^2\beta\frac{c_2^2}{c_1^{2(\gamma+1)}}u''\left(\frac{c_2}{c_1^\gamma}\right) < 0$$

and the determinant is positive if and only if

$$u''\left(\frac{c_2}{c_1^\gamma}\right)u''(c_1) + \gamma(1-\gamma)\beta\frac{c_2}{c_1^{\gamma+2}}u'\left(\frac{c_2}{c_1^\gamma}\right)u''\left(\frac{c_2}{c_1^\gamma}\right) > \gamma^2\beta\frac{1}{c_1^2}\left(u'\left(\frac{c_2}{c_1^\gamma}\right)\right)^2$$

Therefore, the Hessian is negative semi-definite only under certain parameter restrictions. To check that these conditions could be violated in some cases, we need to provide at least one explicit example where the objective is not concave. Consider the HARA instantaneous utility together with MH: $V(c_1, c_2) = \frac{1}{1-\sigma}\left[\sigma\left(A + \frac{\alpha c_1}{\sigma}\right)^{1-\sigma}\right] + \beta\frac{1}{1-\sigma}\left[\sigma\left(A + \frac{\alpha}{\sigma}\frac{c_2}{c_1^\gamma}\right)^{1-\sigma}\right]$. Let $w_1 = 3$, $w_2 = 1$, $\gamma = 0.75$, $\sigma = 0.39$, $A = -1$, $\alpha = 0.79$, and $\beta = 0.55$. Then for $s_t \in [0.974, 2.289]$, concavity is violated. ∎

Proof of Proposition 2

Proof. It it sufficient to show that the von NeumannMorgenstern utility function in the subtractive case is homogeneous of degree n, while the one in the multiplicative case is not. Let $\lambda > 0$ be a scalar. If we multiply the first and second period consumption by λ in the subtractive case we obtain: $u(\lambda c_1) + \beta u(\lambda c_2 - \gamma\lambda c_1) = u(\lambda c_1) + \beta u(\lambda(c_2 - \gamma c_1))$. Assuming that the felicity function $u(.)$ is homogeneous of degree n, then the overall utility function is clearly homothetic, i.e. $\lambda^n u(c_1) + \lambda^n \beta u(c_2 - \gamma c_1) = \lambda^n V(c_1, c_2)$.

For the multiplicative case, instead, we show that the utility function is not homogeneous of degree n regardless of the degree of homogeneity of $u(.)$. Specifically:

$$V(\lambda^n c_1, \lambda^n c_2) = u(\lambda^n c_1) + \beta u\left(\frac{\lambda^n c_2}{(\lambda^n c_1)^\gamma}\right) = u(\lambda^n c_1) + \beta u\left(\lambda^{n(1-\gamma)}\frac{c_2}{c_1^\gamma}\right)$$

$$= \lambda^n u(c_1) + \lambda^{n(1-\gamma)}\beta u(c_2 - \gamma c_1) \neq \lambda^n V(c_1, c_2)$$

∎

Proof of Proposition 3

Proof. The implicit function that defines the optimal level of savings, $\hat{s}$, under SH is given by:

$$\beta(R + \gamma)u'((R + \gamma)s + w_2 - \gamma w_1) = u'(w_1 - s)$$

Since both the left and the right hand side of the first order condition (FOC) are continuous and differentiable, we can study the sign of the derivatives. More precisely, we find the following expressions:

$$\frac{\partial u'(w_1 - s)}{\partial s} = -u''(w_1 - s) > 0$$

$$\frac{\partial \beta(R + \gamma)u'((R + \gamma)s + w_2 - \gamma w_1)}{\partial s} = \beta(R + \gamma)^2 u''((R + \gamma)s + w_2 - \gamma w_1) < 0$$

Note that the second expression (derivative of the left hand side of the FOC) is strictly decreasing in s and the first expression (derivative of the right hand side of the FOC) is strictly increasing in s. By the intermediate value theorem, there exists a unique level of optimal savings, $\hat{s}$. However, the same conclusion cannot be drawn under MH. The implicit function that defines s^*, the optimal level of savings in the MH case, is given by:

$$\beta u'\left(\frac{Rs + w_2}{(w_1 - s)^\gamma}\right)\left(\frac{R(w_1 - s) + \gamma(Rs + w_2)}{(w_1 - s)^\gamma}\right) = u'(w_1 - s)$$

While the right hand side is strictly increasing in s, the sign of the slope for the left hand side depends on the underlying parameters of the economy. After simplifications, we have the following expression:

$$\frac{\partial \beta u'\left(\frac{Rs+w_2}{(w_1-s)^\gamma}\right)\left[\frac{R(w_1-s)+\gamma(Rs+w_2)}{(w_1-s)^\gamma}\right]}{\partial s} = \beta\left(\frac{R(w_1 - s) + (Rs + w_2)\gamma}{(w_1 - s)^{\gamma+1}}\right) \times$$

$$\left[u''\left(\frac{Rs+w_2}{(w_1-s)^\gamma}\right)\left(\frac{R(w_1-s)+\gamma(Rs+w_2)(w_1-s)^{\gamma-1}}{(w_1-s)^{2\gamma}}\right) + u'\left(\frac{Rs+w_2}{(w_1-s)^\gamma}\right)\left(\frac{3\gamma+1}{w_1-s}\right) \right]$$

Thus, whether the left hand side is increasing or decreasing in s depends on the following condition:

$$-\frac{u'\left(\frac{Rs+w_2}{(w_1-s)^\gamma}\right)}{u''\left(\frac{Rs+w_2}{(w_1-s)^\gamma}\right)} \gtrless \frac{R(w_1 - s) + \gamma(Rs + w_2)(w_1 - s)^{\gamma-1}}{(3\gamma + 1)(w_1 - s)^{2\gamma-1}}$$

Therefore, we cannot rule out the possibilities for multiple or a continuum of equilibrium values of s^*. Note that we cannot rule out the possibility of no equilibria either. ∎

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