THE KALAI-SMORODINSKY BARGAINING SOLUTION MANIPULATED BY PRE-DONATIONS IS CONCESSIONARY*

December 2008

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Abstract

This study examines the manipulability of simple *n*-person bargaining problems by pre-donations where the Kalai-Smorodinsky (KS) solution is operant. We extend previous results on the manipulation of two-person bargaining problems to the *n*-person case. We show that with a pre-bargaining stage where agents are allowed to sign contracts that alter the bargaining set, agents with greater ideal payoffs transform the bargaining set into one on which KS distributes payoffs in accordance with the Concessionary Division Rule of disputed property. The resulting payoff distribution is efficient in that every individual is strictly better-off relative to the original payoff allocation.

Key words: Bargaining, Concession, Manipulation, Pre-donation, Kalai-Smorodinsky Solution

JEL Codes: C7, D7

* We are grateful to Ayca Kara, Cagatay Kayi, Benan Z. Orbay, Brennan C. Platt, Ipek Sanver, Remzi Sanver, and William Thomson for their invaluable comments.
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1. Introduction

Suppose that there is a property whose value can be monetarily evaluated. Many individuals have claims on it and each values the item differently. We face a problem, which is to allocate the available value among the claimants. The property may be divisible or indivisible. In the former case, we can allocate pieces of it; if not, one of the individuals can have it all and then monetarily compensate the others.

This division problem can also be analyzed as a bargaining problem, which describes possible non-negative monetary payoff pairs for the bargainers. In a bargaining problem, there is a threat (or a disagreement) point that summarizes the resulting payoff to each bargainer in case of disagreement and a bargaining set that shows all possible payoffs that can be achieved.

In this paper, we examine the behavior of the n-person Kalai-Smorodinsky (KS) bargaining solution under sequential pre-donations. We define a pre-donation as a monetary transfer of a bargainer's would-be payoffs to others before the final bargain is reached. We refer to this stage in which the bargainers make promises of future transfers as the pre-bargaining stage. We limit our attention to simple bargaining problems, where the threat point is fixed at the origin and the set of Pareto Optimal points defines a linear relationship among the possible payoffs. When the threat point is fixed at the origin, a bargaining problem can be identified via its bargaining set only. Having the Pareto set define a linear relationship among the potential payoffs simplifies the computation of the KS solution on the bargaining sets that result from manipulations via monetary transfers.

We show that, when bargainers with higher valuations (or ideal payoffs) are allowed to sequentially sign contracts to transfer shares of their future payoffs to others with lower valuations, the payoffs under KS applied to the altered bargaining set coincide with those of the Concessionary Division Rule. For example, in the two-person case where the valuation of the bargainers are 1 and α, with $\alpha \geq 1$, this division rule concedes his ideal payoff 1 to the bargainer with lower valuation and pays $\alpha - 1$ to the other, unless $\alpha < 2$, in which case α is shared equally. Therefore, for $\alpha > 2$ manipulation by agent two leads to a payoff of 1 to agent one and $\alpha - 1$ to agent two, which strictly dominates the KS allocation of 1/2 to agent one and $\alpha/2$ to agent two. Hence, pre-donations lead to an efficient outcome in which agents are strictly better-off under the new payoff allocation relative to the original one as long as valuations of bargainers are high enough. In particular, we find that in an n-person simple bargaining problem pre-donations lead to a strictly better outcome for all agents when $\alpha_i = \alpha_{i-1}(n - i + 2)$ for $i = 1, .., n$ where α_i is the valuation of agent i.

Our environment is one in which there is perfect information. The valuations of the individuals, the bargaining set, and the bargaining solution are all common knowledge. Therefore, given this information, individuals can calculate their would-be payoffs. Hence, if an individual finds it in his interest to change the set of possible payoffs without hurting others, he will do so by promising monetary transfers out of his would-be payoffs (payoffs that would materialize when KS is applied to the altered set). Our requirement of Pareto optimality is therefore a strong one: a payoff distribution is optimal if there exists no other distribution that will make at least one individual strictly better-off without making anyone worse-off.

This paper relates to a bigger class of problems where manipulations of equilibrium outcomes are analyzed. The most well-known manipulation mechanism is misrepresentation of utility functions by agents in an exchange economy with a competitive allocation: in order to achieve a better outcome for himself, an agent can behave as if his utility function is different than the true one. Hurwicz (1972) shows that any

Pareto optimal and individually rational reallocation scheme suffers from this problem. Another well-known mechanism is manipulation via hiding, transfer, or destruction of endowments. Some applications of this are Postlewaite (1979) to resource reallocation mechanisms, Sertel (1994) to Lindahl Equilibria, Sertel and Sanver (2002) to the men- or women-optimal matching rule, and Atlamaz and Klaus (2007) to exchange markets with indivisible goods. Our paper is most closely related to Sertel (1992), which examines the manipulability of the two-person Nash bargaining solution under pre-donations. He shows that the bargainer with greater valuation generally alters the bargaining set so that the resulting payoffs under the Nash solution, when applied to the altered set, coincide with that of the Talmudic division rule. Another application of this sort appears in Orbay (2003) which examines pre-donations in two-person bargaining problems where the threat point may be in the positive orthant, with an application to collusion in an asymmetric duopoly.

The paper proceeds as follows: Section 2 provides definitions and presents the model. Section 3 illustrates the behavior of the KS solution under manipulation via pre-donations in the three-person bargaining problem and proves that the outcome coincides with that of the Concessionary division rule. Section 4 extends the results to the *n*-person case. Section 5 concludes.

2. The Model

Let $N = \{1,2,\dots,n\}$ be the set of agents. Given any integer $n \geq 2$, an *n-person bargaining problem* is any ordered pair (S, d) where S is a compact and convex subset of R^n, $d = \{d_1, \dots, d_n\} \in S$, and there exists $u \in S$ such that $u_i > d_i$ for every $i \in N$. Here, S is called *the bargaining set* and d is called *the disagreement* or *threat point.* Following Sertel (1992), without loss of generality, we set $d = \underline{0}$, the origin of R^n, so that suppressing d, a bargaining problem is determined by its bargaining set only. We regard the bargaining problem as *simple* if and only if it has a bargaining set of the form:

$$S_\alpha=\left\{u \in R^n \,|\, u_1 \leq 1 \text{ and } u_k \leq \alpha_k \left(1 - \sum_{i=1}^{k-1} \frac{u_i}{\alpha_i}\right) \text{ whenever } k, k-1 \in N\right\}$$

for some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in R^n_+$ with $1 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$.

We can interpret this problem as a property division problem as in Sertel (1992). In this interpretation we assume that there is a certain item of property which can be monetarily valued. Claimants have different valuations on the property, so that agent i's valuation is α_i times the valuation of agent 1 (normalized to one in our problem). The division problem distributes α_n, the highest claimed value, among the claimants.[1] If the property is divisible, we can we can distribute shares of it among our agents. If not, we can give it to one of them and require her to monetarily compensate the others.

Definition 1. An n-person *division rule* is any function D which assigns to each simple bargaining problem S_α a point $D(S_\alpha) = \left(D_1(\alpha), D_2(\alpha), \dots, D_n(\alpha)\right)$ in R^n_+ such that for each $i \in N$ and $\alpha_n \in [1, \infty)$, $D_i(\alpha) \in [0, \alpha_n]$ and $\sum_{i=1}^{n} D_i(\alpha) = \alpha_n$.

[1] A good example of this would be an abstract painting that is inherited by two distant cousins, one living in a modern city and the other in a rural area. The item would be much more valuable to the former, as she has the opportunity to either sell it at a high price to someone else in her community or enjoy the painting in her beautifully decorated home.

In this context where bargainer's valuations are ranked as in a simple bargaining problem, a division rule will be called *Concessionary* if it obeys the following rules:

- Starting from the first agent, each agent $i + 1$ concedes his ideal payoff to agent i if, for each i, agent $i + 1$'s payoff is no less than agent i's payoff.
- Whenever agent $i + 1$'s payoff is less than agent i's payoff for some i, the property is redistributed as follows: the agents with lower valuations are conceded the maximum payoff they can achieve such that each agent's payoff is no more than his ideal and, for each i, agent $i + 1$ receives a payoff that is no less than agent i's.

We will denote the Concessionary division rule by C. Note that when agent $i + 1$'s valuation is high enough relative agent i's, the Concessionary rule will divide the property such that each agent gets his ideal payoff, that is, for each i, $D_i(\alpha) = \alpha_i - \alpha_{i-1}$ whenever $\alpha_{i+1} \geq (n - i + 1)\alpha_i$.

Definition 2. Given a bargaining problem (S, d), a point $u \in S$ is *Pareto Optimal* if and only if $w = u$ for all $w \in S$ with $w \geq u$.[2] Given any set $S \subseteq R^n$, we define its Pareto frontier $P(S)$ as:

$$P(S) = \left\{u \in S \middle| u \leq u' \in S \Rightarrow u = u'\right\}.$$

Definition 3. Given a bargaining problem (S, d) and a point $u \in R^n$, we say that u is individually rational if and only if $u \geq d$.

Definition 4. Let B denote the set of all bargaining problems. A *bargaining solution* is a function $f: B \to R^n$ such that for every $(S, d) \in B$, $f(S, d) \in S$.

Given an *n*-person bargaining problem (S, d), let $\bar{u}_i$ denote the ideal payoff of i, i.e., the maximal payoff $\bar{u}_i = \max_{u \in S, u \geq d} u_i$.The Kalai-Smorodinsky solution is the function *KS* that chooses the unique Pareto optimal point $u = (u_1, \dots, u_n)$ in S such that:

$$\frac{u_i - d_i}{\bar{u}_i - d_i} = \frac{u_{i+1} - d_{i+1}}{\bar{u}_{i+1} - d_{i+1}} \text{ for all } i = 1,2, \dots, n - 1.$$

Figure 1 shows the two-person simple bargaining set, its Pareto frontier (shown in bold) and *KS*. The bargaining set is a triangle and the set of Pareto optimal points is a line. Notice that the solution is on the Pareto frontier and each agent receives one half of his ideal payoff.

Definition 5. Given a bargaining problem (S, d) with d fixed at the origin, a *pre-donation* from agent i to j is any function $\Lambda_i: R^2 \to R^2$, parameterized by some $\lambda_{i,j} \in [0,1)$, which transforms each $(u_i, u_j) \in R^2$ into $\Lambda_i(u_i, u_j) = \left((1 - \lambda_{i,j})u_i,\ u_j + \lambda_{i,j}u_i\right)$.

Given any S_α and pre-donation Λ_i, we write:

$$\Lambda_i(S_\alpha) = \left\{u' \in R^n \middle| \begin{array}{c} u_i' = (1 - \lambda_{i,j})u_i, u_j' = u_j + \lambda_{i,j}u_i \ for\ some\ i, j \in N\ and \\ u_k' = u_k\ for\ all\ k \in N - \{i, j\} \end{array} \right\}$$

[2] $w \geq u$ if and only if $w_i \geq u_i$ for all $i \in N$.

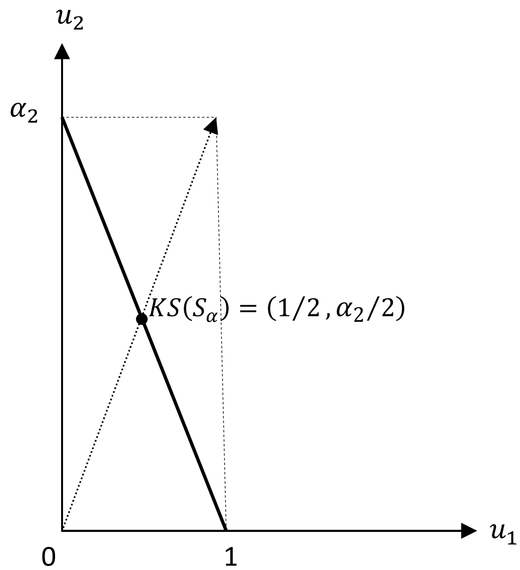

Figure 1: The two-person simple bargaining set, its Pareto frontier and Kalai-Smorodinsky solution.

to denote the bargaining set after agent i's manipulation. Hence, a pre-donation from individual i to j simply augments payoffs of the latter by transferring him a share $\lambda_{i,j}$ of the former's payoff.

In the next section we show the manipulability of a three-person simple bargaining problem. This will fix the ideas for the *n*-person case.

3. A Three-person Bargaining Problem

Consider a three-person simple bargaining problem where the bargaining set is:

$$S_\alpha = \left\{ u \in R_+^3 \middle| u_1 \leq 1,\ u_2 \leq \alpha_2(1 - u_1)\,,\ u_3 \leq \alpha_3 \left(1 - u_1 - \frac{u_2}{\alpha_2}\right) \right\}$$

with $1 \leq \alpha_2 \leq \alpha_3$. This set is a tetrahedron with the disagreement point fixed at the origin, as shown in Figure 2. The Pareto frontier of S_α is the closed convex hull $P(S_\alpha) = H[(1,0,0), (0, \alpha_2, 0), (0,0, \alpha_3)]$. On this set, *KS* picks the point $KS(S_\alpha) = (1/3\,, \alpha_2/3\,, \alpha_3/3)$.

We now show that KS is manipulable via pre-donations. We assume that donations occur sequentially, that is, agents are given a chance to transfer a portion of their would-be payoffs in the order that their valuations are ranked. Therefore, at the first stage of our setting, agent two offers a gift to agent one, and in the next stage agent three makes his offer to agents two and one simultaneously. In Proposition 1 we show that an agent has no incentive to pre-donate to another whose ideal payoff is higher than his own. Therefore, agent one will never offer a transfer to agents two or three, and agent two will never find it profitable to offer a transfer to agent three.

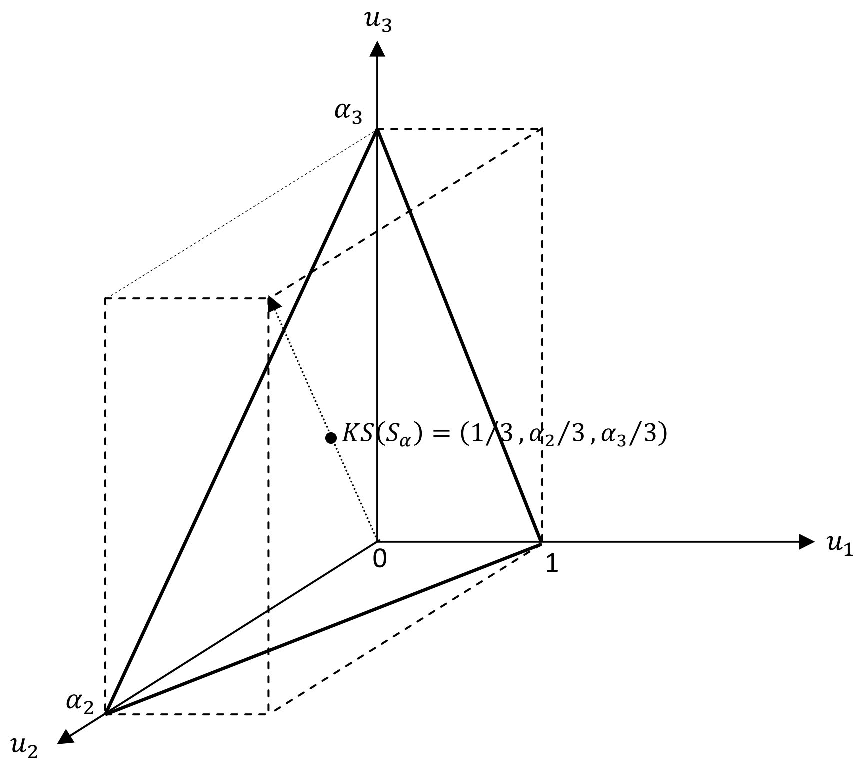

Figure 2: The three-person simple bargaining set and its KS solution

Let agent two pre-donate a share $\lambda_{2,1}$ out of his future payoffs to agent one. We denote the bargaining set after his donation as $\Lambda_2(S_\alpha)$. Figure 3 illustrates this set. Its Pareto frontier is:

$$P(\Lambda_2(S_\alpha)) = \begin{cases} H[(1,0,0), (\alpha_2\lambda_{2,1}, \alpha_2(1-\lambda_{2,1}), 0), (0,0,\alpha_3)] & if\ \alpha_2\lambda_{2,1} < 1 \\ H[(\alpha_2\lambda_{2,1}, \alpha_2(1-\lambda_{2,1}), 0), (0,0,\alpha_3)] & otherwise. \end{cases}$$

Given the Pareto set, we check that the KS solution is:

$$KS(\Lambda_2(S_\alpha)) = \begin{cases} \left(\frac{1}{3-\alpha_2\lambda_{2,1}}, \frac{(1-\lambda_{2,1})\alpha_2}{3-\alpha_2\lambda_{2,1}}, \frac{\alpha_3}{3-\alpha_2\lambda_{2,1}}\right) & if\ \alpha_2\lambda_{2,1} < 1 \\ \left(\frac{\alpha_2\lambda_{2,1}}{2}, \frac{(1-\lambda_{2,1})\alpha_2}{2}, \frac{\alpha_3}{2}\right) & otherwise. \end{cases}$$

It is clear that when $\alpha_2\lambda_{2,1} < 1$, the payoff of the donor (agent two) is increasing in $\lambda_{2,1}$ if and only if $\alpha_2 > 3$. So, we set $\lambda_{2,1}$ as high as possible, which approaches $\frac{1}{\alpha_2}$. Similarly, when $\alpha_2\lambda_{2,1} \geq 1$, his payoff is decreasing in $\lambda_{2,1}$; so we choose $\lambda_{2,1}$ as small as possible. Therefore, the donor's payoff under KS is maximized, as a function of $\lambda_{2,1} \in [0,1)$, at:

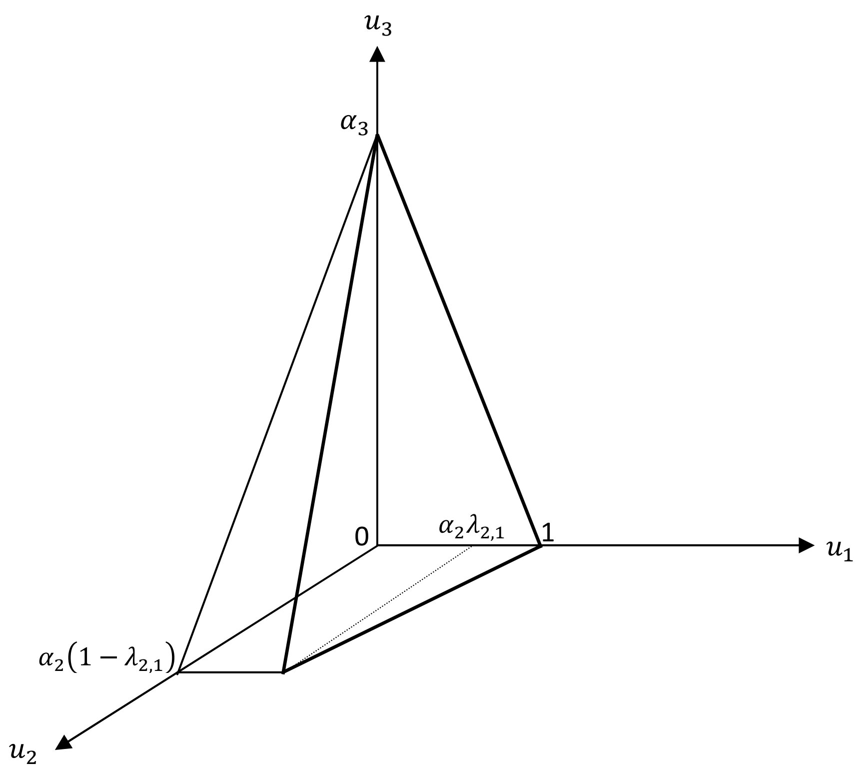

Figure 3: The three person bargaining set manipulated by bargainer two (shown for $\alpha_2\lambda_{2,1} < 1$)

$$\lambda_{2,1}^*(\alpha_2) = \begin{cases} \dfrac{1}{\alpha_2} \; if \; \alpha_2 > 3 \\ \left[0, \dfrac{1}{\alpha_2}\right] \; if \; \alpha_2 = 3 \\ 0 \;\; if \; \alpha_2 < 3. \end{cases}$$

For $\alpha_2 > 3$, the optimal donation shares turn the base of the tetrahedron into a rectangle where two's ideal payoff is now $\alpha_2 - 1$. Denoting the bargaining set after two's optimal donation by $\Lambda_2^*(S_\alpha)$, the payoffs to agents are:

$$KS\big(\Lambda_2^*(S_\alpha)\big) = \begin{cases} \left(\dfrac{1}{2}, \dfrac{(\alpha_2 - 1)}{2}, \dfrac{\alpha_3}{2}\right) \; if \; \alpha_2 > 3 \\ \left[\dfrac{1}{3}, \dfrac{1}{2}\right] \times \left\{\dfrac{\alpha_2}{3}\right\} \times \left[\dfrac{\alpha_2}{3}, \dfrac{\alpha_3}{3}\right] \; if \; \alpha_2 = 3 \\ \left(\dfrac{1}{3}, \dfrac{\alpha_2}{3}, \dfrac{\alpha_3}{3}\right) \;\; if \; \alpha_2 < 3. \end{cases}$$

Notice that agent two achieves a higher payoff by manipulating (for, $\frac{(\alpha_2-1)}{2} > \frac{\alpha_2}{3}$ when $\alpha_2 > 3$).

In the second stage of our setting, the bargainer who has the highest utility per share of the item, agent three, is allowed to offer gifts to agents one and two simultaneously. We do not allow for a pre-donation in which agent three offers a gift to one of the bargainers and then to the other. We further assume that gift offers should be proportional to the agents ideal payoffs on the relevant bargaining set, that is, we require $\lambda_{3,2} = \lambda_{3,1}(\alpha_2 - 1)$. These two assumptions guarantee that the Pareto optimal points lie on a plane; which enables us to calculate the payoffs in a straightforward way.[3] Given $\Lambda_2^*(S_\alpha)$ and any pre-donation shares $\lambda_{3,1}$ and $\lambda_{3,2}$, the bargaining set after agent three's manipulation (illustrated in Figure 4) is:

$$\Lambda_3\big(\Lambda_2^*(S_\alpha)\big) = \big\{\big(u_1 + \lambda_{3,1}u_3, u_2 + \lambda_{3,2}u_3, (1 - \lambda_{3,1} - \lambda_{3,2})u_3\big)\big|(u_1, u_2, u_3) \in \Lambda_2^*(S_\alpha)\big\}.$$

In this case, it can easily be verified that the Pareto set and the KS solution on $\Lambda_3\big(\Lambda_2^*(S_\alpha)\big)$ are (using $\lambda_{3,2} = \lambda_{3,1}(\alpha_2 - 1)$):

$$P\Big(\Lambda_3\big(\Lambda_2^*(S_\alpha)\big)\Big) = \begin{cases} H\Big[\big(\alpha_3\lambda_{3,1},\ \alpha_3\ \lambda_{3,2},\ \alpha_3(1 - \lambda_{3,1} - \lambda_{3,2})\big), (1, \alpha_2 - 1, 0)\Big] & if\ \alpha_3\lambda_{3,1} < 1 \\ \big(\alpha_3\lambda_{3,1},\ \alpha_3\ \lambda_{3,2},\ \alpha_3(1 - \lambda_{3,1} - \lambda_{3,2})\big) & otherwise, \end{cases}$$

$$KS\Big(\Lambda_3\big(\Lambda_2^*(S_\alpha)\big)\Big) = \begin{cases} \left(\dfrac{1}{2 - \alpha_3\lambda_{3,1}}, \dfrac{(\alpha_2 - 1)}{2 - \alpha_3\lambda_{3,1}}, \dfrac{(1 - \alpha_2\lambda_{3,1})\alpha_3}{2 - \alpha_3\lambda_{3,1}}\right) & if\ \alpha_2\lambda_{3,1} < 1 \\ \big(\alpha_3\lambda_{3,1}, \alpha_3\lambda_{3,1}(\alpha_2 - 1), \alpha_3(1 - \alpha_2\lambda_{3,1})\big) & otherwise. \end{cases}$$

Note that agent three's payoff is decreasing in $\lambda_{3,1}$ whenever $\alpha_3\lambda_{3,1} \geq 1$; hence we set $\lambda_{3,1}$ as low as possible, that is, $\lambda_{3,1}^* = \frac{1}{\alpha_3}$. In the other case, his payoff is increasing in $\lambda_{3,1}$ if and only if $\alpha_3 > 2\alpha_2$ (for, the derivative of $\frac{(1-\alpha_2\lambda_{3,1})\alpha_3}{2-\alpha_3\lambda_{3,1}}$ with respect to $\lambda_{3,1}$ has the sign of $\alpha_3 - 2\alpha_2$); so we choose $\lambda_{3,1}$ as high as possible and set $\lambda_{3,1}^* = \frac{1}{\alpha_3}$. Thus, whenever $\alpha_3\lambda_{3,1} < 1$, agent three's payoff is maximized at:

$$\big(\lambda_{3,1}^*, \lambda_{3,2}^*\big) = \begin{cases} \left(\dfrac{1}{\alpha_3}, \dfrac{(\alpha_2 - 1)}{\alpha_3}\right) & if\ \alpha_3 > 2\alpha_2 \\ \left(\left[0, \dfrac{1}{\alpha_3}\right], \left[0, \dfrac{(\alpha_2 - 1)}{\alpha_3}\right]\right) & if\ \alpha_3 = 2\alpha_2 \\ (0,0) & if\ \ \alpha_3 < 2. \end{cases}$$

The resulting the payoffs are:

$$KS\Big(\Lambda_3\big(\Lambda_2^*(S_\alpha)\big)\Big) = \left\{\begin{matrix} (1, \alpha_2 - 1, \alpha_3 - \alpha_2) & if\ \alpha_3 > 2\alpha_2 \\ \left[\dfrac{1}{2}, 1\right] \times \left[\dfrac{\alpha_2 - 1}{2}, \alpha_2 - 1\right] \times \left[\dfrac{\alpha_3}{2}, \alpha_3 - \alpha_2\right] & if\ \alpha_3 = 2\alpha_2 \\ \left(\dfrac{1}{2}, \dfrac{\alpha_2 - 1}{2}, \dfrac{\alpha_3}{2}\right) & otherwise. \end{matrix}\right\}$$

[3]Note that offers are public information here. Therefore, this assumption might also be justified by the fact that any other offer would result in one of the bargainers leaving the table, as that would not obey the spirit of the KS where payoffs are proportional to valuations. A breakdown of negotiations would mean a zero payoff to all.

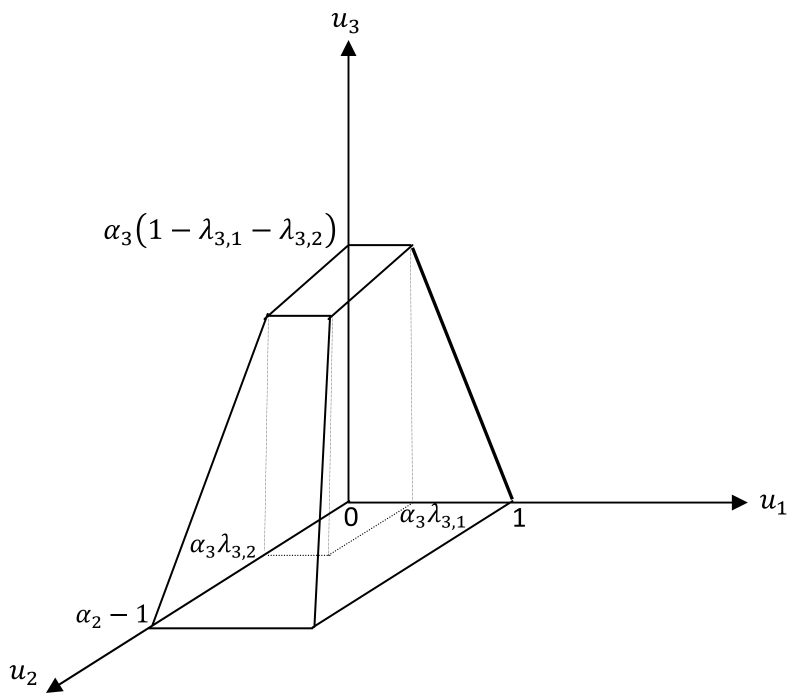

Figure 4: The bargaining set after agent three's donations.

Thus, when $\alpha_3 > 2\alpha_2$, pre-donation results in payoffs that coincide with those of the Concessionary division rule applied to the original set S_α. Note that no agent has an incentive to move away from the final allocation as each receives a payoff which is larger than his payoff had the KS solution been applied to the original bargaining set.

There is a natural question to ask here: what would happen if the sequence in which donations are made would change? That is, what if agent three was the first to make donations to the others, followed by agent two? In the Appendix we show that the final allocation would be $(1, \alpha_2 - 1, \alpha_3 - \alpha_2 - 1)$. In that case both agent three's payoff $(\alpha_3 - \alpha_2 - 1)$ and the total value allocated $(\alpha_3 - 1)$ would be lower compared to those in our original order of pre-donations (where agent three gets $\alpha_3 - \alpha_2$ and the total allocated value is α_3); however, the payoffs after manipulation would still dominate the non-manipulated ones.

4. Extension to the *n*-person Case

In this section we generalize our result to the *n*-person simple bargaining problem. In our setting there are *n* bargainers whose valuations on some divisible disputed property differ. We denote the value of the property for the i^{th} bargainer by α_i $(i \in N)$ with $1 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. In the pre-bargaining stage agents are allowed, sequentially, to give shares out of their future payoffs to their opponents, starting with the second bargainer. After the n^{th} bargainer completes his turn, the KS solution determines the payoffs to all. As in the previous section, $\Lambda_i(.)$ denotes the bargaining set after agent i donates shares $\{\lambda_{i,j}\}_{j=1}^{n}$ of his ideal payoff to his opponents and $\Lambda_i^*(.)$ denotes the resulting set when the optimal pre-donations $\lambda_{i,j}^*$ are applied to the problem.

Lemma: Let $N = \{1,2,\dots,n\}$ be the set of agents and S_α be an *n*-person simple bargaining set. The first bargainer (the bargainer with the lowest ideal payoff) has no incentive to pre-donate to any other bargainer.

Proof. Take a simple bargaining set S_α and any bargainer j such that $\alpha_j > 1$. Assume that agent one manipulates the bargaining set via a pre-donation to j. Let $x_i = (x_{i1},\dots,x_{in})$ denote a generic point in R^n. After agent one's manipulation, the Pareto set will be the closed convex hull of *n* different points in R^n:

$$H[x_1, x_2, \dots, x_n]$$

where $x_{1k} = \begin{cases} 1-\lambda_{1,j} \; for \; k=1 \\ \alpha_j + \lambda_{1,j} \; for \; k=j \\ 0 \qquad otherwise, \end{cases}$

and for $i = 2,\dots,n$, $x_{ik} = \begin{cases} \alpha_i \; for \; k=i \\ 0 \; otherwise. \end{cases}$

KS picks the unique optimal point $u = (u_1, u_2, \dots, u_n)$ with $\frac{u_1}{1-\lambda_{1,j}} = \frac{u_2}{\alpha_2} = \cdots = \frac{u_n}{\alpha_n}$.

Solving this system, we find that the KS payoff for the pre-donor is:

$$u_1 = \frac{\alpha_j(1-\lambda_{1,j})}{n\alpha_j - \lambda_{1,j}}.$$

Maximizing u_1 with respect to $\lambda_{1,j}$, it is apparent that u_1 is increasing in $\lambda_{1,j}$ if and only if $n\alpha_j < 1$, which contradicts our assumption that $\alpha_j > 1$. Hence, agent one has no incentive to give a share out of his future payoffs to any other agent.

■

Corollary. No bargainer prefers to pre-donate another whose ideal payoff is larger than him.

Theorem 1. Let $N = \{1,2,\dots,n\}$ be the set of agents and S_α an *n*-person simple bargaining set. Assume that agents are allowed to pre-donate sequentially, obeying the ascending order of their ideal payoffs. If pre-donations are proportional to the agents' maximum possible payoffs at each stage of the manipulation, then the payoff distribution under the manipulated KS solution coincides with that of the Concessionary division rule whenever $\alpha_i = \alpha_{i-1}(n-i+2)$ for $i \in N$.

Proof. We show, by induction on m, that the optimal behavior of any agent m, $m > 1$, is to choose the shares to be donated to other agents such that $\lambda_{m,k}^* = \frac{\alpha_k - \alpha_{k-1}}{\alpha_m}$ for $k = 1,2,..,m-1$.

We start with agent two. He pre-donates a share $\lambda_{2,1}$ to agent one. When $\alpha_2\lambda_{2,1} < 1$, the Pareto frontier is the closed convex hull $H[x_1, x_2, \dots, x_n]$, where:

$$x_{2k} = \begin{cases} \alpha_2\lambda_{2,k} & for\ k = 1 \\ \alpha_2(1-\lambda_{2,1}) & for\ k = 2 \\ 0 & otherwise, \end{cases}$$

and for $i \in N - \{2\}$, $x_{ik} = \begin{cases} \alpha_k & for\ k = i \\ 0 & otherwise. \end{cases}$

Whenever $\alpha_2\lambda_{2,1} \geq 1$, the optimal set is the hull of n-1 points in the n-dimensional space such that:

$$x_{1k} = \begin{cases} \alpha_2\lambda_{2,k} & for\ k = 1 \\ \alpha_2(1-\lambda_{2,1}) & for\ k = 2 \\ 0 & otherwise, \end{cases}$$

and for $i \in N - \{1,2\}$, $x_{ik} = \begin{cases} \alpha_k & for\ k = i \\ 0 & otherwise. \end{cases}$

The KS solution picks the unique point $u = (u_1, u_2, \dots, u_n)$ on the Pareto frontier with:

$u_1 = \frac{u_2}{\alpha_2(1-\lambda_{2,1})} = \cdots = \frac{u_n}{\alpha_n}$ when $\alpha_2\lambda_{2,1} < 1$ and $\frac{u_1}{\alpha_2\lambda_{2,1}} = \frac{u_2}{\alpha_2(1-\lambda_{2,1})} = \cdots = \frac{u_n}{\alpha_n}$ otherwise. Thus, the payoffs assigned to the bargainers are:

$$KS(\Lambda_2(S_\alpha)) = \begin{cases} \left(\frac{1}{n-\alpha_2\lambda_{2,1}}, \frac{\alpha_2(1-\lambda_{2,1})}{n-\alpha_2\lambda_{2,1}}, \dots, \frac{\alpha_n}{n-\alpha_2\lambda_{2,1}}\right) & if\ \alpha_2\lambda_{2,1} < 1 \\ \left(\frac{\alpha_2\lambda_{2,1}}{n-1}, \frac{\alpha_2(1-\lambda_{2,1})}{n-1}, \dots, \frac{\alpha_n}{n-1}\right) & otherwise. \end{cases}$$

In the case $\alpha_2\lambda_{2,1} < 1$, agent two's payoff is increasing in his donation if and only if $\alpha_2 > n$; so $\lambda^*_{2,1}(\alpha_2) = \frac{1}{\alpha_2}$ gives the highest payoff to him as long as $\alpha_2 > n$. When $\alpha_2\lambda_{2,1} \geq 1$, agent two's payoff is decreasing in his donation; so we set as low as possible, i.e, $\lambda^*_{2,1}(\alpha_2) = \frac{1}{\alpha_2}$. With these optimal shares to be promised, the payoff distribution is:

$$KS(\Lambda^*_2(S_\alpha)) = \left(\frac{1}{n-1}, \frac{\alpha_2 - 1}{n-1}, \dots, \frac{\alpha_n}{n-1}\right).$$

It is important to note here that if $\alpha_2 = n$, our bargainer is indifferent between manipulating or not, whereas if $\alpha_2 < n$, he will not manipulate.

Now, assume that the optimal pre-donation shares for agent m, $m > 2$, are $\lambda^*_{m,k} = \frac{\alpha_k - \alpha_{k-1}}{\alpha_m}$ for $k = 1,2,..,m-1$. We will show that the optimal behavior of agent $m+1$ is to choose the shares to be donated according to our claim.

Assume pre-donations $\{\lambda_{m+1,k}\}_{k=1}^{m}$. As before, we impose $\lambda_{m+1,k+1} = (\alpha_{k+1} - \alpha_k)\lambda_{m+1,k}$ for $k = 1,2,\dots,m$, that is, the shares to be donated must be proportional to the receivers' maximum possible payoffs. As shares are proportional, we can reduce the problem into a much simpler one in which the Pareto set can be defined depending on the size of the promise that bargainer one receives. There are two cases.

First, we analyze the case when $\alpha_{m+1}\lambda_{m+1,1} < 1$. The Pareto frontier of the manipulated bargaining set will be a closed convex hull of $n-m+1$ points in the n-dimensional space. In particular, those points are:

$$x_{1k} = \begin{cases} \alpha_k - \alpha_{k-1} & for\ k \le m \\ 0 & otherwise, \end{cases}$$

$$x_{2k} = \begin{cases} \alpha_{m+1}\lambda_{m+1,k} & for\ k \le m \\ \alpha_{m+1}\left(1 - \sum_{k=1}^{m} \lambda_{m+1,k}\right) & for\ k = m+1 \\ 0 & otherwise, \end{cases}$$

and for $i = 3, 4, \dots, n-m+1$, $x_{ik} = \begin{cases} \alpha_{m-1+i} & for\ k = m-1+i \\ 0 & otherwise. \end{cases}$

The KS picks the optimal point $u = (u_1, u_2, \dots, u_n)$ such that $\frac{u_i}{\alpha_i - \alpha_{i-1}} = \frac{u_{m+1}}{\alpha_{m+1}(1-\sum_{k=1}^{m}\lambda_{m+1,k})} = \frac{u_j}{\alpha_j}$ for $i = 1,2,\dots,m$ and $j = m+2,\dots,n$. The solution of the system yields the following payoffs by KS:

$$u_i = \begin{cases} \dfrac{\alpha_i - \alpha_{i-1}}{n-m+1-\alpha_{m+1}\lambda_{m+1,1}} & for\ i = 1,\dots,m \\ \dfrac{\alpha_{m+1}\left(1 - \sum_{k=1}^{m}\lambda_{m+1,k}\right)}{n-m+1-\alpha_{m+1}\lambda_{m+1,1}} & for\ i = m+1 \\ \dfrac{\alpha_i}{n-m+1-\alpha_{m+1}\lambda_{m+1,1}} & otherwise. \end{cases}$$

It is straightforward to verify that the payoff of agent $m+1$ is increasing in $\lambda_{m+1,1}$ if and only if $\alpha_{m+1} > \alpha_m(n-m+1)$, in which case we set $\lambda^*_{m+1,1}(\alpha_m) = \frac{1}{\alpha_m}$. Plugging this back in the KS solution, we get:

$$KS(\Lambda^*_{m+1}(.)) = \left(\frac{1}{n-m}, \frac{\alpha_2 - 1}{n-m}, \dots, \frac{\alpha_{m+1} - \alpha_m}{n-m}, \dots, \frac{\alpha_n}{n-m}\right).$$

We note that whenever $\alpha_{m+1} = \alpha_m(n-m+1)$, agent $m+1$ will be indifferent between making a pre-donation or not, and when $\alpha_{m+1} < \alpha_m(n-m+1)$, he will choose not to manipulate.

In the remaining case when $\alpha_{m+1}\lambda_{m+1,1} \ge 1$, we verify that KS payoffs are:

$$u_i = \begin{cases} \dfrac{\alpha_{m+1}\lambda_{m+1,i}}{n-m} & for\ i = 1,\dots,m \\ \dfrac{\alpha_{m+1}\left(1 - \sum_{k=1}^{m}\lambda_{m+1,k}\right)}{n-m} & for\ i = m+1 \\ \dfrac{\alpha_i}{n-m} & otherwise. \end{cases}$$

The payoff of agent $m+1$ is decreasing in $\lambda_{m+1,1}$; hence we set $\lambda^*_{m+1,1}(\alpha_m) = \frac{1}{\alpha_m}$. Therefore, the KS payoffs, again, are:

$$KS(\Lambda^*_{m+1}(.)) = \left(\frac{1}{n-m}, \frac{\alpha_2 - 1}{n-m}, \dots, \frac{\alpha_{m+1} - \alpha_m}{n-m}, \dots, \frac{\alpha_n}{n-m}\right).$$

Letting $n = m + 1$, we see that

$$KS\big(\Lambda_n^*(\alpha_n)\big) = (1, \alpha_2 - 1, \alpha_3 - \alpha_2, \dots, \alpha_n - \alpha_{n-1}).$$

■

Next, we demonstrate a rather strong result. When all the pre-donations are done, one might wonder if any of the agents would deviate from the final KS payoff allocation by making a pre-donation to any other agent. In other words, we might ask whether the final payoff allocation in Theorem 1 is stable. We find that no other cycle of pre-donations would make our agents better-off; hence no agent has an incentive to deviate from the final allocation. Note that agents with higher valuations have already given their optimal shares to those with lower valuations.

Theorem 2. For any simple bargaining problem S_α and any pre-donation $\lambda_{i,j} \in (0,1)$ where i and j are two bargainers such that $KS_i\big(\Lambda_n^*(\alpha_n)\big) < KS_j\big(\Lambda_n^*(\alpha_n)\big)$, we have $KS_i\left(\Lambda_i\left(KS_i\big(\Lambda_n^*(\alpha_n)\big)\right)\right) < KS_j\big(\Lambda_n^*(\alpha_n)\big)$, that is, no agent wants to deviate from the final payoff allocation $KS\big(\Lambda_n^*(\alpha_n)\big)$.

Proof. Take any two bargainers i and j such that $KS_i\big(\Lambda_n^*(\alpha_n)\big) < KS_j\big(\Lambda_n^*(\alpha_n)\big)$. Assume that agent i wants to make a reverse donation $\lambda_{i,j} \in (0,1)$ to agent j. Then, the unique Pareto optimal point in the new set $\Lambda_i\left(KS_i\big(\Lambda_n^*(\alpha_n)\big)\right)$ will be $(u_1, u_2, \dots, u_n)$ such that:

$$u_k = \begin{cases} (\alpha_k - \alpha_{k-1})(1 - \lambda_{k,j}) & for\ k = i \\ (\alpha_k - \alpha_{k-1}) + (\alpha_i - \alpha_{i-1})\lambda_{i,k} & for\ k = j \\ (\alpha_k - \alpha_{k-1}) & otherwise. \end{cases}$$

KS will choose this point as it obeys optimality. But, for any $\lambda_{i,j} > 0$, the i^{th} coordinate of the solution $(\alpha_i - \alpha_{i-1})(1 - \lambda_{i,j})$ is lower than $(\alpha_i - \alpha_{i-1}) = KS_i\big(\Lambda_n^*(\alpha_n)\big)$. Thus, agent i would pick $\lambda_{i,j} = 0$.

■

Theorem 1 establishes that any bargainer i has incentive to manipulate the bargaining set when $\alpha_i > \alpha_{i-1}(n - i + 2)$. Thus, although at the early stages of pre-donation (with i low) higher ideal payoffs are necessary for the bargainers to have an incentive to manipulate the problem, this requirement becomes softer as we proceed to later stages. Another observation is that the bargainers do not immediately attain their total concessions. As an example, consider agent one. Without any manipulation, his payoff under KS would be $1/n$. After agent two's donation, his payoff rises to $1/(n-1)$. Agent three's manipulation increases his payoff to $1/(n-2)$ and finally agent *n*'s gift makes him reach his ideal concession $\frac{1}{n-(n-1)} = 1$.

5. Conclusion

In this paper we discussed the manipulability of the many-person KS bargaining solution via pre-donations. We considered simple bargaining problems with monetary payoffs and the disagreement points at the origin. When bargainers sequentially pre-donate (in increasing order of ideal payoffs), the resulting payoff distribution under the KS solution is Concessionary. Thus, under the manipulated solution, the bargainer with the highest ideal payoff takes the whole cake and then monetarily compensates others. We would like to emphasize that manipulation leads to an efficient outcome, where every agent is strictly better-off relative to the original solution.

A transfer of a share of future payoffs is a natural means by which bargainers can alter the allocation of the available value when the solution concept and the bargaining set are public knowledge. As noted by Sertel (1992), no legal obstacle under commercial law can stop the agents from signing contracts under which everybody will be better-off. Thus, "farsighted" individuals might generally manipulate many of the well-known axiomatic bargaining solutions, such as the Nash, Maschler-Perles, Egalitarian, and the Utilitarian solutions.

Appendix

Consider the *three*-person simple bargaining set. Suppose we change the order in which manipulations are made. In particular, agent three goes first and when he is done, agent two follows. We again assume that pre-donations are proportional to the agents' maximum possible payoffs.

Let agent three offer gifts $\lambda_{3,1}$ and $\lambda_{3,2}$ to agents one and two simultaneously, where $\lambda_{3,2} = \alpha_2\lambda_{3,1}$ by assumption. Given S_α, the bargaining set after agent three's manipulation (illustrated in Figure 5) is:

$$\Lambda_3(S_\alpha) = \{(u_1 + \lambda_{3,1}u_3, u_2 + \lambda_{3,2}u_3, (1 - \lambda_{3,1} - \lambda_{3,2})u_3) | (u_1, u_2, u_3) \in S_\alpha\}.$$

To save space, we only analyze the case when $\alpha_3\lambda_{3,1} < 1$. Using $\lambda_{3,2} = \alpha_2\lambda_{3,1}$, the Pareto set and the KS solution on $\Lambda_3(S_\alpha)$ are:

$$P(\Lambda_3(S_\alpha)) = H\left[(1,0,0), (0, \alpha_2, 0), \left(\alpha_3\lambda_{3,1}, 0, \alpha_3\left(1 - (1 + \alpha_2)\lambda_{3,1}\right)\right), \left(0, \alpha_3\alpha_2\lambda_{3,1}, \alpha_3\left(1 - (1 + \alpha_2)\lambda_{3,1}\right)\right)\right]$$

$$KS(\Lambda_3(S_\alpha)) = \left(\frac{1}{3 - \alpha_3\lambda_{3,1}}, \frac{\alpha_2}{3 - \alpha_3\lambda_{3,1}}, \frac{\left(1 - (1 + \alpha_2)\lambda_{3,1}\right)\alpha_3}{3 - \alpha_3\lambda_{3,1}}\right).$$

Agent three's payoff is increasing in $\lambda_{3,1}$ whenever $\alpha_3 > 3(1 + \alpha_2)$; hence we set $\lambda_{3,1}$ as high as possible and set $\lambda_{3,1}^* = \frac{1}{\alpha_3}$. Hence, the KS allocation under this optimal pre-donation is:

$$KS(\Lambda_3^*(S_\alpha)) = \left(\frac{1}{2}, \frac{\alpha_2}{2}, \frac{\alpha_3 - \alpha_2 - 1}{2}\right)$$

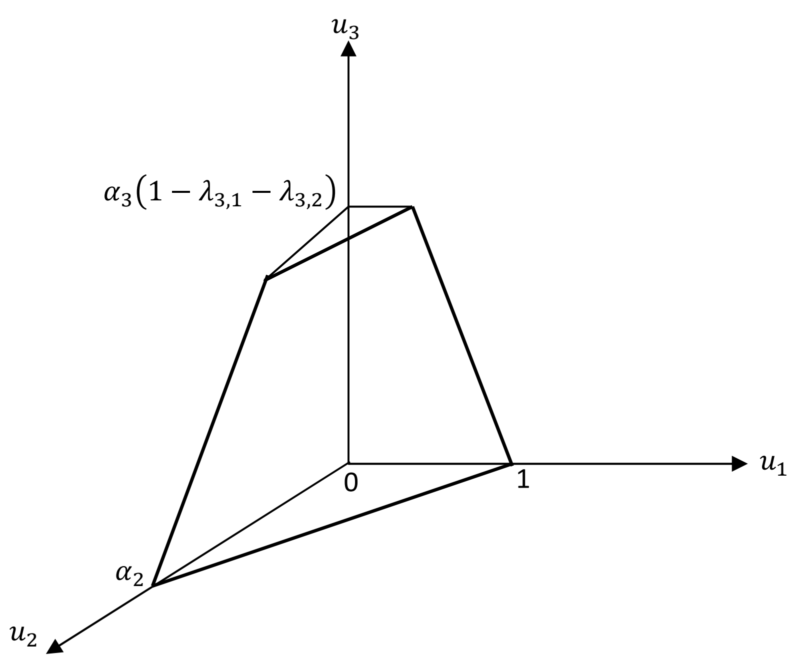

Figure 5: The bargaining set after agent three's donations and its Pareto set (shown in bold).

Next, it is agent two's turn to pre-donate a share $\lambda_{2,1}$ out of his future payoffs to agent one. Figure 6 illustrates the new set. Again, we only illustrate the case when $\alpha_2\lambda_{2,1} < 1$. The Pareto frontier is:

$$P\left(\Lambda_2\left(\Lambda_3^*(S_\alpha)\right)\right) = H\big[(1,0,0), \left(\alpha_2\lambda_{2,1}, \alpha_2(1-\lambda_{2,1}), 0\right), \left(\alpha_2\lambda_{2,1}, \alpha_2(1-\lambda_{2,1}),\ \alpha_3-\alpha_2-1\right), (1,0,\alpha_3-\alpha_2-1)\big].$$

The KS solution requires $u_1 = \frac{u_2}{\alpha_2(1-\lambda_{2,1})} = \frac{u_3}{\alpha_3-\alpha_2-1}$. Therefore, the payoffs are:

$$KS\left(\Lambda_2\left(\Lambda_3^*(S_\alpha)\right)\right) = \left(\frac{1}{2-\alpha_2\lambda_{2,1}}, \frac{(1-\lambda_{2,1})\alpha_2}{2-\alpha_2\lambda_{2,1}}, \frac{\alpha_3-\alpha_2-1}{2-\alpha_2\lambda_{2,1}}\right).$$

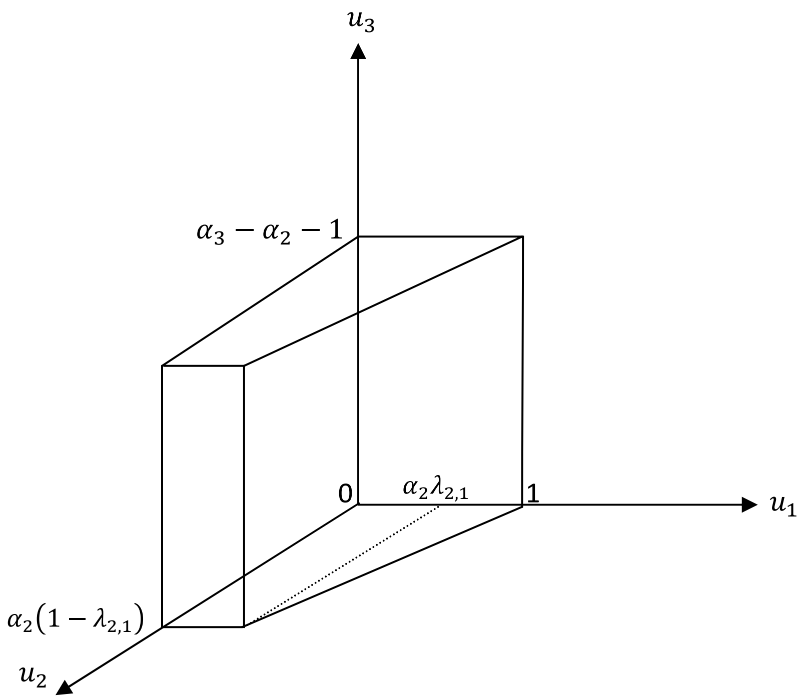

Figure 6: The bargaining set after agent two's manipulation.

Agent two's payoff is increasing in $\lambda_{2,1}$ if $\alpha_2 > 2$; hence we set $\lambda_{2,1}^* = \frac{1}{\alpha_2}$. Inserting this into the KS payoffs above, we get:

$$KS\left(\Lambda_2^*\left(\Lambda_3^*(S_\alpha)\right)\right) = (1, \alpha_2,\ \alpha_3-\alpha_2-1).$$

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