# Contrasting two approaches in real options valuation: contingent claims versus dynamic programming

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## Abstract

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This paper compares two well-known approaches for valuing a risky investment using real options theory: contingent claims (CC) with risk neutral valuation and dynamic programming (DP) using a constant risk adjusted discount rate. Both approaches have been used in valuing forest assets. A proof is presented which shows that, except under certain restrictive assumptions, DP using a constant discount rate and CC will not yield the same answers for investment value. A few special cases are considered for which CC and DP with a constant discount rate are consistent with each other. An optimal tree harvesting example is presented to illustrate that the values obtained using the two approaches can differ when we depart from these special cases to a more realistic scenario. Further, the implied risk adjusted discount rate calculated from CC is found to vary with the stochastic state variable and stand age. We conclude that for real options problems the CC approach should be used.

*Keywords:* optimal tree harvesting, real options, contingent claims, dynamic programming

*Running Title:* Contingent claims theory and dynamic programming

# 1 Introduction

Over the past two decades, developments in the theory and methodology of financial economics have been applied to advantage to general problems of investment under uncertainty. The well known book by Dixit and Pindyck [1994] draws the analogy between valuing financial options and investments in real assets or real options which involve irreversible expenditures and uncertain future payoffs depending on one or more stochastic underlying variables. Natural resource investments, including forestry, provide a good application of real options theory as their value depends on volatile commodity prices and they entail decisions about the timing of large irreversible expenditures.[1]

Two particular approaches used in the real options literature are dynamic programming (DP) and contingent claims (CC). DP is an older approach developed by Bellman and others in the 1950's and used extensively in management science. DP involves formulating the investment problem in terms of a Hamilton-Jacobi-Bellman equation and solving for the value of the asset by backward induction using a discount rate which reflects the opportunity cost of capital for investments of similar risk. In practice dynamic programming typically involves adopting an exogenous constant discount rate.

The contingent claims approach has its origins in the seminal papers of Black and Scholes [1973] and [Merton, 1971, 1973] and is now standard in many finance texts.[2] This approach assumes the existence of a sufficiently rich set of markets in risky assets so that the stochastic component of the risky project under consideration can be exactly replicated. Through appropriate long and short positions, a riskless portfolio can be constructed consisting of the risky project and investment assets which track the project's uncertainty. In equilibrium

[1]Examples of applications of real options theory to natural resources include Paddock et al. [1988], Brennan and Schwartz [1985], Schwartz [1997], Slade [2001] and references therein, Harchaoui and Lasserre [2001], Mackie-Mason [1990], Saphores [2000], and papers contained in Schwartz and Trigeorgis [2001]. A review of the empirical significance of real options in valuing mineral assets is contained in Davis [1996].

[2]See Hull [2006] and Ingersoll [1987] for example. Dixit and Pindyck [1994] and Trigeorgis [1996] present contingent claims in a real options context.

with no arbitrage opportunities, this portfolio must earn the risk free rate of interest, which allows the value of the risky project to be determined. The no-arbitrage assumption avoids the necessity of determining the appropriate risk adjusted discount rate. However if a portion of the return from holding the risky asset is due to an unobservable convenience yield, it is still necessary to estimate either that convenience yield or a market price of risk, which is often problematic.[3]

Both CC and DP have been used in the natural resources literature. For example Slade [2001] and Harchaoui and Lasserre [2001] use a contingent claims approach to value mining investments. In the forestry economics literature, the DP approach has generally dominated. An exception is Morck et al. [1989] who use a CC approach along with an assumed convenience yield for an application in forestry. In those forestry papers that use a DP approach, there is rarely much discussion of the choice of discount rate. Sometimes a risk neutral setting is explicitly assumed allowing use of a riskfree discount rate; other times a rate is adopted without explanation. A selection of papers that use the dynamic programming approach include Clarke and Reed [1989], Haight and Holmes [1991], Thomson [1992], Yin and Newman [1997], Plantinga [1998], Gong [1999], Insley [2002], and Insley and Rollins [2005]. Alvarez and Koskela [2006] and Alvarez and Koskela [2007] deal with risk aversion by explicitly modelling the decision maker's subjective utility function.

One reason for the persistence of the DP approach in the forestry literature is likely the difficulty in estimating the convenience yield. In theory futures prices could be used to obtain an estimate. Futures markets do exist for lumber, however currently the maturity of futures contracts is less than one year, while the typical optimal harvesting problem is applied to a very long lived investment, with stands of trees maturing over 40 to 70 years.[4]

[3]Dixit and Pindyck [1994] discusses the convenience yield in detail. It represents a return that accrues to the holder of the physical asset but not the holder of an option on the asset. For commodities such as copper, oil, or lumber the convenience yield represents the the benefits of holding inventory rather than having to purchase the commodity in the spot market.

[4]There have been papers addressing this issue for other commodities including Gibson and Schwartz [1990] and Schwartz and Smith [2000]. Another difficulty with estimating a convenience yield from a timber

In comparing the two approaches, Dixit and Pindyck [1994] note that each has advantages and disadvantages, but that CC provides a better treatment of risk. They point out that one problem with the investment rule derived from DP is that

> "(...)it is based on an arbitrary and constant discount rate, $\rho$. It is not clear where this discount rate should come from, or even that it should be constant over time" (page 147).

On the other hand, a disadvantage of CC is that it requires a sufficiently rich set of risky assets so that the risky components of the uncertain investment can be exactly replicated. This is not required of dynamic programming;

> "(...) if risk cannot be traded in markets, the objective function can simply reflect the decision maker's subjective valuation of risk. The objective function is usually assumed to have the form of the present value of a flow 'utility' function calculated using a constant discount rate, $\rho$. This is restrictive in its own way, but it too can be generalized. Of course we have no objective or observable knowledge of private preferences, so testing the theory can be harder" (page 121).

In a review of Dixit's and Pindyck's book, Schwartz [1994] disagrees with the common practice of using a discount rate that simply reflects the decision-maker's subjective evaluation of risk. Schwartz states that

> "(...) there is only one way to deal with the problem, which is firmly based on arbitrage or equilibrium in financial markets. If what the decision-maker is trying to get is the market value of the project, then, obviously, a subjective discount rate will not do the job" [Schwartz, 1994, page 1927].

Schwartz notes that when the risk of an investment is not spanned by existing assets the value of the option should be estimated by adjusting the drift of the stochastic process for the state variable using an equilibrium model of asset prices.

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investment is that a stand of trees produces several different products such as lumber and paper whereas futures are traded in lumber only.

An intuitive explanation for why the results of CC and DP with a constant discount rate will differ is provided in Ingersoll [1987, pages 311-313]. Trigeorgis [1996, chap 2] shows that using a constant risk adjusted discount rate implies that the market risk born per period is constant or, in other words, the total risk increases at a constant rate through time. Trigeorgis [1996] draws on the work of an earlier paper by Fama [1977] which deals with the valuation of multi-period cash flows using a Capital Asset Pricing Model (CAPM) framework. Fama [1977] shows that the correct risk adjusted discount rates implied by the CAPM model will not in general be constant, but must evolve deterministically through time. However Fama notes that the use of a constant risk adjusted discount rate may be a reasonable approximation in certain cases for "an investment project of a given type or for a firm whose activities are not anticipated to change much in nature through time" [Fama, 1977, page 23]. As is pointed out by Trigeorgis [1996], it is questionable whether this will be the case when a decision maker is faced with choices such as the potential to delay, expand, or contract an investment - i.e. in the presence of embedded options.

Although CC is judged preferable because of its better treatment of risk, it may be asked whether the DP approach is good enough in practical applications, particularly when it is difficult to obtain a reliable estimate of the convenience yield or market price of risk. In this paper we derive the condition that must hold for CC and DP, with a constant risk adjusted discount rate, to give the same result. We show that this condition will hold for certain simple cases, one of which is of particular interest because of its appearance in the literature in stylized real options models. This special case is an infinitely-lived American option with zero exercise cost and underlying state variable(s) that follows geometric Brownian motion. We argue that in more realistic real options problems, it is unwise to assume that the DP approach will give an adequate result. To demonstrate this point, we provide an example where the use of DP or CC makes a significant difference to the estimated value ofa a real option. The example is an optimal tree harvesting problem which has been examined previously in the literature. In this problem, the value and optimal harvest time of a stand of trees depend on the price of timber, which is assumed to be stochastic and mean reverting, and on the age of the stand.

In summary, the contributions of this paper to the literature are as follows.

- We present a proof of the conditions which must hold in order for the CC and DP approaches to give identical results. Although this proof is developed in the context of an optimal harvesting problem, it applies to a large class of real options problems in which the underlying stochastic variable follows a fairly general Ito process.
- We show that the condition for the equivalence of CC and DP will hold for some simple cases.
- We provide an example of the empirical significance of using DP versus CC in an optimal tree harvesting problem.[5] We show numerically how the risk adjusted discount rate, implied by the CC approach, changes with the stochastic state variable.

In the next section we derive the condition which must hold for CC and DP to be consistent and examine several cases for which the condition is met. In Section 3 we present the empirical example of the optimal tree harvesting problem to demonstrate that the difference between CC and DP may be significant. In Section 4 we discuss the results and lastly in Section 5 we provide some concluding comments.

# 2 CC and DP approaches to a real options problem

For concreteness, we use an optimal tree harvesting problem to compare the CC and DP approaches. However the resulting partial differential equations that describe the value of the option can be easily adapted to the valuation of other investment problems that depend on a single stochastic state variable. The extension to additional stochastic state variables is also straight forward.

[5]The particular harvesting problem presented was analyzed in Insley and Lei [2007].

## 2.1 Dynamic Programming

In this section we describe the optimal tree harvesting model using the dynamic programming approach. The objective is to value the right to harvest a stand of trees on land that will be harvested over an infinite number of future rotations. We are using the model presented in Insley and Rollins [2005] and reproduce the details here for the convenience of the reader.

We denote the value of this asset as $W$, which depends on the price of timber ($P$), the age of the stand ($\alpha$), and time ($t$). The price of timber is assumed to follow a known Ito process:

$$dP = a(P,t)dt + b(P,t)dz. \tag{1}$$

In Equation (1), $a(P,t)$ and $b(P,t)$ represent known functions and $dz$ is the increment of a Wiener process.

The age of the stand, or time since the last harvest, $\alpha$, is given as

$$\alpha = t - t_h \tag{2}$$

where $t$ is the current time and $t_h$ is the time of the last harvest. Wood volume is assumed to be a deterministic function of age:

$$Q = g(\alpha). \tag{3}$$

Age is used as a state variable, along with price, $P$. It follows that:

$$d\alpha = dt. \tag{4}$$

Using the dynamic programming approach the decision to harvest the stand of trees is formulated as an optimal stopping problem where the owner must decide in each period whether it is better to harvest immediately or delay until the next period. This decision process can be expressed as a Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} W(P,t,\alpha) =& max\big\{(P-C)Q + W(P,t,0); \\ & A(Q)\Delta t + (1+\rho\Delta t)^{-1}E[W(P+\Delta P, t+\Delta t, \alpha+\Delta\alpha)]\big\} \end{aligned} \tag{5}$$

where

| | | |
|---|---|---|
| $E$ | $=$ | expectation operator |
| $W$ | $=$ | value of the opportunity to harvest using DP |
| $P$ | $=$ | price of timber |
| $C$ | $=$ | per unit harvesting cost |
| $Q$ | $=$ | current volume of timber |
| $\alpha$ | $=$ | age of stand |
| $A(Q)$ | $=$ | per period amenity value of standing forest less any management costs |
| $\rho$ | $=$ | risk adjusted annual discount rate |
| $t$ | $=$ | time. |

The first expression in the curly brackets represents the return if harvesting occurs in the current period, $t$. It includes the net revenue from harvesting the trees plus the value of the land after harvesting, $W(P, t, 0)$. This is the value that could be attained if the land were sold subsequent to the harvest, assuming that the land will remain in forestry.

The second expression in the curly brackets is the value of continuing to hold the asset (the continuation region) by delaying the decision to harvest for another period. It includes any amenity value of the standing forest, such as its value as a recreation area, less any forest management costs, $A(Q)$. In this paper amenity benefits are set to zero so that $A(Q)$ reflects only management costs. The value in the continuation region also includes the expected value of the option to harvest in the next period, discounted to the current period. The discount rate is set exogenously, but is intended to reflect the return required by an investor to hold the asset over $\Delta t$.

Following standard arguments (Dixit and Pindyck [1994], Wilmott et al. [1993]), we can derive the following partial differential equation that describes $W(P, t, \alpha)$ in the continuation region. We denote $W(P, t, \alpha)$ as $W$ when there is no confusion. Subscripts $t$, $P$ and $\alpha$ indicate partial derivatives with respect to those variables.

$$W_t + \frac{1}{2}b^2(P,t)W_{PP} + a(P,t)W_P - \rho W + A(Q) + W_\alpha = 0. \tag{6}$$

## 2.2 Contingent Claims

We start with the assumption that markets are sufficiently complete that project risk can be eliminated through hedging with another risky asset. We also assume that there are no arbitrage opportunities in the economy. Denoting our project of interest as $V_1$, we can find a traded asset, $V_2$, that also depends on the stochastic underlying variable $P$. $V_2$ is not the physical commodity, lumber, but rather it is a traded contract that depends on the price of lumber - perhaps the shares of a firm with harvesting rights to nearby stands of trees. By Ito's lemma we know that $V_1$ and $V_2$ will follow stochastic processes as follows:

$$\frac{dV_j}{V_j} = \mu_j dt + s_j dz, \quad j = 1, 2 \tag{7}$$

where $\mu_j$ and $s_j$ are functions of $P$, $t$ and $\alpha$. In particular,

$$\begin{aligned} \mu_j &= \left[ (V_j)_t + a(P,t)(V_j)_P + (V_j)_\alpha + \frac{1}{2}b^2(P,t)(V_j)_{PP} \right] \frac{1}{V_j} \\ s_j &= \frac{b(P,t)}{V_j}(V_j)_P \end{aligned} \tag{8}$$

where

$$(V_j)_P \equiv \frac{\partial V_j}{\partial P}; (V_j)_{PP} \equiv \frac{\partial^2 V_j}{\partial P^2}; (V_j)_t \equiv \frac{\partial V_j}{\partial t}; (V_j)_\alpha \equiv \frac{\partial V_j}{\partial \alpha}. \tag{9}$$

Note that $s_j$ is the volatility of asset $j$.

We can form an instantaneously riskless portfolio of $V_1$ and $V_2$ which under our no-arbitrage assumption must earn the riskfree rate of interest. Following standard arguments (presented in Appendix A) the following relationship will hold:

$$\frac{\mu_1 + \frac{A(Q)}{V_1} - r}{s_1} = \frac{\mu_2 - r}{s_2} \equiv \lambda. \tag{10}$$

$\mu_j$ is the capital gain on the contingent claim $V_j$. We also introduce the notation $\mu^T$ to refer to the total return on an asset from all sources. For our tree stand, $\mu_1^T = \mu_1 + \frac{A(Q)}{V}$. $\lambda$, called the market price of risk of P, represents the excess total return over the risk free rate per unit of variability. By the no arbitrage assumption, it must be the same for all contingent claims that depend on $P$ and $t$. $\lambda s_j$ is the risk premium for contingent claim $j$. Dropping the $j$ subscript for our forest stand of interest,

$$\frac{\mu + \frac{A(Q)}{V} - r}{s} = \lambda. \tag{11}$$

Substituting for $\mu$ and $s$ from Equation (8), we rearrange Equation (11) to get the partial differential equation that must be satisfied the contingent claim if it is to be held by a willing investor.

$$V_t + \frac{1}{2} b^2(P,t) V_{PP} + [a(P,t) - \lambda b(P,t)]\, V_P - rV + A(Q) + V_\alpha = 0. \tag{12}$$

According to Equation (12), we are able to value our contingent claim using the risk free rate as the discount rate, and reducing the drift rate $a(P,t)$ of the stochastic state variable by a factor $\lambda b(P,t)$ that reflects the extra return required to compensate for risk. Any asset dependent on $P$ can be valued by reducing the expected growth rate of $P$ by $\lambda b(P,t)$ to $[a(P,t) - \lambda b(P,t)]$ and discounting the resulting net benefits by the risk free rate. This result called equivalent risk neutral valuation is due to Cox et al. [1985].

The adjustment of the drift term $a(P,t)$ by $\lambda$ results from the fact that we have hedged the price risk in $V_1$ with another contract, $V_2$. If, instead, we were able to trade lumber in financial markets then we could hedge price risk directly by buying and selling timber, and our hedging asset, which we will call $V_3$, would be $V_3 = P$. The return from holding lumber, $\mu_3^T$, would be the sum of the capital gain, $\frac{a(P,t)}{P}$, and any convenience yield that results from holding an inventory of lumber, $\delta$. In this case, instead of Equation (12), the fundamental partial differential equation is of the alternate form (see Appendix A):

$$V_t + \frac{1}{2} b^2(P,t) V_{PP} + (r - \delta) P V_P - rV + A(Q) + V_\alpha = 0. \tag{13}$$

If the convenience yield is zero then we are left with the riskless rate associated with the term $V_P$. This is the well known result that if the underlying stochastic variable can be traded in financial markets represents and the convenience yield is zero then $a(P,t)$ can be replaced by the riskfree rate and there is no need to estimate a market price of risk.

In general, however, we would not expect the convenience yield to be zero, particularly for a storable commodity. Thus with the CC approach it is required to estimate either the market price of risk or the convenience yield, which can be problematic. Both of these parameters may be non-constant. For some natural resource investments it is possible to estimate the market price of risk from futures contracts on underlying traded commodities.[6]

## 2.3 Comparing CC and DP

In this section we derive a necessary and sufficient condition for CC and DP to yield the same result. Let $\tau$ be defined as time remaining in the option's life, i.e. $\tau \equiv T - t$. Subtracting $\rho V$ from both sides of Equation (12), converting from $t$ to $\tau$, and rearranging terms, we get

$$-V_\tau + \frac{1}{2}b^2(P,\tau)V_{PP} + a(P,\tau)V_P - \rho V + V_\alpha + A(Q) = \lambda b(P,\tau)V_P - (\rho - r)V. \tag{14}$$

Let $Z = V - W$ where $V$ is the known solution to Equation (14). Recall that $V$ refers to the value of the investment using CC and $W$ refers to the value using DP. Subtract Equation (6) (expressed in terms of $\tau$) from Equation (14):

$$-Z_\tau + \frac{1}{2}b^2(P,\tau)Z_{PP} + a(P,\tau)Z_P - \rho Z + Z_\alpha = \lambda b(P,\tau)V_P - (\rho - r)V. \tag{15}$$

It is obvious that if $Z = 0$, the left hand side of Equation (15) will be zero implying on the right hand side $\lambda b(P,\tau)V_P - (\rho - r)V = 0$. If the right hand side is zero, the risk adjusted discount rate can be expressed as

$$\rho = r + \frac{\lambda b(P,t)V_P}{V} = r + \lambda s. \tag{16}$$

Equation (16) provides a sufficient condition for DP and CC to give the same result. What is not obvious is that Equation (16) is also a necessary condition. A proof is given in Appendix

[6]Eduardo Schwartz has done extensive work in this area. See for example Schwartz and Smith [2000], Schwartz [1998], and Schwartz [1997].

B that $Z = 0$ **if and only if** Equation (16) holds. We conclude that the risk adjusted discount rate, $\rho$, will be constant if the volatility of the asset $s$ (as defined in Equation (8)) is constant. The asset's volatility depends on $V$, $V_P$, and $b(P,t)$ and we would not expect it to be constant except in special cases.

## 2.4 Special cases where the risk adjusted discount rate, $\rho$ is constant

We derive special cases in which $\rho$ will be constant so that with the appropriate choice of $\rho$ CC and DP will give the same result. These cases are simplistic and generally not realistic for most applied real options problems. However these have been used in the literature in stylized real options problems because an analytical solution can be obtained.

From Equation (16) we can observe that in the trivial case when the market price of risk, $\lambda$, is zero,then $\rho = r$ and CC and DP will give the same result. From Equation (10), we observe that $\lambda = 0$ implies that the total asset's total return from all sources equals the riskless rate $\mu^T = r$.[7]

If $\lambda \neq 0$, then for a constant $\rho$ we require that

$$\frac{\lambda b(P,t)V_P}{V} = K_1 \tag{17}$$

for some constant $K_1$. If we assume that $\lambda$ is constant then we can derive a more general expression of the form that the solution to $V$ must take to imply a constant $\rho$. For Equation (17) to hold, the variance rate $b(P,t)$ will need to be time invariant, so we rewrite $b(P,t)$ as $b(P)$ and write Equation (17) as:

$$\frac{dV}{V} = K_1 \frac{dP}{b(P)}. \tag{18}$$

[7]Knudsen et al. [1999] showed the equivalence of CC and DP when $\rho = r$.

It follows that:

$$\int \frac{dV}{V} = K_1 \int \frac{dP}{b(P)}$$

or

$$V = e^{K_1 \int \frac{dP}{b(P)} + K_2} \tag{19}$$

with constants $K_1$ and $K_2$. When $b(P)$ takes the simple form used in this paper $b(P) = \sigma P$, $V$ will have a solution of the form

$$V = K_3 P^{K_4} \tag{20}$$

with constants $K_3$ and $K_4$.

An example of a solution of this form is $V$ as a simple linear function of $P$. $K_4 = 1$ and $V = K_3 P$. Substituting for $V$, $V_P$ and $b(P)$ into Equation (16) gives

$$\begin{aligned} \rho &= r + \frac{\lambda \sigma P g}{gP} \\ &= r + \lambda\sigma \end{aligned} \tag{21}$$

In this case, DP with a constant discount rate as specified in Equation (21) will be consistent with the CC approach.

Equation (20) is the form of the solution for the problem presented in Dixit and Pindyck [1994, pages 136-144] which asks at what point it is optimal to pay a sunk cost $I$ in return for a project with a value $V$ that evolves according to geometric Brownian motion and is infinitely lived.[8] The problem presented in Dixit and Pindyck [1994] is a variation of the example in the much cited work by McDonald and Siegel [1986] which addresses a similar question, but both project value and investment cost evolve according to geometric Brownian motion. We briefly review the McDonald and Siegel [1986] example here as it demonstrates the extension of condition in Equation (16) to two stochastic variables.

Let the value of an investment project be denoted by $P$ and investment cost by $C$ and

[8] Dixit and Pindyck denote $V$ as the stochastic variable, and the value of the option as $F$.

assume, as in McDonald and Siegel [1986], that these factors behave according to:

$$dP = \alpha_P P dt + \sigma_P P dz_P$$
$$dC = \alpha_C C dt + \sigma_C C dz_C \tag{22}$$

The firm has the opportunity to pay $C_t$ to install an investment project with present value $P_t$. For an infinitely lived investment opportunity the problem is to find the boundary $B^* = \frac{P_t}{C_t}$ at which the investment should occur to maximize

$$E_0[(P_t - C_t)e^{-\rho t}] \tag{23}$$

where $\rho$ is the appropriate discount rate, which will be specified below, and $E_0$ refers to the expectation at time zero. The value of this investment opportunity can be solved analytically as shown in McDonald and Siegel [1986] to obtain the following expression:

$$V(P, C) = \frac{(B^* - 1)}{B^{*\beta}} P^\beta C^{1-\beta}. \tag{24}$$

where

$$\beta = \sqrt{\left(\frac{\alpha_P - \alpha_C}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(\rho - \alpha_C)}{\sigma^2}} + \left(\frac{1}{2} - \frac{\alpha_P - \alpha_C}{\sigma^2}\right)$$
$$B^* = \frac{\beta}{\beta - 1}$$
$$\sigma^2 = \sigma_P^2 + \sigma_C^2 - 2\varrho_{PC}\sigma_P\sigma_C \tag{25}$$

where $\varrho_{PC}$ refers to the correlation between the rates of increase of $P$ and $C$.[9]

It can be shown using arguments similar to those in the previous section that for the case of two stochastic factors which follow GBM, as in Equation (22), the condition for DP and CC to give the same answer is:

$$\rho = r + \frac{\lambda_P \sigma_P P V_P + \lambda_C \sigma_C C V_C}{V}. \tag{26}$$

[9]Equation (24) is equivalent to Equation (4) in McDonald and Siegel [1986], while Equation (25) is equivalent to Equation (5).

where $\lambda_P$ refers to the market price of risk for assets dependent on $P$ and $\lambda_C$ refers to the market price of risk for assets dependent on $C$. This condition is similar to the condition with one stochastic factor, Equation (16).

We derive expressions for $V_P$ and $V_C$ from Equation (24) and substitute, along with the expression for $V$, into Equation (26). Denoting $D = (B^* - 1)/B^{*\beta}$, we get

$$\begin{aligned} \rho &= r + \frac{\lambda_P \sigma_P P D \beta P^{\beta-1} C^{1-\beta} + \lambda_C \sigma_C C D P^{\beta}(1-\beta) C^{-\beta}}{D P^{\beta} C^{(1-\beta)}} \\ &= r + \lambda_P \sigma_P \beta + \lambda_C \sigma_C (1-\beta) \end{aligned} \quad (27)$$

This last expression agrees with the expression for the discount rate in Equation (10)(page 716) of McDonald and Siegel. Thus we see that an infinitely lived American option type problem with two stochastic variables following geometric Brownian motion is another special case where the DP and CC approaches will give the same result, if the risk adjusted discount rate satisfies Equation (27). Note that Equation (27) does not represent an explicit solution for $\rho$ since $\beta$ also depends on $\rho$.

# 3 Empirical Example: Comparing CC and DP in an optimal harvesting problem

In this section we consider an optimal harvesting problem that was addressed in Insley and Lei [2007] using a CC approach.[10] We solve that problem using DP with a constant $\rho$ and compare the results for asset value and optimal harvest age. We also calculate the (non-constant) values of $\rho$ that would ensure consistency between CC and DP.

The typical tree harvesting problem will not have a simple solution such as given by Equation (20), which represents an infinitely lived asset whose value evolves according to GBM. Factors which may cause the tree harvesting problem to depart from the simple case include the presence of fixed management costs to maintain the stand, and the separate

[10] Note that the example in Insley and Lei [2007] is similar to the case studied in Insley and Rollins [2005] with updated timber yield estimates and cost estimates.

modelling of price and quantity of timber. In addition it is generally agreed that commodity prices, such as timber, are better characterized by a process that exhibits some mean reversion, as many commodity prices have been fairly flat in real terms over the long term [Schwartz, 1997]. So if we depart from the assumption of GBM prices and include other realistic characteristics, such as management costs, we would not expect that DP and CC would give the same result.

The empirical example that will be analyzed is the valuation of a stand of Jack Pine in Ontario's boreal forest. It is assumed that timber prices follow a mean reverting process of a very simple form:

$$dP = \eta(\bar{P} - P)dt + \sigma P dz. \tag{28}$$

where $\bar{P}$ is the long run average price of timber, $\eta$ is the constant speed of mean reversion and $\sigma P$ is the variance rate.

## 3.1 Formulating the Linear Complementarity Problem

The tree harvesting problem is akin to an American option which can be exercised at any time. The problem can be solved using a linear complementarity approach as in Insley [2002] and Insley and Rollins [2005]. We set up the linear complementarity problem (LCP) for the CC version of our problem, but the LCP for the dynamic programming approach would be expressed in a parallel fashion.[11] $T$ denotes the terminal time. Rearranging Equation (12) and substituting $\tau$ for $t$, we define an expression, $HV$, as follows:

$$HV \equiv rV - \left[\frac{1}{2}\sigma^2 P^2 V_{PP} + [a(P,\tau) - \lambda b(P,\tau)]V_P + A(Q) + V_\alpha - V_\tau\right] \tag{29}$$

In Equation (29), $rV$ represents the return required on the investment opportunity for the risk neutral investor to continue to hold the option. The expression within square brackets represents the (certainty equivalent) return over the infinitesimal time interval $d\tau$.[12]

[11] See Wilmott et al. [1993] and Tavella [2002] for further discussion on the linear complementarity problem.

[12] The certainty equivalent return refers to the return on an investment when the growth rate of cash flows

Then the LCP is given as:

$$
\begin{aligned}
&(i) \quad HV \geq 0 \\
&(ii) \quad V(P,\tau,\alpha) - [(P-C)Q + V(P,\tau,0)] \geq 0 \\
&(iii) \quad HV\Big[V(P,\tau,\alpha) - [(P-C)Q + V(P,\tau,0)]\Big] = 0
\end{aligned}
\tag{30}
$$

As noted in Insley and Rollins [2005], the LCP expresses the rational individual's strategy with regards to holding versus exercising the option to harvest the stand of trees. Part (i) of Equation (30) states that the certainty equivalent return from holding the asset will be no more than the riskfree return. As long as the asset is earning the riskfree rate, it is worthwhile continuing to hold, which means delaying the harvest of the stand of trees. As the trees age and their growth rate slows, the certainty equivalent return will slip below the risk free rate, at which point it would be optimal to harvest the stand. Hence when part (i) holds as a strict equality, harvesting is delayed. When it holds as an inequality, harvesting is optimal.

Part (ii) states that the value of the option, $V$, must be at least as great as the return from harvesting immediately. The return from harvesting immediately is the sum of the net revenue from selling the logs, $(P-C)Q$, plus the value of the land immediately after harvesting, $V(P,t,0)$. When trees are young and growing rapidly we expect $V$ to exceed the value of harvesting immediately (Part (ii) holds as an inequality) and it is optimal to delay harvesting the stand. As the trees age, their growth rate falls and $V$ approaches $[(P-C)Q + V(P,\tau,0)]$. Harvesting is optimal when (ii) holds as a strict equality.

Part (iii) states that at least one of statements (i) or (ii) must hold as a strict equality. If both expressions hold as strict equalities then the investor is indifferent between harvesting and continuing to hold the asset.

The LCP is solved numerically which involves discretizing the relevant partial differential equation including a penalty term that enforces the American constraint (Equation (30), ii).

$(a(P,\tau))$ has been reduced by an appropriate risk premium, $(\lambda b(P,\tau))$. After this adjustment is made we can value the resulting cash flows as if investors are risk neutral.

Using a fully implicit numerical scheme, we are left with a series of nonlinear algebraic equations which must be solved iteratively.

Boundary conditions can then be specified as folows.

1. **As $\boldsymbol{P} \to \mathbf{0}$**, we observe from Equation(28) no special boundary conditions are needed to prevent negative prices.

2. **As $\boldsymbol{P} \to \boldsymbol{\infty}$**, we follow Wilmott [1998] and set $V_{PP} = 0$.

3. **As $\boldsymbol{\alpha} \to \mathbf{0}$**, we require no boundary condition since the PDE is first order hyperbolic in the $\alpha$ direction, with outgoing characteristic in the negative $\alpha$ direction.

4. **As $\boldsymbol{\alpha} \to \boldsymbol{\infty}$**, we assume $V_\alpha \to 0$. This means that as stand age gets very large, the value of the option to harvest, $V$, does not change with $\alpha$. In essence we are presuming the wood volume in the stand has reached some sort of steady state.

5. **Terminal condition.** As $T$ gets large it is assumed that $V = 0$. $T$ is made large enough that this assumption has a negligible effect on $V$ today.

## 3.2 Parameter Values: drift, diffusion, and market price of risk

We use the same values for the drift and diffusion terms of the price process as in Insley and Lei [2007]. We provide some details here (not given in Insley and Lei [2007]) on their estimation.

The historical price series used for parameter estimation is the price of spruce-pine-fir random length 2X4's in Toronto.[13] The deflated lumber price series is shown in Figure 1.[14]

A discrete time approximation of Equation (28) is as follows:

$$P_t - P_{t-1} = \eta \bar{P} \Delta t - \eta \Delta t P_{t-1} + \sigma P_{t-1} \sqrt{\Delta t} \epsilon_t \tag{31}$$

[13]Data was obtained from Madison's Canadian Lumber Reporter.

[14]The original data is weekly and quoted in U.S. \$ per mbf (thousand board feet). It is converted to Canadian \$ per cubic metre and deflated by the Canadian consumer price index (CPI). The monthly CPI was interpolated using a cubic spline procedure to generate a weekly index.

where $\epsilon_t$ is $N(0,1)$. We have weekly data, so $\Delta t = (1/52)year$. We performed ordinary least squares on the following equation:

$$\frac{P_t - P_{t-1}}{P_{t-1}} = c(1) + c(2)\frac{1}{P_{t-1}}. \tag{32}$$

Our estimation results are given in Table 1.[15]

The contingent claims approach requires an estimate of the market price of risk of the project, which is not directly observable. Ideally this estimate would be derived from futures markets, but lumber futures markets trade in only very short term contracts. An analysis using lumber futures is beyond the scope of this paper, and is the subject of future research. For the purposes of this paper, we will solve for stand value using a reasonable range of different values for the market price of risk.

To get a sense of what would be a reasonable value for the market price of risk, we appeal to the approach of Hull [2006, pages 716-77] which is based on the knowledge that all assets depending on the same stochastic underlying variable(s) will have the same market price of risk. An estimate can be obtained for the market price of risk for a hypothetical contract that depends linearly on the stochastic underlying variable, $P$. This approach is detailed in Insley and Lei [2007] and the resulting estimate for $\lambda$ on the hypothetical contract is 0.01. For this paper, we use $\lambda = 0.01$ as a base case, and also consider the impact of $\lambda = 0.03$ and $\lambda = 0.05$.

## 3.3 Risk Adjusted Discount Rate

As noted above, the correct risk adjusted discount rate will, in general, change with the value of the investment. However we wish to consider the impact of a constant discount rate as is standard practice in DP applications. One way to choose a constant discount rate would be

[15] The estimates of $\eta$ $\sigma$ and $\bar{P}$ are calculated from the OLS coefficients as follows:

$$\hat{\eta} = \frac{-c(1)}{1/52}; \quad \hat{\sigma} = \frac{se}{\sqrt{1/52}}; \quad \hat{\bar{P}} = \frac{c(2)}{\hat{\eta}(1/52)} \tag{33}$$

to use a risk premium consistent with Capital Asset Pricing Model. This amounts to using the expected return on the hypothetical contract, used to calculate the market price of risk for the project. The value of this contract depends linearly on price. From Equation (21)

$$\rho \equiv \mu^T = r + \lambda\sigma = 0.03 + 0.01 \times 0.27 = 0.0327. \tag{34}$$

Note the assumption here that the volatility of this asset is constant, hence $s = \sigma$. Similarly, for $\lambda = 0.03$ and 0.05, the risk adjusted discount rates are 0.0381 and 0.0435 respectively.

## 3.4 Timber Yield, Product Prices, and Silviculture and Harvesting Costs

The empirical example used in this paper is for a stand of Jack Pine in Ontario's boreal forest. We include a so-called basic level of silvicultural investment which represents the current level of spending on many stands in Ontario's boreal forest. Silvicultural costs[16] (in $/hectare) are $200 for site preparation and $360 to purchase nursery stock in year 1, $360 for planting in year 2, $120 for tending in year 5, and finally $10 for monitoring in year 35. Amenity value is assumed to be zero, so that $A$ in Equation (29) reflects only silvicultural costs. The timber yield curves for Jack Pine saw logs and pulp under basic management in the boreal forest are provided in Table 2.[17]

Assumptions for harvesting costs and the different log prices are given in Table 3. These prices are considered representative for 2003 prices at the millgate in Ontario's boreal forest. Average delivered wood costs to the mill for 2003 are reported as $55 per cubic meter in a recent Ontario government report [Ontario Ministry of Natural Resources, May, 2005]. From this is subtracted $8 per cubic meter as an average stumpage charge in 2003 giving $47 per cubic metre.[18] It will be noted the lower valued items (SPF3 and poplar/birch) are harvested

[16] Kindly provided by Tembec Inc.

[17] Timber yield curves were estimated by M. Penner of Forest Analysis Ltd., Huntsville,Ontario, for Tembec.

[18] This consists of $35 per cubic meter for harvesting and $12 per cubic meter for transportation. Average stumpage charges are available from the Canadian Council of Forest Ministers. Land value is estimated before any stumpage charges.

at a loss. These items must be harvested according to Ontario government regulation. The price for poplar/birch is at roadside, so there is no transportation cost to the mill. In the empirical application SPF1 is modelled as the key stochastic variable, with the prices of other products maintaining the same relationship with SPF1 as is shown in Table 3.

# 4 Empirical Results

## 4.1 Bare Land Value

Using the parameters described in the previous section, the linear complementarity problem, Equation (30), plus boundary conditions were solved using a fully implicit finite difference approach as describe in Insley and Rollins [2005]. We estimate the value of a stand of trees at the beginning of the first rotation (bare land value) using the CC approach and compare it with the value estimated using a DP approach with our naive risk adjusted discount rate. The results are given in Figure 2 for the three values of the market price of risk. We report values for an initial price of \$60 per cubic metre for SPF1 logs.[19] [20]

We observe from Figure 2 that, consistent with the theoretical discussions in Section 2, CC and DP with a constant discount rate do not give the same land values. The differences are quite significant with DP 16% below the CC value for $\lambda = 0.01$ and 55% below for $\lambda = 0.05$. Also notice that for CC, land value is quite insensitive to the tripling of the market price of risk.

If we compare the PDE's which hold in the continuation region for CC and DP, it is evident why the value computed using CC is so much larger than for DP value in the mean reverting model and also why the CC value is fairly insensitive to $\lambda$. We rewrite these PDE's

[19]For the mean reverting price model, land value is very insensitive to the initial price.

[20]The accuracy of these results was checked by successive refinement of the solution grid. In addition a Richardson extrapolation was used to improve accuracy. (See Wilmott [1998] for an explanation of Richardson extrapolation.) The results indicate a numerical error of less that 1% of the value of the stand or approximatly \$5 for the value of the bare land.

for convenience. For DP the relevant PDE is:

$$W_t + \frac{1}{2}b^2(P,t)W_{PP} + a(P,t)W_P - \rho W + A + W_\alpha = 0 \tag{35}$$

while for CC:

$$V_t + \frac{1}{2}b^2(P,t)V_{PP} + [a(P,t) - \lambda b(P,t)]\, V_P - rV + A + V_\alpha = 0. \tag{36}$$

Going from Equation (35) to Equation (36) the required return associated with $V$ is reduced from $\rho$ to the risk free rate $r$, which will raise the value of $V$, *certeris paribus*. However this is offset by the reduction of the drift rate of the stochastic price by a risk premium, which will tend to lower $V$. For the mean reverting model we have $a(P,t) \equiv \eta(\bar{P} - P)$ and $b(P,t) = \sigma P$, so that the term associated with $V_P$ is $[\eta(\bar{P} - P) - \lambda\sigma P]$. If the speed of mean reversion $\eta$ is large relative to $\lambda\sigma$ then the risk premium will not have a large impact on the drift term when $P$ deviates from $\bar{P}$. The biggest effect of going from DP to CC in this case is the reduction in the discount rate to the risk free rate. Hence going from DP to CC we observe an increase in $V$.

## 4.2 Critical Harvesting Prices

Besides the value of the bare land, it is also of interest to estimate critical harvesting prices, which are determined at the point where the value of continuing to hold the option to harvest the stand of trees equals the value from harvesting immediately. Smooth pasting and value matching conditions hold at the critical points, although these do not need to be imposed explicitly in the numerical solution. If the current price of timber equals or exceeds the critical price for a particular stand age, then it is optimal to harvest the stand.

Critical prices for $\lambda = 0.01$ and $0.05$ are given in Figure 3. Interestingly the critical prices track quite closely for DP and CC, remaining within 2% of each other even for the higher value of $\lambda$. So in this example, the apparent optimal action is basically unchanged whether CC or DP is used.

### 4.3 Implied Risk Adjusted Discount Rates

We can use our numerical results for the CC analysis to estimate the implied risk adjusted discount rate as determined by Equation (16). Figure 4 (left hand graph) shows implied discount rates versus price for $\lambda = 0.03$. We observe that $\rho$ varies with both price and stand age, with different curves shown for stands of different ages. The largest variation in $\rho$ is for a stand of 50 years where $\rho$ ranges from is from 3 % to about 4.7% . The comparable figure for the case where $\lambda = 0.01$ has a similar shape, but with a much smaller variation in $\rho$ - from 3% to 3.5%. For $\lambda = 0.5$, $\rho$ varies from 3% to 6.9%.

Further intuition may be gained from the right hand graph in Figure 4 which shows $V_P/V$ for stands of age 35 and 50 (vertical left hand axis) and $\lambda\sigma P$ (vertical right hand axis). Since $\lambda\sigma P$ increases with $P$, a constant $\rho$ would require $V_P/V$ to decrease with $P$ at an offsetting rate. Beyond a price of about \$150 the the two are offsetting and we see that the implied risk adjusted discount rate settles at around 4 %. This makes sense since at very high prices it is optimal to harvest immediately and the ability to delay harvesting the stand has no value.

## 5 Summary and Concluding Remarks

Use of a constant risk adjusted discount rate with a dynamic programming approach is a common practice in problems of investment under uncertainty in forestry. However we have shown that a constant risk adjusted discount rate implies the value of the risky investment in question has a constant volatility over its lifetime. We presented a theorem and proof which specifies a necessary and sufficient condition for CC and DP, with a constant risk adjusted discount rate, to yield the same answer. We argued that this condition is too restrictive for most practical problems of investment under uncertainty.

We presented several special cases for which the risk adjusted discount rate will be constant. One that has appeared in the literature is the case of an infinitely lived simple American-type option in which investment payoff follows GBM.

For illustrative purposes, we examined the extent of the difference between CC and DP estimates of option value for an optimal harvesting problem with the stochastic underlying variable following a simple mean reverting process. We compared investment values for CC and DP using a constant discount rate equal to $\rho = r + \lambda\sigma$ and found non-trivial differences ranging from 16% to 55% of land value, depending on the assumed market price of risk. However, critical harvesting prices were quite close using the two approaches.

Lastly, we calculated the risk adjusted discount rates that are implied by contingent claims analysis for the forestry example. We found that the implied discount rates vary with price and stand age. This variation increases with the assumed market price of risk ranging from 0.5% point for the the lowest market price of risk ($\lambda = 0.01$) to 3.9% points for $\lambda = 0.05$.

We conclude that real options-type problems should be analyzed using contingent claims analysis. Some might suggest that given the difficulty of estimating the market price of risk it does not matter whether CC or DP is used. However, we would argue that it is preferable to use CC and put some effort into determining a reasonable market price of risk rather than adopting DP with a constant risk adjusted discount rate, which we know is incorrect except under special assumptions. In addition we observed that with mean reverting prices the value of the investment calculated using CC was quite insensitive to a tripling of the market price of risk. This is encouraging for real options problems that deal with commodities with mean reverting prices.

# 6 Appendix A: Contingent Claims Arguments

This appendix presents the steps for deriving Equation (10). This is now standard in finance texts, but we summarize the arguments used in Hull [2006] for the convenience of readers.

We can form an instantaneously riskless portfolio by purchasing $n_1 = (s_2 V_2)$ of $V_1$ and $n_2 = -(s_1 V_1)$ of $V_2$. These amounts are constant over the hedging interval, $dt$. The value of this riskless portfolio is:

$$\Pi = n_1 V_1 + n_2 V_2. \tag{37}$$

In our forestry example there is a cashflow each period that represents amenity benefits less management costs of the forest, $A(Q)$. We assume that there is no such cashflow involved with $V_2$. The change in value of the portfolio over the interval is then any capital gains in $V_1$ or $V_2$ plus the cashflow term:

$$d\Pi = n_1 dV_1 + n_1 A(Q) dt + n_2 dV_2. \tag{38}$$

Substituting for $dV_1$ and $dV_2$ from Equation (7) and for $n_1$ and $n_2$ as defined above into Equation (38), we find that

$$d\Pi = [(s_2 V_1 V_2 \mu_1) - (s_1 V_1 V_2 \mu_2) + s_2 V_2 A(Q)]dt \tag{39}$$

We know that our portfolio is riskless as we have eliminated the stochastic component $dz$. Our portfolio must therefore earn the riskless rate of return, $r$, which implies that

$$d\Pi = r\Pi dt \tag{40}$$

Substituting from Equations (37) and (39) into Equation (40)

$$\left[(s_2 V_1 V_2 \mu_1) + s_2 V_2 A(Q) - (s_1 V_1 V_2 \mu_2)\right] dt = r \left[(s_2 V_2) V_1 - (s_1 V_1) V_2\right] dt \tag{41}$$

This simplifies to

$$\frac{\mu_1 - \frac{A(Q)}{V_1} - r}{s_1} = \frac{\mu_2 - r}{s_2} \equiv \lambda \tag{42}$$

which is Equation (11).

Equation (13) uses the assumption that the underlying stochastic asset can be traded in financial markets. The return from holding timber would be that due to the capital gain, $\frac{a(P,t)}{P}$, plus any convenience yield that results from holding a physical inventory of timber., $\delta$. In this case the hedging asset $V_3 = P$ and $\mu_3 \equiv a(P,t)/P$. From Equation (10),

$$\begin{aligned} \lambda &= \frac{\mu_3 + \delta - r}{s_2} \\ &= \frac{a(P,t)/P + \delta - r}{b(P,t)/P}. \end{aligned} \tag{43}$$

where the convenience yield is added in as another component of the total return. Substituting for $\lambda$ in Equation (12) gives Equation (13).

# 7 Appendix B: Theorem and Proof regarding the Equivalence of CC and DP

Let $Z = V - W$ where $V$ is the known solution to Equation (14). Recall that $V$ refers to the value of the investment using CC and $W$ refers to the value using DP. Subtract Equation (6) (expressed in terms of $\tau$) from Equation (14):

$$-Z_\tau + \frac{1}{2}b^2(P,\tau)Z_{PP} + a(P,\tau)Z_P - \rho Z + Z_\alpha = \lambda b(P,\tau)V_P - (\rho - r)V \tag{44}$$

Let

$$\frac{1}{2}b^2(P,\tau)Z_{PP} + a(P,\tau)Z_P - \rho Z \equiv \mathcal{L}Z. \tag{45}$$

Let

$$\lambda b(P,\tau)V_P - (\rho - r)V \equiv f(P,\tau,\alpha). \tag{46}$$

Equation (44) can then be expressed as

$$-Z_\tau + Z_\alpha + \mathcal{L}Z = f(P,\tau,\alpha). \tag{47}$$

We assume that the boundary conditions are the same for $V$ and $W$ so that:

$$\begin{aligned} V(0,\tau,\alpha) &= W(0,\tau,\alpha) \\ V(P\to\infty,\tau,\alpha) &= W(P\to\infty,\tau,\alpha) \\ V(P,0,\alpha) &= W(P,0,\alpha) \end{aligned} \tag{48}$$

Equation (47) is therefore completely specified in the domain $[0,\infty] \times [0,T]$. Following from Equation (48), boundary conditions for $Z$ are

$$\begin{aligned} Z(0,\tau,\alpha) &= 0 \\ Z(P\to\infty,\tau,\alpha) &= 0 \\ Z(P,\tau=0,\alpha) &= 0. \end{aligned} \tag{49}$$

We can use a Green's Function to write the unique solution to Equation (47), but we must first go through some steps to deal with the term $Z_\alpha$ which does not contain a second derivative with respect to $\alpha$.[21]

Define an arbitrary function $M(P, \tau, \theta)$ that satisfies

$$-M_\tau + \mathcal{L}M = f(P, \tau, \theta - e\tau) \tag{50}$$

where $e$ is a constant, $P$ and $\tau$ are as previously defined and $\theta$ is an arbitrary variable. The solution to Equation (50) can be written in terms of the Green's function, $G$:

$$M(P, \tau, \theta) = \int_0^\infty \int_0^\tau G(P, \tau, P', \tau') f(P', \tau', \theta - e\tau) d\tau' dP'. \tag{51}$$

We shift the function $M$ by an amount $e\tau$ along the $\theta$ axis. Let $x = \theta - e\tau$. It follows that $M(P, \tau, x)$ satisfies

$$-M_\tau + eM_x + \mathcal{L}M = f(P, \tau, \theta). \tag{52}$$

Again using the Green's Function, the solution to (52) may be written as:

$$M(P, \tau, x) = \int_0^\infty \int_0^\tau G(P, \tau, P', \tau') f(P', \tau', \theta - e(\tau - \tau')) d\tau' dP'. \tag{53}$$

Noting the similarity between Equation (52) and Equation (47), we assume:

$$Z(P, \tau, \alpha) = M(P, \tau, x). \tag{54}$$

It then follows from Equation (53) that $Z$ can be expressed using the Green's Function as:

$$Z(P, \tau, \alpha) = \int_0^\infty \int_0^\tau G(P, \tau, P', \tau') f(P', \tau', \alpha - e(\tau - \tau')) d\tau' dP'. \tag{55}$$

**Theorem: Equivalence of DP and CC.** $W$ in Equation (6) (DP approach) and $V$ in Equation (12) (CC approach) will be the same value (and hence $Z = 0$) if and only if $f(P, \tau, \alpha) = 0$ in $[0, \infty] \times [0, T]$

*Proof.* From Equation (47) we have that if $Z = 0$ then $f(P, \tau, \alpha) = 0$. Conversely from Equation (55), if $f(P', \tau', \alpha - e(\tau - \tau')) = 0$ then $Z = 0$. □

[21] See Garroni and Menaldi [1992] for an explanation of the Green's Function.

From the definition of $f$ in Equation (46) our theorem implies that $Z = 0$ so that $V = W$ if and only if $\lambda b(P, \tau)V_P - (\rho - r)V = 0$. For given values of $\lambda$, $b(P, t)$ and $V$, this implies that

$$\rho = r + \frac{\lambda b(P, t)V_P}{V}. \tag{56}$$

| Parameter | Estimates (t-statistic) | Parameter | Estimates |
|---|---|---|---|
| c(1) | -.0147 (-2.71) | $\hat{\eta}$ | 0.8 |
| c(2) | 3.5089 (2.82) | $\hat{\bar{P}}$ | $230* |
| se of regression | 0.0369 | $\hat{\sigma}$ | 0.27 |

TABLE 1: *Parameter Estimates for Mean Reverting Price Process, Sample: weekly observations from January 1980 to July 2005. *Note that this estimated price is in $ per cubic metre at Toronto, which had to be translated to a price at the mill gate. Since the price of $230 dollars was close to the actual Toronto price in 2003, we adopted our estimated 2003 mill gate price of $60 per cubic metre for SPF1 logs as* $\bar{P}$ *at the mill gate.*

Table 2: Timber volume estimates for a Jack Pine stand in Ontario's boreal forest, $m^3/ha$ by product.

| Age | NMV | SPF1 | SPF2 | SPF3 | other[22] |
|---|---|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 5 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 10 | 0.2 | 0.0 | 0.0 | 0.2 | 0.0 |
| 15 | 2.4 | 0.0 | 0.0 | 2.3 | 0.1 |
| 20 | 12.2 | 0.0 | 0.0 | 11.5 | 0.6 |
| 25 | 40.0 | 0.0 | 0.0 | 37.8 | 2.2 |
| 30 | 91.4 | 0.0 | 26.7 | 59.4 | 5.4 |
| 35 | 146.8 | 0.0 | 53.0 | 84.7 | 9.1 |
| 40 | 190.7 | 0.0 | 80.1 | 98.4 | 12.2 |
| 45 | 222.4 | 49.7 | 80.2 | 77.8 | 14.7 |
| 50 | 245.6 | 63.4 | 93.0 | 72.7 | 16.7 |
| 55 | 264.1 | 76.9 | 103.0 | 66.0 | 18.2 |
| 60 | 280.3 | 90.3 | 110.9 | 59.4 | 19.6 |
| 65 | 295.1 | 103.6 | 117.2 | 53.4 | 20.8 |
| 70 | 308.7 | 116.4 | 122.1 | 48.3 | 21.9 |
| 75 | 321.3 | 128.3 | 125.9 | 44.2 | 22.9 |
| 80 | 332.8 | 139.1 | 129.0 | 41.0 | 23.7 |
| 85 | 343.3 | 148.7 | 131.5 | 38.6 | 24.4 |
| 90 | 351.8 | 156.2 | 133.4 | 37.1 | 25.0 |
| 95 | 356.2 | 160.8 | 134.1 | 36.0 | 25.2 |
| 100 | 358.2 | 163.5 | 134.1 | 35.3 | 25.3 |
| 105 | 358.7 | 164.9 | 133.7 | 34.8 | 25.3 |

Continued on next page

**Table 2 – continued from previous page**

| Age | NMV | SPF1 | SPF2 | SPF3 | other |
|---|---|---|---|---|---|
| 110 | 357.6 | 165.2 | 132.9 | 34.4 | 25.1 |
| 115 | 355.2 | 164.6 | 131.6 | 34.1 | 24.9 |
| 120 | 351.7 | 163.2 | 130.1 | 33.7 | 24.6 |
| 125 | 347.2 | 161.3 | 128.3 | 33.4 | 24.3 |
| 130 | 342.0 | 158.8 | 126.2 | 33.0 | 24.0 |
| 135 | 336.0 | 156.0 | 123.9 | 32.6 | 23.6 |
| 140 | 329.5 | 152.7 | 121.4 | 32.2 | 23.2 |
| 145 | 322.4 | 149.2 | 118.7 | 31.7 | 22.8 |
| 150 | 314.8 | 145.4 | 115.8 | 31.2 | 22.4 |
| 155 | 306.9 | 141.4 | 112.8 | 30.7 | 22.0 |
| 160 | 298.7 | 137.2 | 109.8 | 30.2 | 21.6 |
| 165 | 290.2 | 132.9 | 106.6 | 29.6 | 21.1 |
| 170 | 281.6 | 128.5 | 103.4 | 29.0 | 20.7 |
| 175 | 272.8 | 124.1 | 100.1 | 28.3 | 20.3 |
| 180 | 264.0 | 119.6 | 96.8 | 27.7 | 19.9 |
| 185 | 255.1 | 115.2 | 93.5 | 27.0 | 19.5 |
| 190 | 246.3 | 110.7 | 90.2 | 26.3 | 19.1 |
| 195 | 237.6 | 106.3 | 86.9 | 25.6 | 18.7 |
| 200 | 228.9 | 102.0 | 83.6 | 24.9 | 18.3 |
| 205 | 220.4 | 97.8 | 80.4 | 24.2 | 18.0 |
| 210 | 212.0 | 93.6 | 77.3 | 23.5 | 17.6 |
| 215 | 203.8 | 89.6 | 74.2 | 22.7 | 17.3 |
| 220 | 195.8 | 85.7 | 71.2 | 22.0 | 16.9 |
| 225 | 188.0 | 81.9 | 68.2 | 21.3 | 16.6 |

Continued on next page

**Table 2 – continued from previous page**

| Age | NMV | SPF1 | SPF2 | SPF3 | other |
|---|---|---|---|---|---|
| 230 | 180.4 | 78.2 | 65.3 | 20.6 | 16.2 |
| 235 | 173.0 | 74.6 | 62.6 | 19.9 | 15.9 |
| 240 | 165.8 | 71.2 | 59.9 | 19.2 | 15.6 |
| 245 | 158.9 | 67.9 | 57.3 | 18.5 | 15.3 |
| 250 | 152.3 | 64.7 | 54.7 | 17.8 | 15.0 |
| 255 | 145.8 | 61.7 | 52.3 | 17.2 | 14.7 |

[22]NMV is net merchantable volume. SPF refers to spruce, pine, fir logs. SPF1 is greater than 16 cm diameter at the small end, SPF2 is 12 to 16 cm, and SPF3 is less than 12 cm. 'Other' refers to poplar and birch. SPF3 and 'other' are used for pulp.

| | |
|---|---|
| Harvest and transportation cost | \$47 |
| Price of SPF1 | \$60 |
| Price of SPF2 | \$55 |
| Price of SPF3 | \$30 |
| Price of poplar/birch | \$20 |

TABLE 3: *Assumed values for log prices and cost of delivering logs to the mill in \$ per cubic meter*

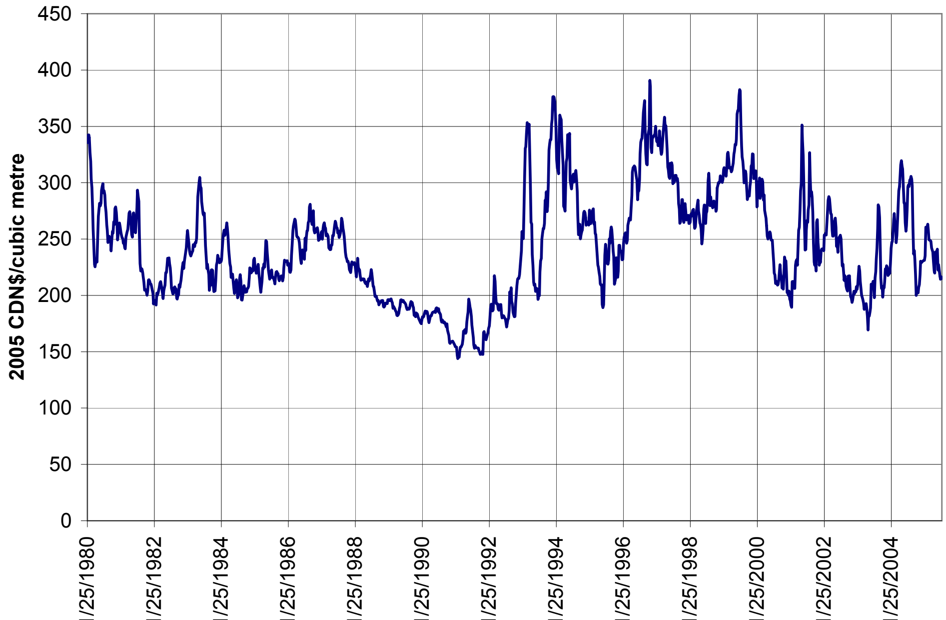



FIGURE 1: *Real price of softwood lumber, Toronto, Ontario, 2005 Canadian $ per cubic metre. (Source: Madison's Canadian Lumber Reporter, weekly data from January 1980 to July 2005, Eastern Spruce-Pine-Fir Std #2&Better, Kiln-dried, Random Length - 2x4, Deflated by the Canadian consumer price index and converted to Canadian $.)*

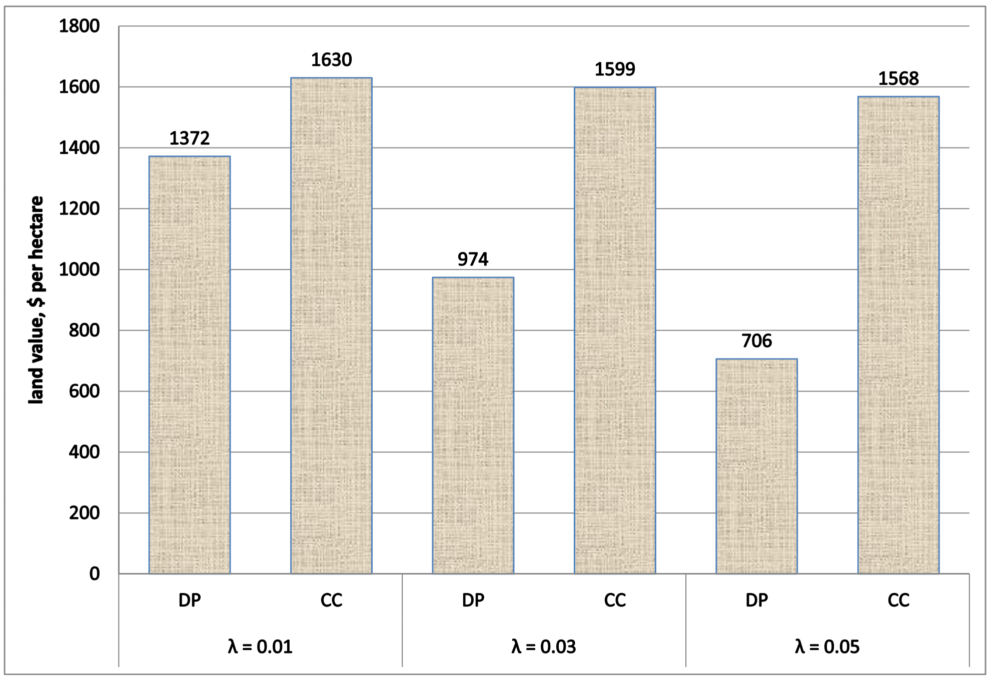
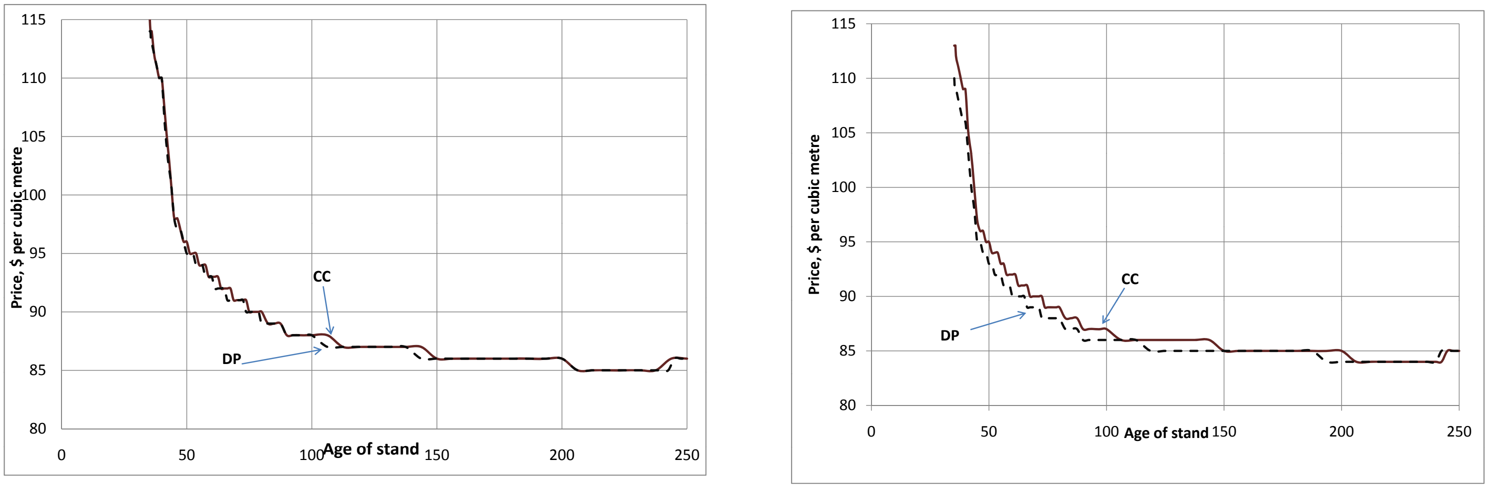


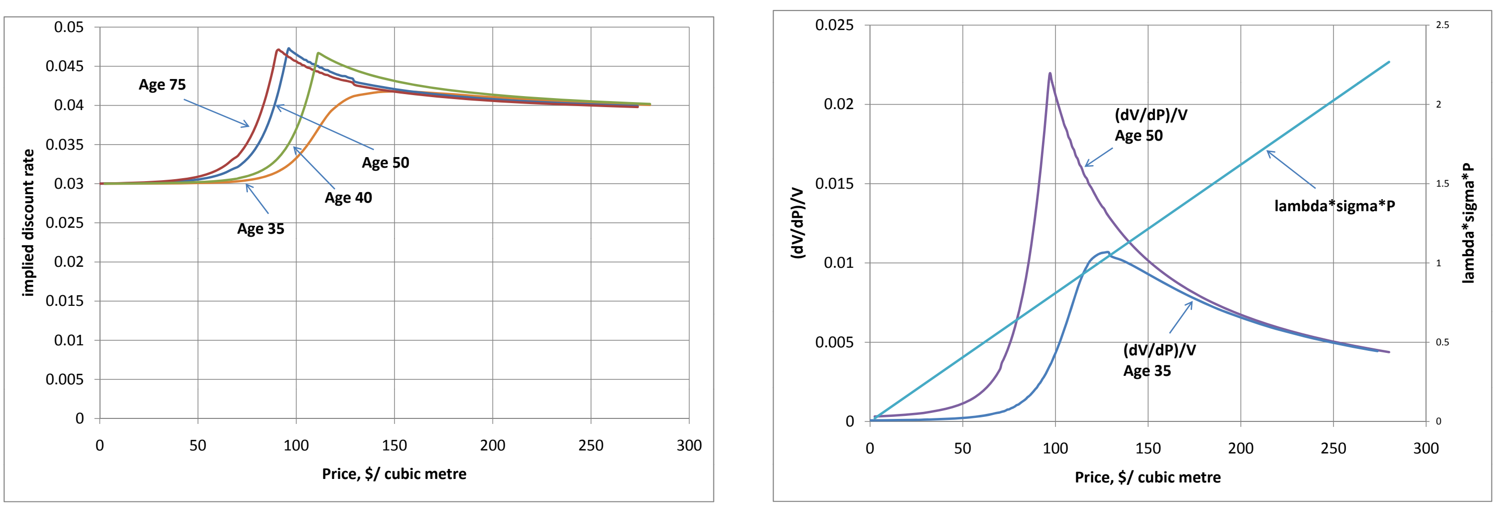

Figure 2: *Comparison of land values for dynamic programming (DP) and contingent claims (CC) approaches assuming an initial price of \$60 per cubic metre for SPF1 logs.*




(A) $\lambda = 0.01$

(B) $\lambda = 0.05$

Figure 3: *Critical Harvesting Prices, Comparing DP and CC for Mean Reverting Process*




(A) *Implied Discount Rate,* $\rho = r + \lambda\sigma P\frac{V_P}{V}$*, versus price for various stand ages.*

(B) *Components of the implied discount rate; left axis:* $V_P/V$ *and right axis:* $\lambda\sigma P$*.*

Figure 4: *Examining the implied risk adjusted discount rate for the mean reverting process and* $\lambda = 0.03$

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