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On the Accuracy of Surface Spline Interpolation on the Unit Sphere

Simon Hubbert and Tanya M. Morton

Abstract. This paper considers a novel modification to the surface splines that have previously been used on the unit sphere. The surface splines considered are a natural analogue of surface splines in $I\!R^d$ and possess a unique Fourier expansion in terms of an orthonormal basis of spherical harmonics. Knowing the decay of the associated Fourier coefficients is important because they enable error estimates for spherical interpolation. In this paper we explicitly compute the Fourier coefficients of the surface splines and employ a recent theoretical result [8] to provide a useful error bound. We illuminate our theoretical findings by performing numerical experiments on the sphere and also on the hemisphere.

§1. Surface Spline Interpolation in Euclidean Space

Let $m, d \in \mathbb{N} := \{1, 2, \ldots\}$ be such that $m > d/2$. The surface spline basis function $\phi : [0, \infty) \to \mathbb{R}$ is defined to be

$$\phi(r) = \begin{cases} (-1)^n r^{2m-d} \log r, & \text{if } d \text{ is even;} \\ (-1)^n r^{2m-d}, & \text{if } d \text{ is odd,} \end{cases} \tag{1}$$

where n is a positive integer defined by

$$n := \begin{cases} m - \frac{d-2}{2} & \text{if } d \text{ is even;} \\ m - \frac{d-1}{2} & \text{if } d \text{ is odd.} \end{cases} \tag{2}$$

Let $X = \{x_1, \ldots, x_N\}$ be a finite subset of distinct points in $\mathbb{R}^d$ that satisfy the following condition:

$$\text{if } \ p \in \Pi_{n-1}(\mathbb{R}^d) \ \text{ satisfies } \ p(x_j) = 0 \ \ (1 \le j \le N) \ \text{ then } \ p = 0, \tag{3}$$

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where $\Pi_{n-1}(\mathbb{R}^d)$ denotes the space of $d-$variate polynomials of degree at most $n-1$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary function. The unique *surface spline interpolant* to f at X, denoted by s_f, has the form

$$s_f(x) = \sum_{j=1}^{N} \lambda_j \phi(\|x - x_j\|) + p(x), \tag{4}$$

where $p \in \Pi_{n-1}(\mathbb{R}^d)$ and where the λ_j $(1 \leq j \leq N)$ satisfy

$$\sum_{j=1}^{N} \lambda_j p(x_j) = 0, \quad \text{for all } p \in \Pi_{n-1}(\mathbb{R}^d). \tag{5}$$

The surface spline basis function (1) has the property that it is *conditionally positive definite* of order n (2) on $\mathbb{R}^d$. This means that, for any set $X = \{x_1, \ldots, x_N\}$ of distinct data points in $\mathbb{R}^d$, the matrix $A \in \mathbb{R}^{N \times N}$ given by

$$A_{jk} = \phi(\|x_j - x_k\|), \quad 1 \leq j, k \leq N,$$

is positive definite on the following subspace

$$V_{n-1} = \left\{ \lambda \in \mathbb{R}^N : \sum_{i=1}^{N} \lambda_i p(x_i) = 0 \text{ for all } p \in \Pi_{n-1}(\mathbb{R}^d) \right\}. \tag{6}$$

For most practical applications we assume that we have we have a bounded open domain $\Omega \subset \mathbb{R}^d$ for which $X \subset \text{closure}(\Omega)$. We measure the density of X in Ω by assigning the mesh norm

$$h := \sup_{x \in \Omega} \inf_{x_j \in X} \|x - x_j\|. \tag{7}$$

Let $p \in [1, \infty]$ and let $f : \Omega \to \mathbb{R}$ be chosen from a sufficiently smooth class of functions. The surface spline interpolant is said to provide an L_p-approximation order of $\lambda > 0$ if

$$\|f - s_f\|_{L_p(\Omega)} = O(h^\lambda), \text{ as } h \to 0. \tag{8}$$

The largest possible value of λ such that (8) holds is called the optimal approximation order.

In the late 1980s, Buhmann [2] investigated the special case of surface spline interpolation where $\Omega = \mathbb{R}^d$ and $X = h \cdot \mathbb{Z}^d$, i.e., the data points are the vertices of a scaled integer lattice. In this framework it is shown that, given a function $f \in C^{2m}(\mathbb{R}^d)$, its unique surface spline interpolant s_f at $h \cdot \mathbb{Z}^d$ satisfies

$$\|s_f - f\|_{L_p(\mathbb{R}^d)} = O(h^{2m}), \quad p \in [1, \infty]. \tag{9}$$

Furthermore, $2m$ is the optimal approximation order.

At the present time, the optimal L_p-approximation orders ($p \in [1,\infty]$) are not known for the case of surface spline interpolation on a smooth bounded domain $\Omega \subset \mathbb{R}^d$. Indeed, numerical evidence suggests that, in such cases, the accuracy is badly affected by the presence of a boundary. There has been a great deal of research into understanding (i.e., quantifying) the boundary effect, most notably by Michael Johnson. Indeed, a combination of Johnson's results from [9], [10] and [11] demonstrates that, for a suitably smooth domain $\Omega \subset \mathbb{R}^d$, the surface spline interpolant to a smooth enough target function f over a data set $X \subset \Omega$ with mesh morm h satisfies

$$\|f - s_f\|_{L_p(\Omega)} = O(h^\lambda), \text{ as } h \to 0,$$

$$\text{where } \begin{cases} \lambda \in [m - \frac{d}{2} + \frac{d}{p} + \frac{1}{2}, m + \frac{1}{p}], & p \in [2,\infty]; \\ \lambda = m + \frac{1}{p}, & p \in [1,2]. \end{cases}$$

Thus, the optimal L_p-approximation orders are known for $p \in [1,2]$. Furthermore, the numerical evidence presented in [11] strongly suggests that $m + \frac{1}{p}$ is the optimal L_p-approximation order for $p > 2$.

§2. Surface Spline Interpolation on the Unit Sphere

Suppose that we wish to interpolate scattered data which are known to lie on a smooth compact $(d-1)-$dimensional differentiable manifold $\mathcal{M} \subset \mathbb{R}^d$. One possible scheme would be to use a modified surface spline, that is, we choose $m, d \in \mathbb{N}$ such that $m > \frac{d-1}{2}$, and consider

$$\widetilde{\phi}(r) = \begin{cases} (-1)^{m-\frac{d-3}{2}} r^{2m-(d-1)} \log r, & \text{if } d \text{ is odd;} \\ (-1)^{m-\frac{d-2}{2}} r^{2m-(d-1)}, & \text{if } d \text{ is even.} \end{cases} \tag{10}$$

In the special case of the unit sphere $\mathcal{M} = S^{d-1}$ we can use the relation

$$\|\xi - \eta\| = \sqrt{2 - 2\xi^T\eta}, \qquad \xi, \eta \in S^{d-1}, \tag{11}$$

and completely specialise (10) to the sphere to give $\psi : [-1,1] \to \mathbb{R}$ as

$$\psi(t) = \begin{cases} (-1)^{m-\frac{d-3}{2}} \frac{1}{2} (2-2t)^{m-\frac{d-1}{2}} \log(2-2t), & \text{if } d \text{ is odd;} \\ (-1)^{m-\frac{d-2}{2}} (2-2t)^{m-\frac{d-1}{2}}, & \text{if } d \text{ is even.} \end{cases} \tag{12}$$

This introduces a new function that has not been considered previously in the literature of basis function inteterpolation on spheres. We will see that this basis function has several properties that make it a natural analogue of surface splines in $\mathbb{R}^d$. Taking the surface spline analogy one step further we dispense with the notion of polynomial reproduction and replace it with *spherical harmonic* polynomial reproduction. A spherical harmonic of order k on S^{d-1} is the restriction to S^{d-1} of a $d-$dimensional homogeneous harmonic polynomial of degree k. For a good account of the theory of spherical harmonics see [14], we present here a brief review.

We let $\mathcal{H}_k^*(S^{d-1})$ denote the space of spherical harmonics of order k on S^{d-1}. This space has a useful intrinsic characterisation. If we let Δ_{d-1} denote the Laplace-Beltrami operator on S^{d-1} then the eigenvalues for the eigenvalue problem

$$(\Delta_{d-1} + \lambda)u = 0 \tag{13}$$

are $\lambda_k = k(k + d - 2)$ $k \geq 0$, and $\mathcal{H}_k^*(S^{d-1})$ is precisely the eigenspace of Δ_{d-1} corresponding to λ_k. The dimension $N_{k,d}$ of $\mathcal{H}_k^*(S^{d-1})$ is given by the multiplicity of λ_k in (13), specifically

$$N_{0,d} = 1, \quad \text{and } N_{k,d} = \frac{2k + d - 2}{k}\binom{k + d - 3}{k - 1}, \ \ k \geq 1.$$

Given an orthonormal basis $\{\mathcal{Y}_{k,l} : l = 1, \ldots, N_{k,d}\}$ for $\mathcal{H}_k^*(S^{d-1})$ the collection

$$\{\mathcal{Y}_{j,l} : l = 1, \ldots, N_{j,d} : j = 0, 1, \ldots k\}$$

is an orthonormal basis for the space of spherical harmonics of order at most k, which we denote by $\mathcal{H}_k(S^{d-1})$. Furthermore, the collection

$$\{\mathcal{Y}_{j,l} : l = 1, \ldots, N_{j,d} : j \geq 0\}$$

forms an orthonormal basis for $L_2(S^{d-1})$. According to the celebrated addition theorem [14]

$$P_{k,d}(\xi^T\eta) := \frac{\omega_{d-1}}{N_{k,d}} \sum_{l=1}^{N_{k,d}} \mathcal{Y}_{k,l}(\xi)\mathcal{Y}_{k,l}(\eta), \quad \xi, \eta \in S^{d-1}, \tag{14}$$

where ω_{d-1} denotes the surface area of S^{d-1} and $P_{k,d}$ is the $d-$dimensional Legendre polynomial of degree k which are defined on the interval $[-1, 1]$ via the Rodrigues formula

$$P_{k,d}(t) = \frac{(-1)^k\Gamma(\frac{d-1}{2})}{2^k\Gamma(k + \frac{d-1}{2})}(1 - t^2)^{\frac{3-d}{2}} \frac{d^k}{dt^k}(1 - t^2)^{k+\frac{d-3}{2}}, \tag{15}$$

Spherical harmonics can be used to give a *Fourier analysis* for the sphere. In particular, every function $f \in L_2(S^{d-1})$ has an associated Fourier series

$$f = \sum_{k=0}^{\infty} \sum_{l=1}^{N_{k,d}} \hat{f}_{k,l}\mathcal{Y}_{k,l}. \tag{16}$$

The Fourier coefficients are obtained by

$$\hat{f}_{k,l} = \int_{S^{d-1}} f(\xi)\mathcal{Y}_{k,l}(\xi)dS(\xi), \tag{17}$$

where dS represents a surface element of S^{d-1}. The square of L_2-norm of f is given by

$$\|f\|^2_{L_2(S^{d-1})} = \int_{S^{d-1}} |f(\xi)|^2 dS(\xi) = \sum_{k=0}^{\infty}\sum_{l=1}^{N_{k,d}} |\hat{f}_{k,l}|^2, \tag{18}$$

where the second equality is Parseval's identity.

For a real number $\beta \geq 0$ the Sobolev space $W_2^\beta(S^{d-1})$ of order β is defined as

$$\Big\{ f \in L_2(S^{d-1}) : \|f\|^2_{W_2^\beta(S^{d-1})} = \sum_{k=0}^{\infty}\sum_{l=1}^{N_{k,d}} (1+\lambda_k)^\beta |\hat{f}_{k,l}|^2 < \infty \Big\}. \tag{19}$$

The Sobolev embedding theorem holds true on the sphere, and asserts that whenever $\beta > \frac{d-1}{2}$ then $W_2^\beta(S^{d-1})$ is continuously embedded in $C(S^{d-1})$.

Let $\Xi = \{\xi_1, \ldots, \xi_N\}$ be a set of distinct points on S^{d-1} that satisfy the following condition

$$\text{if } \ \mathcal{Y} \in \mathcal{H}_{n-1}(S^{d-1}) \ \text{ satisfies } \ \mathcal{Y}(\xi_j) = 0 \ \ (1 \leq j \leq N) \ \text{ then } \ \mathcal{Y} = 0. \tag{20}$$

Let $f : S^{d-1} \to \mathbb{R}$ be an arbitrary function. The unique *specialised* surface spline interpolant to f at Ξ, denoted by s_f, has the form

$$s_f(\xi) = \sum_{j=1}^{N} \alpha_j \psi(\xi^T \xi_j) + \mathcal{Y}(\xi), \tag{21}$$

where $\mathcal{Y} \in \mathcal{H}_{n-1}(S^{d-1})$ and where the α_j $(1 \leq j \leq N)$ satisfy

$$\sum_{j=1}^{N} \alpha_j \mathcal{Y}(\xi_j) = 0, \quad \text{for all } \mathcal{Y} \in \mathcal{H}_{n-1}(S^{d-1}). \tag{22}$$

The new restricted surface spline (12) is conditionally positive definite of order n on S^{d-1}. This means, in analogy to the notion in $\mathbb{R}^d$, that for any set $\Xi = \{\xi_1, \ldots, \xi_N\}$ of distinct data points on S^{d-1}, the matrix $B \in \mathbb{R}^{N\times N}$ given by

$$B_{jk} = \psi(\xi_j^T \xi_k), \quad 1 \leq j,k \leq N,$$

is positive definite on the following subspace

$$W_{n-1} = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i \mathcal{Y}(\xi_i) = 0 \ \text{ for all } \ \mathcal{Y} \in \mathcal{H}_{n-1}(S^{d-1}) \right\}. \qquad (23)$$

Using the work of Schoenberg [16], and extensions thereof [6], we can deduce that ψ has the following unique representation

$$\psi(t) = \sum_{k=0}^{\infty} a_k P_{k,d}(t), \quad \text{with } a_k \geq 0 \text{ for } k \geq n, \qquad (24)$$

where $\{P_{k,d}\}_{k\geq 0}$ denote the $d-$dimensional Legendre polynomials.

In recent years, a more general theory of interpolation using so-called *zonal basis functions* has emerged [3], [5]; these are functions $\psi : [-1,1] \to \mathbb{R}$ which have unique Legendre expansions as in (24). Since the Legendre polynomials satisfy the following orthogonality relation,

$$\int_{-1}^{1} P_{j,d}(t) P_{k,d}(t)(1-t^2)^{\frac{d-3}{2}} dt = \frac{\omega_{d-1}}{\omega_{d-2} N_{k,d}} \delta_{jk}, \qquad (25)$$

it immediately follows that the associated expansion coefficients are given by

$$a_k = \frac{N_{k,d}\omega_{d-2}}{\omega_{d-1}} \int_{-1}^{1} P_{k,d}(t)\psi(t)(1-t^2)^{\frac{d-3}{2}} dt, \quad k \geq 0 \qquad (26)$$

To any zonal basis function the corresponding zonal kernel $\Psi(\xi,\eta) = \psi(\xi^T \eta)$ has a spherical Fourier expansion

$$\Psi(\xi,\eta) = \sum_{k=0}^{\infty} \sum_{l=1}^{N_{k,d}} \hat{c}_k \mathcal{Y}_{k,l}(\xi) \mathcal{Y}_{k,l}(\eta). \qquad (27)$$

Applying the addition theorem and (26) we deduce that the Fourier coefficients are given by

$$\hat{c}_k = \omega_{d-2} \int_{-1}^{1} P_{k,d}(t)\psi(t)(1-t^2)^{\frac{d-3}{2}} dt, \quad k \geq 0 \qquad (28).$$

Definition 1. *A zonal basis function ψ is said to have $\alpha-$Fourier decay whenever there exists positive constants A_1, A_2 such that the Fourier coefficients (28) of the zonal kernel $\Psi(\xi,\eta) = \psi(\xi^T \eta)$ satisfy*

$$A_1(1+k)^{-(d-1+\alpha)} \ \leq \ \hat{c}_k \ \leq \ A_2(1+k)^{-(d-1+\alpha)}, \ \ \alpha > 0, \ \ k \geq n. \qquad (29)$$

The accuracy of the method of interpolation with zonal basis functions has received a great deal of theoretical attention (cf. [12],[7], [8]). A typical error estimate takes into account 3 factors; the density of the data set $\Xi \subset S^{d-1}$, the smoothness of the basis function ψ, and the smoothness of the target function f to be interpolated.

To measure the density of the data points we assign the mesh norm

$$h := h(\Xi, S^{d-1}) := \sup_{\eta \in S^{d-1}} \min\{\cos^{-1}(\eta^T \xi_i) : \xi_i \in \Xi\}. \tag{30}$$

The smoothness of the underlying basis function is measured by the decay rate of the corresponding Legendre coefficients; see Definition 1. Finally, the smoothness of the target function f is captured by restricting attention to functions which belong to a certain Sobolev space $W_2^\beta(S^{d-1})$, see (19). The following result is taken from [8].

Theorem 2. *Let ψ be a zonal basis function which has $\alpha-$Fourier decay. Let $f \in W_2^\beta(S^{d-1})$ where $\beta = \alpha + d - 1$. Let s_f denote the unique $\psi-$based ZBF interpolant to f over a set $\Xi \subset S^{d-1}$ of distinct data points with mesh-norm h. Then, we have*

$$\|s_f - f\|_{L_p(S^{d-1})} = \begin{cases} O(h^{\alpha + \frac{d-1}{2} + \frac{d-1}{p}}), & p \in [2, \infty]; \\ O(h^{\alpha + d - 1}), & p \in [1, 2]. \end{cases}$$

For historical reasons, the research community tends to view the surface splines as the prototype basis functions. In view of Theorem 2, it is clear that in order to estimate the accuracy of surface spline interpolation on the sphere, we only need to determine the decay rate of their Fourier expansion coefficients.

§3. The Fourier Coefficients of the Surface Splines

We can use equation (28) to compute the Fourier coefficients of the zonal kernel $\Psi(\xi, \eta) = \psi(\xi^T \eta)$ induced by a zonal basis function ψ. In particular, substituting the Rodrigues representation (15) for the $P_{k,d}$ $(k \geq 0)$, and integrating by parts k times gives

$$\hat{c}_k = \kappa_d \int_{-1}^{1} \psi^{(k)}(t)(1 - t^2)^{k + \frac{d-3}{2}} \, dt, \tag{31}$$

where

$$\kappa_d = \frac{\omega_{d-2} \Gamma(\frac{d-1}{2})}{2^k \Gamma(k + \frac{d-1}{2})} = \frac{\pi^{\frac{d-1}{2}}}{2^{k-1} \Gamma(k + \frac{d-1}{2})}. \tag{32}$$

We can now use this formula to compute the Fourier coefficient of the restricted surface splines and we begin with the case where d is even and so, by (12), consider $\psi(t) = (2-2t)^{m-\frac{d-1}{2}}$. Substituting

$$\psi^{(k)}(t) = \frac{(-1)^k 2^{m-\frac{d-1}{2}}\Gamma(m-\frac{d-1}{2}+1)}{\Gamma(m-\frac{d-1}{2}-k+1)}(1-t)^{m-\frac{d-1}{2}-k},$$

into (31) yields

$$\hat{c}_k = \frac{(-1)^k 2^{m-\frac{d-1}{2}}\Gamma(m-\frac{d-1}{2}+1)}{\Gamma(m-\frac{d-1}{2}-k+1)}\kappa_d \int_{-1}^{1}(1-t)^{m-\frac{d-1}{2}-k}(1-t^2)^{k+\frac{d-3}{2}}dt.$$

Letting $2u = 1+t$ allows us to write

$$\hat{c}_k = \frac{(-1)^k 2^{2m+k-1}\Gamma(m-\frac{d-1}{2}+1)}{\Gamma(m-\frac{d-1}{2}-k+1)}\kappa_d \int_{0}^{1}(1-u)^{m-1}u^{k+\frac{d-3}{2}}du.$$

The integral in the above expression is the *beta function*

$$B(x,y) = \int_0^1 (1-u)^{y-1}u^{x-1}du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

evaluated at $x = m$ and $y = k + \frac{d-1}{2}$. Thus we have that

$$\hat{c}_k = \frac{(-1)^k 2^{2m+k-1}\Gamma(m-\frac{d-1}{2}+1)\Gamma(m)\Gamma(k+\frac{d-1}{2})}{\Gamma(m-\frac{d-1}{2}-k+1)\Gamma(m+k+\frac{d-1}{2})}\kappa_d. \tag{33}$$

This can be simplified further by using the reflection formula ([1], 6.1.17)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{34}$$

In particular setting $z = m - \frac{d-1}{2} - k + 1$ yields

$$\Gamma(m - \frac{d-1}{2} - k + 1)\Gamma(-m + k + \frac{d-1}{2}) = (-1)^{k-m+\frac{d-2}{2}}\pi,$$

furthermore, expressing $\Gamma(m+k+\frac{d-1}{2})$ as

$$(m+k+\frac{d-1}{2}-1)\cdots(m+k+\frac{d-1}{2}-2m)\Gamma(-m+k+\frac{d-1}{2})$$

allows us to rewrite (33) as

$$\hat{c}_k = \frac{(-1)^{m-\frac{d-2}{2}}2^{2m+k-1}\Gamma(m-\frac{d-1}{2}+1)\Gamma(m)\Gamma(k+\frac{d-1}{2})}{(m+k+\frac{d-1}{2}-1)\cdots(m+k+\frac{d-1}{2}-2m)\pi}\kappa_d.$$

Substituting the value for κ_d yields

$$\hat{c}_k = \frac{(-1)^{m-\frac{d-2}{2}} 2^{2m} \pi^{\frac{d-3}{2}} \Gamma(m - \frac{d-1}{2} + 1)\Gamma(m)}{(m+k+\frac{d-1}{2} - 1) \cdots (m+k+\frac{d-1}{2} - 2m).} \tag{35}$$

We now turn attention to the case where d is odd and so, by (12), we consider $\psi(t) = \frac{1}{2}(2-2t)^{m-\frac{d-1}{2}} \log(2-2t)$. This function can be represented as follows

$$\psi(t) = \frac{1}{2}\frac{\partial}{\partial\beta}(2-2t)^{\beta}\Big|_{\beta = m - \frac{d-1}{2}.} \tag{36}$$

The Fourier coefficients of the more general function $t \mapsto (2-2t)^{\beta}$ can be computed in the same way as above. Indeed, they are given by

$$\hat{c}_k(\beta) = (-1)^k \pi^{\frac{d-1}{2}} 2^{d-1} h(\beta) \tag{37}$$

where

$$h(\beta) = \frac{2^{2\beta}\Gamma(\beta+1)\Gamma(\beta+\frac{d-1}{2})}{\Gamma(k+\beta+d-1)\Gamma(-k+\beta+1)}. \tag{38}$$

The Fourier coefficients $\{\hat{d}_k\}_{k\geq 0}$ of the zonal kernel induced by (36) are given by

$$\hat{d}_k = \frac{1}{2}\frac{\partial}{\partial\beta}\hat{c}_k(\beta)\Big|_{\beta = m - \frac{d-1}{2}.}$$

Since the function $(-1)^{m-\frac{d-3}{2}}\psi$ is conditionally positive definite of order $m - \frac{d-3}{2}$ we shall restrict attention to the coefficients $\hat{d}_k$ where $k > m - \frac{d-1}{2}$. In order to differentiate $h(\beta)$ it is useful to express it as

$$h(\beta) = 2^{2\beta}\Gamma(\beta + \frac{d-1}{2})\frac{\beta(\beta-1)\cdots(\beta-k+1)}{\Gamma(\beta+k+d-1)} = u(\beta)\frac{v(\beta)}{w(\beta)}.$$

Differentiating with respect to β gives

$$h'(\beta) = \frac{w(\beta)\{u'(\beta)v(\beta) + v'(\beta)u(\beta)\} - u(\beta)v(\beta)w'(\beta)}{w(\beta)^2}.$$

We observe that $v(m - \frac{d-1}{2}) = 0$ whenever $k > m - \frac{d-1}{2}$ and so the expression above, evaluated at $\beta = m - \frac{d-1}{2}$ simplifies to

$$h'(m - \frac{d-1}{2}) = \frac{v'(m-\frac{d-1}{2})u(m-\frac{d-1}{2})}{w(m-\frac{d-1}{2})}.$$

Furthermore,

$$v'(m - \frac{d-1}{2}) = (-1)^{k-m+\frac{d-3}{2}}\Gamma(m - \frac{d-1}{2} + 1)\Gamma(k - m + \frac{d-1}{2}),$$

from which we can see $h'(m-\frac{d-1}{2})$ equals

$$(-1)^{k+m-\frac{d-3}{2}}2^{2m-(d-1)}\Gamma(m)\Gamma(m-\frac{d-1}{2}+1)\frac{\Gamma(k-m+\frac{d-1}{2})}{\Gamma(k+m+\frac{d-1}{2})}.$$

Using the fact that

$$\frac{\Gamma(k-m+\frac{d-1}{2})}{\Gamma(k+m+\frac{d-1}{2})}=\frac{1}{(m+k+\frac{d-1}{2}-1)\cdots(m+k+\frac{d-1}{2}-2m)},$$

we can deduce that, for $k>m-\frac{d-1}{2}$, we have

$$\hat{d}_k=\frac{(-1)^{m-\frac{d-3}{2}}2^{2m-1}\pi^{\frac{d-1}{2}}\Gamma(m-\frac{d-1}{2}+1)\Gamma(m)}{(m+k+\frac{d-1}{2}-1)\cdots(m+k+\frac{d-1}{2}-2m)}. \tag{39}$$

In summary we have proved the following result.

Lemma 3. *For d even the restricted surface spline*

$$\psi(t)=(-1)^{m-\frac{d-2}{2}}(2-2t)^{m-\frac{d-1}{2}}$$

is conditionally positive definite of order $m-\frac{d-2}{2}$ on S^{d-1} and the Fourier coefficients of its associated kernel are given by

$$\hat{c}_k=\frac{2^{2m}\pi^{\frac{d-3}{2}}\Gamma(m-\frac{d-1}{2}+1)\Gamma(m)}{(m+k+\frac{d-1}{2}-1)\cdots(m+k+\frac{d-1}{2}-2m)}.$$

Similarly, for d odd the restricted surface spline

$$\psi(t)=(-1)^{m-\frac{d-3}{2}}\frac{1}{2}(2-2t)^{m-\frac{d-1}{2}}\log(2-2t)$$

is conditionally positive definite of order $m-\frac{d-3}{2}$ on S^{d-1} and the Fourier coefficients of its associated kernel are given by

$$\hat{d}_k=\frac{2^{2m-1}\pi^{\frac{d-1}{2}}\Gamma(m-\frac{d-1}{2}+1)\Gamma(m)}{(m+k+\frac{d-1}{2}-1)\cdots(m+k+\frac{d-1}{2}-2m)}.$$

In both cases the restricted splines have $\alpha-$Fourier decay where $\alpha=2m-(d-1)$.

With the decay rates established, we can appeal to Theorem 2 and formulate the following result concerning the accuracy of surface spline interpolants on the sphere.

Theorem 4. *Let $m,d\in\mathbb{N}$ be such that $m>\frac{d-1}{2}$. Let $f\in W_2^{2m}(S^{d-1})$ and let s_f denote the unique surface spline interpolant to f over a set $\Xi\subset S^{d-1}$ of distinct data points with mesh-norm h. Then, we have*

$$\|s_f-f\|_{L_p(S^{d-1})}=\begin{cases}O(h^{2m-\frac{d-1}{2}+\frac{d-1}{p}}), & p\in[2,\infty];\\ O(h^{2m}), & p\in[1,2].\end{cases} \tag{40}$$

§4. A Numerical Investigation

In this section we investigate the numerical performance of employing surface splines to fit scattered data on the sphere. Attention is devoted to the circle S^1, where we shall use the linear and cubic splines, that is,

$$\psi(t) = -(2-2t)^{1/2} \quad \text{and} \quad \psi(t) = (2-2t)^{3/2}, \tag{41}$$

and the 2-sphere S^2, where we shall use the thin plate spline

$$\psi(t) = (1-t)\log(2-2t). \tag{42}$$

For a thorough investigation it is important that we have some control on the distribution of the data points on either S^1 or S^2. The important quantity to consider is the mesh norm h (30). In view of this we perform numerical testing on S^1 with equally spaced points, where doubling the number of points causes h to halve. For S^2, we use the spiral points of Saff and Kuijlaars [15]. Here the data points uniformly fill up the sphere by tracing out an imaginary spiral from the south pole x_1 to the north pole x_N. Numerical experiments suggest that doubling the number of spiral points causes h to decrease by a factor of approximately $1/\sqrt{2}$.

For testing purposes we choose to interpolate the following infinitely smooth target functions

$$f(x,y) = 1 + x^8 + e^{2y^3}, \quad \text{for} \quad S^1, \tag{43}$$

and

$$f(x,y,z) = \sin x \cdot \sin y \cdot \sin z, \quad \text{for} \quad S^2. \tag{44}$$

In order to measure the interpolation error we generate 10,000 points (equally spaced for S^1 and randomly distributed for S^2) and approximate as follows,

$$\|s-f\|_{L_\infty(S^{d-1})} \approx \max\{|s(\xi_i) - f(\xi_i)| : 1 \le i \le 10{,}000\},$$

and

$$\|s-f\|^p_{L_p(S^{d-1})} \approx \frac{1}{10{,}000} \sum_{i=1}^{10{,}000} |s(\xi_i) - f(\xi_i)|^p, \quad p \in [1,\infty).$$

With this test environment in place, we ask the following question.
Question: *How close are the theoretical L_p−convergence orders, given by Theorem 4 to those observed by experiment ?*

Let $\epsilon_{p,N}$ denote the L_p−error measure achieved using N data points with mesh norm h_N. The aim of our experiment is to examine how the

Tab. 1. Accuracy of linear spline interpolation on S^1.

Nodes	L_1−error	L_2−error	L_∞−error
64	4.24_{-02}	2.73_{-02}	4.91_{-02}
128	1.07_{-02}	6.84_{-03}	1.26_{-02}
256	2.68_{-03}	1.71_{-03}	3.17_{-03}
512	6.69_{-04}	4.28_{-04}	7.94_{-04}
1024	1.67_{-04}	1.07_{-04}	1.99_{-04}
Ratio	L_1−order	L_2−order	L_∞−order
64/128	1.99	2.00	1.96
128/256	2.00	2.00	1.99
256/512	2.00	2.00	2.00
512/1024	2.00	2.00	2.00

Tab. 2. Accuracy of cubic spline interpolation on S^1.

Nodes	L_1−error	L_2−error	L_∞−error
64	2.00_{-05}	4.30_{-05}	2.57_{-04}
128	1.14_{-06}	2.45_{-06}	1.54_{-05}
256	7.05_{-08}	1.49_{-07}	9.50_{-07}
512	4.38_{-09}	9.29_{-09}	5.92_{-08}
1024	2.73_{-10}	5.80_{-10}	3.70_{-09}
Ratio	L_1−order	L_2−order	L_∞−order
64/128	4.12	4.13	4.06
128/256	4.03	4.03	4.02
256/512	4.01	4.01	4.00
512/1024	4.00	4.00	4.00

error measure changes as the interpolation nodes double. The theory predicts that

$$\frac{\epsilon_{p,N}}{\epsilon_{p,2N}} \approx \begin{cases} (h_N/h_{2N})^{2m-\frac{d-1}{2}+\frac{d-1}{p}}, & p \in [2,\infty]; \\ (h_N/h_{2N})^{2m}, & p \in [1,2]. \end{cases}$$

Using our data point distributions we know that $h_N/h_{2N} = 2$ for the circle, and $h_N/h_{2N} \approx \sqrt{2}$ for the 2-sphere. Thus we can use our numerical results to predict the optimal convergence orders. The results are displayed in Tables 1–3.

Conclusions: The numerical results suggest that the optimal $L_p(S^{d-1})$ convergence order ($p \in [1,\infty]$), for restricted surface spline interpolation is $2m$. This implies that Theorem 4 predicts the optimal orders for $p \in [1,2]$. We summarise our findings in the following conjecture.

Tab. 3. Accuracy of thin plate spline interpolation on S^2.

Nodes	L_1−error	L_2−error	L_∞−error
128	3.39_{-04}	4.38_{-04}	1.27_{-03}
256	7.99_{-05}	1.03_{-04}	2.72_{-04}
512	1.90_{-05}	2.46_{-05}	6.73_{-05}
1024	4.68_{-06}	6.08_{-06}	1.67_{-05}
2048	1.14_{-06}	1.49_{-06}	4.22_{-06}
Ratio	L_1−order	L_2−order	L_∞−order
64/128	4.17	4.17	4.45
128/256	4.15	4.14	4.02
256/512	4.04	4.04	4.02
512/1024	4.07	4.05	3.97

Conjecture 5. *Let $m, d \in \mathbb{N}$ be such that $m > \frac{d-1}{2}$. Let $f \in W_2^{2m}(S^{d-1})$ and let s_f denote the unique surface spline interpolant to f over a set $\Xi \subset S^{d-1}$ of distinct data points with mesh-norm h. Then, we have*

$$\|s_f - f\|_{L_p(S^{d-1})} = O(h^{2m}) \quad \textit{for all } p \in [1, \infty].$$

Moreover, the number $2m$ cannot be improved.

We now begin to investigate the impact of a boundary upon the accuracy of surface spline interpolation. Specifically we consider the following question.

Question: *For S^1 and S^2 consider the interpolation problem set on the semi-circle and hemisphere respectively. In both cases, how does the presence of the boundary affect the convergence of the restricted surface spline interpolants?*

We tackle this question in the same way as for the global analysis of the previous question. The results are displayed in Tables 4–6.

Conclusions: The results clearly show that the optimal orders are not achieved for interpolation on the semi-circle and hemisphere and this corroborates with the conjecture that the presence of a boundary causes a deterioration in convergence orders. What is most interesting is that the deterioration can be quantified. We express this as a conjecture.

Conjecture 6. *Let s_f denote the unique restricted surface spline interpolant to a target function $f \in W_2^{2m}(S^{d-1})$ over a set Ξ of distinct points on a hemisphere $H^{d-1} \subset S^{d-1}$, (semi-circle for $d = 2$) then*

$$\|s_f - f\|_{L_p(S^{d-1})} = O(h^{m+\frac{1}{p}}) \quad \text{for all } p \in [1, \infty],$$

where h is the appropriate density measure of Ξ in H^{d-1}.

Tab. 4. Accuracy of linear spline interpolation on the semi-circle.

Nodes	L_1	L_2	L_3	L_4	L_∞
128	3.76_{-06}	3.51_{-05}	7.78_{-05}	1.17_{-04}	3.93_{-04}
256	4.71_{-07}	6.21_{-06}	1.54_{-05}	2.45_{-05}	9.80_{-05}
512	5.88_{-08}	1.10_{-06}	3.06_{-06}	5.14_{-06}	2.45_{-05}
1024	7.36_{-09}	1.94_{-07}	6.08_{-07}	1.08_{-06}	6.12_{-06}
Ratio	L_1	L_2	L_3	L_4	L_∞
128/256	3.00	2.50	2.33	2.25	2.00
256/512	3.00	2.50	2.33	2.25	2.00
512/1024	3.00	2.50	2.33	2.25	2.00

Tab. 5. Accuracy of cubic spline interpolation on the semi-circle.

Nodes	L_1	L_2	L_3	L_4	L_∞
128	3.76_{-06}	3.51_{-05}	7.78_{-05}	1.17_{-04}	3.93_{-04}
256	4.71_{-07}	6.21_{-06}	1.54_{-05}	2.45_{-05}	9.80_{-05}
512	5.88_{-08}	1.10_{-06}	3.06_{-06}	5.14_{-06}	2.45_{-05}
1024	7.36_{-09}	1.94_{-07}	6.08_{-07}	1.08_{-06}	6.12_{-06}
Ratio	L_1	L_2	L_3	L_4	L_∞
128/256	3.00	2.50	2.33	2.25	2.00
256/512	3.00	2.50	2.33	2.25	2.00
512/1024	3.00	2.50	2.33	2.25	2.00

Tab. 6. Accuracy of thin plate spline interpolation on the hemisphere.

Nodes	L_1	L_2	L_3	L_4	L_∞
128	1.13_{-03}	3.46_{-03}	6.26_{-03}	8.90_{-03}	3.87_{-02}
256	4.53_{-04}	1.68_{-03}	3.17_{-03}	4.58_{-03}	2.09_{-02}
512	1.60_{-04}	6.89_{-04}	1.39_{-03}	2.07_{-03}	1.02_{-02}
1024	5.50_{-05}	2.86_{-04}	6.14_{-04}	9.46_{-04}	5.05_{-03}
Ratio	L_1	L_2	L_3	L_4	L_∞
128/256	2.64	2.09	1.96	1.92	1.78
256/512	3.00	2.57	2.39	2.29	2.07
512/1024	3.08	2.54	2.35	2.26	2.02

§5. Final Remarks

The idea of using generalized surface splines on the sphere has also been studied by Levesley [13], where the so-called ultraspherical expansion coefficients are computed. The ultraspherical coefficients are related to the Fourier coefficients that we have computed in this paper, however the approach taken in [13] is distinct from the one given here.

The surface splines (1) for Euclidean space induce a $d-$dimensional function $\Phi(x) = \phi(\|x\|)$, $x \in \mathbb{R}^d$. Depending upon the parity of the dimension, the function Φ has a generalised Fourier transform which is given by

$$\hat{\Phi}(\xi) = \frac{2^{2m}\pi^{\frac{d-2}{2}}\Gamma(m - \frac{d}{2} + 1)\Gamma(m)}{\|\xi\|^{2m}}, \quad d \textit{ odd},$$

and

$$\hat{\Phi}(\xi) = \frac{2^{2m-1}\pi^{\frac{d}{2}}\Gamma(m - \frac{d}{2} + 1)\Gamma(m)}{\|\xi\|^{2m}}, \quad d \textit{ even}.$$

We observe that the numerators appearing in the expressions of the spherical Fourier coefficients (see Lemma 3) have the same form as the numerators appearing above, where d is replaced by $d-1$. This reveals an interesting link between Euclidean and spherical Fourier analysis. This may extend to more general basis function restrictions or even to more general manifolds, and is an obvious topic for further research.

It is interesting to note that the known results for surface spline interpolation in Euclidean space can be directly transfered to the sphere. The optimal $L_p(S^{d-1})$ approximation order is conjectured to be $2m$, this is known to be true on $\mathbb{R}^d$. The optimal L_p-approximation order on a hemisphere (and so for any suitable bounded open subdomain of S^{d-1}) is conjectured to be $m + \frac{1}{p}$. This is also the current conjecture on $\mathbb{R}^d$. The authors believe that the task of settling the $2m$ conjecture; that is, replacing the factor $\frac{d-1}{p}$ in (40) with $\frac{d-1}{2}$, is a challenging puzzle and one which deserves further investigation.

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