Afriat's Theorem and Negative Cycles

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Abstract

This note provides a short proof of Afriat's theorem and shows that the problem is equivalent to the problem of identifying a negative length cycle in a related graph.

1 Introduction

Afriat's (1967) theorem is an answer to the question of when a sequence of purchase decisions is consistent with the purchaser maximizing a concave utility function $u(\cdot)$. Suppose a sequence of purchase decisions (p_i, x_i), $i = 1, \ldots, n$, where $p_i \in \Re^n_+$ and $x_i \in \Re^n_+$ are price vectors and purchased quantity vectors respectively. Asume the purchaser makes purchase decision based on utility maximization. If $p_i \cdot (x_j - x_i) \leq 0$, the utility function u must satisfy $u(x_j) \leq u(x_i)$, otherwise, with purchase prices of p_i, bundle x_j costs less but provides higher utility to the purchaser. If we have a sequence of decisions $(p_i, x_i), (p_j, x_j), (p_k, x_k), \ldots, (p_r, x_r)$, with

$$p_i \cdot (x_j - x_i) \leq 0, \quad p_j \cdot (x_k - x_j) \leq 0, \ldots, \quad p_r \cdot (x_i - x_r) \leq 0,$$

then $u(x_i) = u(x_j) = \ldots = u(x_r)$, and

$$p_i \cdot (x_j - x_i) = 0, \quad p_j \cdot (x_k - x_j) = 0, \ldots, \quad p_r \cdot (x_i - x_r) = 0.$$

The above $n$ecessary condition can be described in graph theoretic terms as follows: let A be a $n \times n$ matrix of real numbers with all zero's on the diagonals. We associate with the matrix A a directed graph $D(A)$ as follows: introduce a vertex for each index and for each ordered pair (i, j) an edge with length $a_{ij} = p_i \cdot (x_j - x_i)$. The matrix A will be said to satisfy the Afriat condition (AC) if every negative length cycle in $D(A)$ contains at least one edge of positive weight.

Associated with A is an inequality system:

$$y_j \leq y_i + s_i a_{ij} \ \forall i \neq j, \ 1 \leq i, j \leq n$$

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$$s_i > 0 \ \forall 1 \leq i \leq n$$

which we label $L(A)$. We now state Afriats theorem:

Theorem 1 *$L(A)$ is feasible iff. $D(A)$ satisfies AC.*

Whenever $D(A)$ satisfies AC, we use the solution to $L(A)$ to construct a utility function $u(\cdot)$ consistent with the sequence of purchase decisions (p_i, x_i) by setting:

$$u(x) = \min\{y_1 + s_1 p_1(x - x_1), y_2 + s_2 p_2(x - x_2), \ldots, s_n p_n(x - x_n)\}.$$

A number of proofs of the theorem exist.[1] This note provides a new proof that makes explicit the network structure inherent in $L(A)$. This structure appears to have been overlooked (but used implicitly) in previous theorems. In fact, the most recent paper on the subject by Fostel, Scarf and Todd (2003) describes the system $L(A)$ as being unusual.

To each $s \in \Re^n_+$ and matrix A with zeros on the diagonals, we associate a directed graph $D(A, s)$ as follows: introduce a vertex for each index and for each ordered pair (i, j) an edge with length $s_i a_{ij}$. Notice that $D(A) = D(A, e)$ where e is the n-vector of all 1's.

Now fix $s \in \Re^n_+$. Then feasibility of $L(A)$ reduces to identifying $y \in \Re^n$ such that $y_j - y_i \leq s_i a_{ij}$ for all $i \neq j$. Readers familiar with network flows will recognize this system as the constraint set to the dual of a shortest path problem. A standard result is that this system is feasible iff. $D(A, s)$ contains no negative cycles. Assuming feasibility, we can choose the y's as follows: set $y_1 = 0$ and y_j to be the length of the shortest path from 1 to i in $D(A, s)$. Afriat's theorem can be rephrased as;

Theorem 2 *There is an $s \in \Re^n_+$ such that $D(A, s)$ contains no negative cycles iff. $D(A, e)$ satisfies AC.*

Proof: If there is an $s \in \Re^n_+$ such that $D(A, s)$ contains no negative cycles, the system of inequalities $L(A)$ (with s fixed) is feasible. So we can construct a utility function $u(\cdot)$ consistent with the sequence of purchase decisions. Therefore, $D(A, e)$ must satisfy AC. We next prove the non-trivial direction. Suppose $D(A, e)$ satisfies AC. We prove there exists $s \in \Re^n_+$ such that $D(A, s)$ has no negative cycles.

Let $S = \{(i, j) : a_{i,j} < 0\}$, $E = \{(i, j) : a_{i,j} = 0\}$, and $T = \{(i, j) : a_{i,j} > 0\}$. Consider the weighted digraph G with edges in $S \cup E$, where arcs $(i, j) \in S$ are given weight $w_{ij} = -1$, and arcs $(i, j) \in E$ are given weight $w_{ij} = 0$. Since $D(A, e)$ satisfies AC, G does not contain a negative length cycle. Hence there exists a set of potentials $\{\phi_j\}$ on the nodes such that

$$\phi_j \leq \phi_i + w_{ij}, \quad \forall \quad (i, j) \in E(G).$$

Without loss of generality, we relabel the vertices so that $\phi_n \leq \phi_{n-1} \leq \ldots \leq \phi_1$. Choose $\{s_i\}$ non-decreasing so that

$$s_i \times \min_{(i,j)\in T} a_{ij} \geq (n-1) \times s_{i-1} \max_{(i,j)\in S} (-a_{ij}) \text{ if } \phi_i < \phi_{i-1},$$

[1] Afriat's orginal proof assumed that $a_{ij} \neq 0 \ \forall i \neq j$. This was relaxed by Diewert (1973) and Varian (1982).

and

$$s_i = s_{i-1} \text{ if } \phi_i = \phi_{i-1}$$

for all $i > 2$, with $s_1 = 1$.

For any cycle C in the digraph $D(A, s)$, let (v, u) be an edge in C such that (i) v has the smallest potential among all vertices in C, and (ii) $\phi_u > \phi_v$. Such an edge exists, otherwise ϕ_i is identical for all vertices i in C. In this case, all edges in C have non-negative edge weight in $D(A, s)$.

By selection, $\phi_u > \phi_v$. If $(v, u) \in S \cup E$, then we have $\phi_u \leq \phi_v + w_{vu} \leq \phi_v$, a contradiction. Hence $(v, w) \in T$. Now, note that all vertices q in C with the same potential as v must be incident to an edge (q, t) in C such that $\phi_t \geq \phi_q$. Hence the edge (q, t) must have non-negative length. i.e., $a_{q,t} \geq 0$. Let p denote a vertex in C with the second smallest potential. Now, C has length

$$s_v a_{vu} + \sum_{(k,l) \in C, (k,l) \neq (v,u)} s_k a_{k,l} \geq s_v a_{v,u} + s_p (n-1) \max_{(i,j) \in S} \{a_{ij}\} \geq 0,$$

i.e., C has non-negative length.

Since $D(A, s)$ is a digraph without any negative length cycles, $L(A)$ is feasible. ■

Given the matrix A, we can verify if condition AC is violated by checking if the graph G has a negative length cycle. This can be done in $O(n^3)$ time (see Ahuja, Magnanti and Orlin (1993)) using standard algorithms.[2] Assuming AC holds, the node potentials can be computed by a standard shortest path algorithm in $O(n^2)$ (again see Ahuja, Magnanti and Orlin (1993)). This gives the s's in $O(n^2)$ steps and another $O(n^2)$ steps to determine the value of the y variables.

References:

1. Afriat, S. N. 'The Construction of a Utility Function from Expenditure Data', **International Economic Review**, 8, 1967, 67-77.

2. Ahuja, R., T. Magnanti and J. Orlin, 'Network Flows: Theory, Algorithms, and Applications', Prentice-Hall, NJ, 1993.

3. Diewert, E. 'Afriat and Revealed Preference Theory', **Review of Economic Studies**, 40, 1973, 419-426.

4. Fostel, A., H. Scarf and M. J. Todd. 'Two New Proofs of Afriat's Theorem', Cowles foundation discussion paper, 1415, 2003.

5. Varian, H. 'The Non-parametric Approach to Demand Analysis', **Econometrica**, 50, 1982, 945-974.

[2] An improvement to $O(n^{2.5})$ using a more spohisticated algorithm due to Gabow is possible.