Rationalizable outcomes of large independent private-value first-price discrete auctions.[1]

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Abstract

We consider discrete versions of independent, private-value, first-price auctions. We show for any fixed finite set of possible bids, if the number of participants is large enough, then the set of rationalizable bids involves all players bidding the highest bid that is lower than their private value.

1 Introduction

Most results on auctions rely on Nash equilibria as the solution concept. These equilibria in turn rely on the assumption that the values among the bidders are determined according to some commonly known probability distribution. Moreover, the equilibria are typically sensitive to this distribution. Because of this sensitivity, and more generally the strong assumptions underlying Nash equilibrium, theorists often doubt the power of auction theory in providing fine predictions of behavior in actual auctions. Recently, the popularity of the auction mechanism in both private and public sales has increased. As positive as this development might be for the actual conduct of business affairs, it does not provide support for the conclusions that equilibrium theory yields in the analysis of auctions. An important task for this literature is therefore to identify robust results that can be obtained under less demanding assumptions than those employed in standard equilibrium models.

The work reported here is a small step within this agenda. We consider first-price auctions with private and independent values and with many players. It is well known that in the unique equilibrium of the symmetric model the bids converge to the true values as the number of bidders is made large and hence the price converges to the highest value. Our analysis here presents a sense in which this result is robust to relaxing the solution concept. We assume that the set of valuations and the set of allowable bids are finite and show that the result that bidders bid (almost) their true value holds for all interim rationalizable outcomes when it is only common knowledge that all values have likelihood bounded above zero. Thus, with many bidders (in this discrete environment), the object goes to the bidder with the highest value (efficiency), and almost surely the price is (almost) the highest value, even without imposing the equilibrium assumptions.

The most closely related work is Battigalli and Siniscalchi (1999). They also study interim rationalizable outcomes in a first price auction with private independent values. Unlike our model, they adopt the standard (for auction theory) set-up of continuum sets of bids and values. They show that any positive bid up to some level above the Nash equilibrium bid is interim rationalizable. Therefore, in particular, the set of interim rationalizable strategies in their model does not approach the competitive equilibrium when the number of bidders becomes large. Thus, their result stands in sharp contrast to ours. We will explain the reason for the difference between these results in the discussion section at the end.

A more distantly related literature explores the eductive justification of the competitive Equilibrium. Guesnerie (1992) looks at the set of rationalizable equilibria in a game in which a continuum of suppliers decide simultaneously on the quantities of a homogenous product that they supply and then the price is determined by an exogenously given demand function. He shows that when the supply curve is steeper than the demand curve (in the traditional labeling of price on the vertical axis), then the rationalizable set contains only the competitive equilibrium. One may think of course of the mirror image of that model in which the supply curve is fixed and the buyers decide strategically on their quantities. The corresponding condition in that variation is that the demand curve is steeper than the supply curve. The auction model is not a special case of that variation, since it designates prices rather than quantities as the strategic variables. But, in any case, the condition on the relative slopes does not hold in the auction model, since the supply curve is inelastic at one unit. Thus, the competitive prediction of Guesnerie's model does not apply in the auction model.

2 The Model

As mentioned, we consider a first-price auction with independent and private values. Each player $i \in \{1, 2, ..., n\}$ is informed of her private value, v_i, of the object, and then submits a bid. The object is awarded to the highest bidder who then pays his or her bid; in the case of ties, the object is awarded with equal probability to one of the tied highest bidders (and only the winner pays the winning bid). We assume that values and bids are on a discrete grid, say $V = \{0, 1/m, 2/m, ..., 1 - 1/m, 1\}$, and that values are believed to be drawn independently according to some distribution that is *not* necessarily commonly known. We do assume, however, that it is commonly known that the distribution assigns each value a positive probability. An *ex ante* strategy for a player in this environment is then a function from a player's possible values, V, into the possible bids, V, and a strategy profile is an n–tuple of such functions. For our purposes it is more useful to think of *interim* strategies that specify the bid of a player with a particular value, so it is an element of V, and an interim strategy profile is then a $(m + 1) \times n$–tuple specifying what bid each type of each player chooses. As is well known, *interim* rationalizability is a weaker solution concept (i.e., allows for larger sets) than *ex ante* rationalizability (since the latter imposes the same beliefs on all types of a given player, while the former does not). We therefore consider interim rationalizability as then our result that the set is a singleton is stronger.

We say that an $(m+1) \times n$–tuple of sets of interim strategies is *interim correlated rationalizable with weight* δ, or *rationalizable*(δ) for short, if the bid b specified for type v_i of player i is a best reply to some *admissable* belief by that type over bids by other players. The belief is *admissable* if it can be derived as follows. Each type v_i can have any belief over the strategies chosen by each possible $n-1$–tuple of other players' types, restricted of course to their rationalizable(δ) sets. Each type v_i also has a belief over the likelihood of each such profile of $n-1$ types, which is obtained from a belief that players' types are drawn independently according to some probability distribution over V that assigns weight at least δ to each type[1]. The distribution over bids is then the sum, over all possible profiles of types, of the beliefs on the bids chosen by each profile of types, weighted by the probability of that profile of types. Since we only use bounds on these probabilities, we do not

[1] Obviously $\frac{1}{m+1} \geq \delta \geq 0$; we consider only the case where $\delta > 0$. Note that the set of rationalizable(δ) outcomes is *decreasing* in δ; for $\delta = 0$ there are no restrictions on players beliefs over opponents' types, whereas for $\delta = \frac{1}{m+1}$ it is the same as interim correlated rationalizability in the game of incomplete information in which the prior assigns probability δ to all types.

develop notation for stating the above formally, and present within the proof below only the notation needed for our bounds.

The symbol δ in the term *rationalizability*(δ) is meant to emphasize that this notion is not standard; it allows players substantial freedom in forming their beliefs. First, not only may a player have correlated beliefs about the opponents, but she may even believe that the play of every opponent depends on the realization of the types of all opponents. It is not clear whether the latter feature has an interesting interpretation, but our result will just be stronger with it.[2] Second, we do not have a common prior (or even a commonly known but different prior for each player) over the type space. All we require is (that it be common knowledge) that there is a lower bound of δ on the probability of each type. Extensions of rationalizability that do not impose common priors are given in Battigalli (1998).[3]

[2] However, we do not claim that allowing this generality is interesting in itself. At first glance one might think that it allows for communication among the players, which could create correlation in their actions. However such communication might reveal types, whereupon players' beliefs need not correspond to assigning probability at least δ to every type. We believe the result will hold even when allowing for cominucation, but have not proven this.

[3] We believe that our notion is equivalent to the correlated extension of his weak Δ-rationalizability where Δ denotes the restriction to beliefs that assign probability at least δ to all types (see Battigalli (1998, Section 3.1 and 4).

3 The Result

Theorem 1 *For any m and any δ there exists $N(m, \delta)$ such that for any $n > N(m, \delta)$ the set of rationalizable(δ) strategies for any type v is $\{v - 1/m\}$.*

The intuition for this result is as follows. Consider the type $v = 1$ and assume that some bids below $1 - 1/m$ are rationalizable(δ) for this type. Let b be the lowest such bid. To justify this bid, the player with type $v = 1$ making it must believe it to be best. It is clearly not best if other players of type $v = 1$ are around and are bidding more than b. It is also not best if there are many other players of type $v = 1$ who are bidding b. It may be best otherwise. We show that, for n large enough, the loss in expected payoff from bidding $1 - 1/m$ instead of b in the otherwise event is smaller than the gain in expected payoff from bidding $1 - 1/m$ instead of b in the preceding two events.

Proof: The proof is via a sequence of steps which we now develop. Each step describes strategies that are dominated in the game that remains after the dominated strategies described in preceding steps are deleted. We do not repeat the caveat that the domination is in this reduced game. Also, since the game is symmetric, we consider bids of types of a generic player with value $v \in V$, dropping the subscript i. Finally, to simplify notation, let $\Delta \equiv 1/m$.

Bidding 1 is dominated by bidding 0 for all types $v < 1$ since a bid of 1 may win, and then such a type will end up with a negative payoff.[4] Next, bidding 1 is dominated by bidding $1 - \Delta$ for $v = 1$, because bidding 1 yields a payoff of 0 and bidding $1 - \Delta$ can yield a positive payoff. (It is possible that all other players have types less than 1 in which case they bid less than 1.) Now bidding $1-\Delta$ is dominated by bidding zero for all types $v < 1-\Delta$, and therefore bidding $1 - \Delta$ is dominated by bidding $1 - 2\Delta$ for $v = 1 - \Delta$. Iterating we conclude that it is dominated for any type v to bid more than $v - \Delta$, except type zero who bids zero.

Let b_n be the lowest rationalizable strategy for type $v = 1$. We now argue that for n large enough $b_n = 1 - \Delta$. Assuming not, we show that for any belief it is better to bid $1 - \Delta$ than to bid $b_n < 1 - \Delta$ for n large. Let $q(j|\ell)$ denote the probability that j players with value $v = 1$ bid b_n conditional on there being ℓ players of type $v = 1$. For now, assume that $\Pr(v = 1) = \delta$; below we explain why our argument extends the result to any F with $\Pr(v = 1) \geq \delta$.

[4] Bidding more than v is not necessarily dominated since one can believe that all types are bidding even more, so that one gets a payoff of zero in any case.

The profit to type $v = 1$ from bidding $1 - \Delta$ is at least

$$L \equiv \Delta \times \left((1-\delta)^{n-1} + \sum_{\ell=1}^{n-1} \binom{n-1}{\ell} (1-\delta)^{n-1-\ell} \delta^{\ell} \left(\sum_{j=0}^{\ell} q\left(j|\ell\right) \frac{1}{\ell - j + 1} \right) \right). \tag{1}$$

This is the benefit from winning with bid $1 - \Delta$ times a lower bound of the probability of winning with this bid. The probability of winning is at least the probability $(1-\delta)^{n-1}$of everyone else having value $v < 1$ plus a lower bound on the probability of winning in the event that there are some players with type $v = 1$. The latter bound is a sum of probabilities of the form $\binom{n-1}{\ell} (1-\delta)^{n-1-\ell} \delta^{\ell} q\left(j|\ell\right) \frac{1}{\ell-j+1}$. This is the probability of there being ℓ players with type $v = 1$ times the probability $q\left(j|\ell\right)$ that j of those players bid b_n times the probability of winning if the remaining $\ell - j$ are also bidding $1 - \Delta$. This is a lower bound since some of those $\ell - j$ players who bid above b_n may still bid below $1 - \Delta$.

The profit from bidding b_n is at most

$$U \equiv (1 - b_n) \times \left((1-\delta)^{n-1} + \sum_{\ell=1}^{n-1} \binom{n-1}{\ell} (1-\delta)^{n-1-\ell} \delta^{\ell} q\left(\ell|\ell\right) \frac{1}{\ell+1} \right). \tag{2}$$

Again this is the benefit of winning times an upper bound of the probability of winning. The probability of winning is at most the probability that everyone else has value $v < 1$ plus the probability of there being ℓ players with type $v = 1$ times the probability that all those players bid b_n, divided by $\ell + 1$ and summed over all possible values of ℓ. This is an upper bound because even when everyone has value $v < 1$, they may bid more than b_n.

We want to argue that $L > U$ for large n.

Let k be the largest integer j that solves $1-(1-\Delta) \square (1-b_n)/j$. When everyone bids no more than b_n and no more than k players bid exactly b_n, then it is better to bid b_n and share it than to bid $1-\Delta$ and win for sure. We now partition the summations in (1) and (2) into ℓ's that are no more than k, and those that are greater than k, and weaken the bounds further. First, since $q(\ell|\ell)\frac{1}{\ell+1} < \sum_{j=0}^{\ell} q(j|\ell)\frac{1}{\ell-j+1}$, we have

$$\begin{aligned} & \Delta\left((1-\delta)^{n-1} + \sum_{\ell=0}^{k}\binom{n-1}{\ell}(1-\delta)^{n-1-\ell}\,\delta^{\ell}\left(\sum_{j=0}^{\ell} q(j|\ell)\,\frac{1}{\ell-j+1}\right)\right) > \\ L_1 \;\equiv\; & \Delta\left((1-\delta)^{n-1} + \sum_{\ell=0}^{k}\binom{n-1}{\ell}(1-\delta)^{n-1-\ell}\,\delta^{\ell} q(\ell|\ell)\,\frac{1}{\ell+1}\right). \end{aligned} \tag{3}$$

Second, since $q\left(\ell|\ell\right)+\left(1-q\left(\ell|\ell\right)\right)\frac{1}{\ell+1}<\sum_{j=0}^{\ell}q\left(j|\ell\right)\frac{1}{\ell-j+1}$,

$$\begin{aligned}
&\Delta\left(\sum_{\ell=k+1}^{n-1}\binom{n-1}{\ell}\left(1-\delta\right)^{n-1-\ell}\delta^{\ell}\left(\sum_{j=0}^{\ell}q\left(j|\ell\right)\frac{1}{\ell-j+1}\right)\right)> \\
L_2 \equiv\ &\Delta\left(\sum_{\ell=k+1}^{n-1}\binom{n-1}{\ell}\left(1-\delta\right)^{n-1-\ell}\delta^{\ell}\left(q\left(\ell|\ell\right)+\left(1-q\left(\ell|\ell\right)\right)\frac{1}{\ell+1}\right)\right)
\end{aligned} \tag{4}$$

Define

$$U_1\equiv\left(1-b_n\right)\times\left(\left(1-\delta\right)^{n-1}+\sum_{\ell=0}^{k}\binom{n-1}{\ell}\left(1-\delta\right)^{n-1-\ell}\delta^{\ell}q\left(\ell|\ell\right)\frac{1}{\ell+1}\right) \tag{5}$$

and

$$U_2\equiv\left(1-b_n\right)\times\left(\sum_{\ell=k+1}^{n-1}\binom{n-1}{\ell}\left(1-\delta\right)^{n-1-\ell}\delta^{\ell}q\left(\ell|\ell\right)\frac{1}{\ell+1}\right). \tag{6}$$

Clearly $L-U>\left(L_1-U_1\right)+\left(L_2-U_2\right)$. Moreover, if $b_n<1-\Delta$, then $L_1-U_1<0$. On the other hand, we now show that $L_2-U_2>0$. Moreover, L_2-U_2 is minimized when $q\left(\ell|\ell\right)=0$, and even in this case it outweighs the negative term L_1-U_1 for n large.

$$
\begin{aligned}
L_2 - U_2 \;\; &= \;\; \Delta\left(\sum_{\ell=k+1}^{n-1}\binom{n-1}{\ell}(1-\delta)^{n-1-\ell}\,\delta^{\ell}\left(q\left(\ell|\ell\right)+\left(1-q\left(\ell|\ell\right)\right)\frac{1}{\ell+1}\right)\right) - \\
&\quad (1-b_n)\times\left(\sum_{\ell=k+1}^{n-1}\binom{n-1}{\ell}(1-\delta)^{n-1-\ell}\,\delta^{\ell}q\left(\ell|\ell\right)\frac{1}{\ell+1}\right) = \\
&\quad \sum_{\ell=k+1}^{n-1}\binom{n-1}{\ell}(1-\delta)^{n-1-\ell}\,\delta^{\ell}\frac{1}{\ell+1}\left(\Delta+\ell q\left(\ell|\ell\right)\left(\Delta-\frac{1-b_n}{\ell}\right)\right).
\end{aligned}
$$

Since $\ell > k$, it follows from the choice of k that $\Delta - \frac{1-b_n}{\ell} > 0$. Therefore, $L_2 - U_2 > 0$ and $L_2 - U_2$ is minimized when $q\left(\ell|\ell\right) = 0$. So we have

$$
\begin{aligned}
0 \;\; &< \;\; \Delta\left(\sum_{\ell=k+1}^{n-1}\binom{n-1}{\ell}(1-\delta)^{n-1-\ell}\,\delta^{\ell}\frac{1}{\ell+1}\right) < L_2 - U_2 \\
0 \;\; &> \;\; L_1 - U_1 > \left[-1+b_n+\Delta\right]\left((1-\delta)^{n-1}+\sum_{\ell=0}^{k}\binom{n-1}{\ell}(1-\delta)^{n-1-\ell}\,\delta^{\ell}\right)
\end{aligned}
$$

We want to show that if $b_n < 1-\Delta$ then the deviation to $1-\Delta$ is profitable, i.e., that the $L_2 - U_2$ term dominates. To do this we show that the ratio of the terms in large parentheses converges to ∞ as n grows. Let

$$
P_1 = (1-\delta)^{n-1} + \sum_{\ell=0}^{k}\binom{n-1}{\ell}(1-\delta)^{n-1-\ell}\,\delta^{\ell} < (k+2)\,n^{k+1}\,(1-\delta)^{n-k}
$$

and

$$\begin{aligned}
P_2 &= \sum_{\ell=k+1}^{n-1} \binom{n-1}{\ell} (1-\delta)^{n-1-\ell} \delta^{\ell} \frac{1}{\ell+1} = \frac{1}{\delta n} \sum_{\ell=k+2}^{n} \binom{n}{\ell} (1-\delta)^{n-\ell} \delta^{\ell} \\
&> \frac{1}{\delta n} \left(1 - (k+2)\, n^{k+1} (1-\delta)^{n-k}\right).
\end{aligned}$$

Therefore

$$\frac{P_2}{P_1} > \frac{\frac{1}{\delta n}\left(1-(k+2)\, n^{k+1}(1-\delta)^{n-k}\right)}{(k+2)\, n^{k+1}(1-\delta)^{n-k}.} = \frac{1}{\delta}\left(\frac{1}{(k+2)\, n^{k+2}(1-\delta)^{n-k}} - \frac{1}{n}\right)$$

and $$\lim_{n\to\infty}\left(\frac{1}{(k+2)\, n^{k+2}(1-\delta)^{n-k}} - \frac{1}{n}\right) = \infty$$

To verify this limit, observe that $n^{k+2}(1-\delta)^{n-k}$ can be rewritten as $n^{k+2}/[1/(1-\delta)]^{n-k}$. By treating n as a continuous variable and applying L'Hopital rule repeatedly $k+2$ times, we get $\lim_{n\to\infty} n^{k+2}(1-\delta)^{n-k} = 0$ and hence the desired limit. Hence, for n large $L_2 - U_2 > |L_1 - U_1|$.

Therefore, assuming $\Pr(v=1) = \delta$, if $n > N(m,\delta)$ then bidding $1-\Delta$ dominates bidding any b □ $1-2\Delta$. Moreover, it can be shown (by taking derivatives and simple manipulations[5]) that, for sufficiently large n, $\frac{P_2}{P_1}$ is

[5] $sign\{D_{\varepsilon}\frac{1}{\varepsilon}\left(\frac{1}{(k+2)\, n^{k+2}(1-\varepsilon)^{n-k}} - \frac{1}{n}\right) = sign\left\{n[(n-k)\varepsilon - 1 + \varepsilon] + (k+2)\, n^{k+2}(1-\varepsilon)^{n-k+1}\right\}$ which is positive for a large n.

increasing in δ. So, by choosing $N(m, \delta)$ appropriately, for any belief resulting from an F in which $\Pr(v = 1) > \delta$, if $n > N(m, \delta)$ then $1 - \Delta$ is better than $b \leq 1 - 2\Delta$. Hence, iterated deletion of dominated strategies as above results in types $v = 1$ bidding $1 - \Delta$.

Consider next type $v = 1 - \Delta$. As this type bids less than $1 - \Delta$, this type only wins if no players are of type $v = 1$, so their bidding behavior can be analyzed conditional on their being no players of type $v = 1$. But then the analysis above implies that for n large enough this type will bid $v - 2\Delta$. Continuing in this way shows that iterated deletion yields the sets described in the theorem.

In this static game of incomplete information the equivalence between the iterated deletion process used above and our notion of rationalizable(δ) outcomes is standard; for a related result see Battigalli (1998, Theorem 3.11(a)).

■

4 Discussion

We now discuss some of the assumptions in this paper and the relation with the previous literature. Perhaps the key assumption for our result is the finiteness of the set of possible bids. To understand the role of finiteness, consider the case where bids must be in $B = \{1/i : i = 1, 2, ...\}$, and let the values be distributed uniformly on the unit interval. In this case it is easy to see that for *any* m large enough, it is interim rationalizable for all types with $v > 1/(m-1)$ to bid $1/m$. (Such a type can believe that everyone with $v > 1/m$ bids $1/(m+1)$, and so on.)

Battigalli and Siniscalchi (1999) analyze the case where the bids and values are not on a grid (thus are any number in $[0, 1]$) and allow for any n (not necessarily large). Using the idea captured by the above example, they show that any small positive bid is rationalizable. They also go beyond this intuition and show that the rationalizable set includes any bid between 0 and some bid that is strictly greater than the Nash equilibrium bid, and they provide methods for calculating the upper bound precisely.

Thus, the finiteness of the possible bids is crucial. However, the finiteness of the type space does not seem crucial. It seems obvious, though we have not verified all the details, that our analysis carries through also when only the bids are restricted to a finite grid, and it is commonly known that the values are distributed according to some distribution function with density at least δ on [0,1]. The result would then be that for any m, $\eta \in (0, 1/m)$, and $\delta > 0$ there exists $N(m, \eta, \delta)$such that for any $n > N(m, \eta, \delta)$ the set of rationalizable(δ) strategies for any type $v \in [k/m + \eta, (k+1)/m]$ is (k/m).

The symmetry assumption that all bidders' types are drawn from the same distribution is also not crucial for the argument. If we assumed instead that each player's type is drawn from a different distribution, then as long as we assume that the probability of each type is bounded away from zero uniformly for all players, the analysis will be similar. In such a case, the probabilities of different configurations of the bidders' types will not be expressions like$\binom{n-1}{\ell}(1-\delta)^{n-1-\ell}\delta^{\ell}$ but rather sums of products involving different δ's for the different players. But then the appropriate bounds can be used to continue the argument as above.

Recall that we impose few restrictions on players' beliefs, even allowing a player to believe that the play of every opponent depends on the realization of the types of all opponents. This means that a player may believe, for example, that his opponents are sharing information. We also do not assume commonly known priors on the values. This might be more freedom than is commonly assumed, but as it makes the result stronger, there is no reason to limit it.

5 References

References

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