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A NOTE ON POLYMATRIX GAMES

by

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Abstract: This work is concerned with the class of n-person games called polymatrix games (Janovskaya (1968)). The structure of the set of Nash equilibrium points in a polymatrix game is studied and characterizations of these games are given.

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1-Introduction

The polymatrix games (Janovskaya (1968)) form a subclass of the n-person noncooperative games. The equilibria of polymatrix games was studied by Janovskaya (1968) as a linear complementary problem. Howson (1972) proved the existence of Nash equilibria based on a generalization of the almost complementary paths used for the linear complementary problems. He also proved that every non degenerate polymatrix game has an odd number of equilibrium points.

In this note we will study the structure of the set of equilibrium points of a polymatrix game and we will present characterizations of these games.

The organization of the paper is as follows. After some definitions and notations (section 2), we will describe the set of Nash equilibrium points. Every Nash equilibrium point will be obtained from a finite set of extreme equilibrium points (section 3). We will give conditions which will allow us to recognize if a given n-person games is a polymatrix game (section 4).

2.- Some Notations and Definitions

A polymatrix game (presented by Janovskaya (1968)) is a n-person non cooperative normal form game : $\Gamma = \{ \Sigma_i , A_i , i \in N \}$. Here N is the finite set $\{1,..,n\}$, with $n \geq 2$. Every player $i \in N$ has a finite set of pure strategies:

$$\Sigma_i = \{ \sigma^i_1, \ldots, \sigma^i_{m_i} \}$$

If player i chooses strategy $\sigma^i \in \Sigma_i$ and player j chooses pure strategy $\sigma^j \in \Sigma_j$ it is possible to assign a partial payoff $a^{ij}(\sigma^i, \sigma^j)$ such that for any choice of pure strategies $(\sigma^1, \ldots, \sigma^n)$ by the n players, the payoff to player i is given by:

(2.1) $A_i(\sigma^1,\ldots,\sigma^n) = \sum_{j\neq i} a^{ij}(\sigma^i,\sigma^j)$

Let $A^{ij} = (a^{ij}_s)$ be the $m_i \times m_j$ matrix of partial payoffs to player i resulting from the choices of pure strategies by player i and player j.

The scalars and all entries in matrices and vectors will be taken from an ordered field F.

A mixed strategy for player i is a probability distribution over the pure strategies. That is a column vector $x_i = (x_i(\sigma^i_1),\ldots,x_i(\sigma^i_{m_i})) = (x^i_1,\ldots,x^i_{m_i})$ where x^i_s is the probability of i playing his strategy $\sigma^i_s \in \Sigma_i$

Let $\tilde{\Sigma}_i$ be the set of mixed strategies for player i:

$$\tilde{\Sigma}_i = \{ x_i : e^T x_i = 1 \text{ and } x_i \geq 0 \}$$

Here e and 0 are column vectors of dimension m_i consisting of 1's and 0's respectively. The inequality $x_i \geq 0$ means inequality between the respective elements of the vectors. We denote with $(.)^T$ the transposed of the corresponding vector or matrix.

Let $x = (x_1,\ldots,x_n) \in \prod_{i=1}^{n} \tilde{\Sigma}_i$ be a n-tuple of mixed strategies of the n players. The expected payoff to player i is:

(2.2) $E_i(x) = (x_i)^T \sum_{j\neq i} A^{ij} x_j = \sum_{j\neq i} \sum_{r=1}^{m_i} \sum_{s=1}^{m_j} a^{ij}_{rs} x^j_s x^i_r$

For a n-person non cooperative normal form game $\Gamma = \{ \Sigma_i , A_i , i \in N \}$, we have the following definition of Nash equilibrium points:

An n-tuple $x = (\overset{*}{x}_1,\ldots,\overset{*}{x}_n) \in \prod_{i=1}^{n} \tilde{\Sigma}_i$ is a Nash Equilibrium point if and only if :

(2.3) $E_i(\overset{*}{x}) \geq E_i(\overset{*}{x}_{N-\{i\}},x_i)$ for each $x_i \in \tilde{\Sigma}_i$ and for each $i \in N$.

Here we denoted: $(\overset{*}{x}_{N-\{i\}},x_i) = (\overset{*}{x}_1,\ldots,\overset{*}{x}_{i-1},x_i,\overset{*}{x}_{i+1},\ldots,\overset{*}{x}_n)$

We say that $\tilde{x}_i$ is a best reply of player i against x if and only if:

$$E_i(x_{N-\{i\}},\tilde{x}_i) = \max_{\bar{x}_i \in \tilde{\Sigma}_i} E_i(x_{N-\{i\}},\bar{x}_i)$$

We denote with $J_i(x_{N-\{i\}})$ the set of all pure best replies against x:

$$(2.4) \qquad J_i(x_{N-\{i\}}) = \{ \sigma_r^i \in \Sigma_i : E_i(x_{N-\{i\}},\sigma_r^i) = \max_{\sigma_t^i \in \Sigma_i} E_i(x_{N-\{i\}},\sigma_t^i) \}$$

We used the notation:

$$(x_{N-\{i\}},\sigma_p^i) = (x_1,\ldots,x_{i-1},e_p,x_{i+1},\ldots,x_n)$$

where e_p is a column vector of dimension m_i with 1 in the place p and zero in other places.

Let $S(x_j)$ be the support of the mixed strategy x_j, that is:

$$S(x_j) = \{ \sigma_p^j \in \Sigma_j : x_p^j > 0 \}$$

It is well known that x_i is a best reply against x if and only if:

$$(2.5) \qquad S(x_i) \subseteq J_i(x_{N-\{i\}})$$

Therefore, an n-tuple x is a Nash Equilibrium if and only if

$$(2.6) \qquad S(x_i) \subseteq J_i(x_{N-\{i\}}) \quad \text{for each } i \in N.$$

Now for the case of Polymatrix games, the definition of Nash equilibrium point (2.1) becomes :

An n-tuple $\tilde{x}$ is a Nash Equilibrium point of the polymatrix game Γ if and only if:

$$(x_i)^T \sum_{j\neq i} A^{ij} \overset{*}{x}_j \leq (\overset{*}{x}_i)^T \sum_{j\neq i} A^{ij} \overset{*}{x}_j$$

for each $x_i \in \tilde{\Sigma}_i$ and each $i \in N$.

Let $E(\Gamma)$ be the set of Nash equilibrium points of a Polymatrix game Γ.

For Each player $i \in N$ and each non-empty set $S_i \subseteq \Sigma_i$ we define:

$$(2.7) \qquad \tilde{H}_i(S_i) = \{ x \in \prod_{i=1}^{n} \tilde{\Sigma} : S_i \subseteq J_i(x_{N-\{i\}}) \}$$

We remark that in view of (2.2) :

$$(2.8) \qquad E_i(x_{N-\{i\}},\sigma_p^i) = \sum_{j\neq i} \sum_{s=1}^{m_j} a_{ps}^{ij} x_s^j = \sum_{j\neq i} F_i^j (\sigma_p^i,x_j)$$

where F_i^j is a linear function in the variable x_j , and then the sets $\tilde{H}$ are convex polytopes.

3.- The structure of the set of Equilibrium Points

We are going to describe the set of Equilibrium points $E(\Gamma)$ of a polymatrix game. We will show that it is a finite union of convex polytopes. Each equilibria will be a suitable convex combination of a finite set of equilibrium points. *[1]

We define :

$$(3.1) \qquad C(S_1,\dots,S_n) = \{\, x \in \prod_{i=1}^{n} \tilde{\Sigma}_i \;:\; S(x_i) \subseteq S_i \text{ for each } i \in N \,\}$$

It is a convex polytope.

For each $x \in \prod_{i=1}^{n} \tilde{\Sigma}_i$ we have that:

$$x \in \bigcap_{i\in N} \tilde{H}_i(J_i(x_{N-\{i\}})) \cap C(S(x_1),\dots,S(x_n))$$

Moreover, given $x \in E(\Gamma)$, by (2.6),(2.7) and (3.1) we have:

$$(3.2) \qquad x \in \bigcap_{i\in N} \tilde{H}_i(S_i) \cap C(S_1,\dots,S_n) \qquad \text{with } S_i = J_i(x_{N-\{i\}})$$

For each $(S_1,\dots,S_n) \in \prod_{i=1}^{n} \Sigma_i$ we define:

$$(3.3) \qquad H_{C(S_1,\dots,S_n)} = \bigcap_{i\in N} \tilde{H}_i(S_i) \cap C(S_1,\dots,S_n)$$

It is also a convex polytope. We note that for some choices of n-tuples $(S_1,\dots,S_n)$ the set $H_{C(S_1,\dots,S_n)}$ could be the empty set $\emptyset$. However,(by the Nash Theorem) $E(\Gamma)\neq \emptyset$. This and (3.2) imply that at least for some n-tuple $(S_1,\dots,S_n)$: $H_{C(S_1,\dots,S_n)} \neq \emptyset$

We also note that for each $x \in H_{C(S_1,\dots,S_n)}$ we have :

$$S(x_i) \subseteq J_i(x_{N-\{i\}}) \quad \text{for each } i \in N$$

Then, in view of (2.6), x is an equilibrium point. We will denote with V(.) the set of vertices of each convex polytope. Thus, we denote the finite set:

$$VH_{C(S_1,\dots,S_n)} = \{\, x^1,\dots,x^q \,\}$$

If $H_{C(S_1,\dots,S_n)}$ is such that $\nexists$ $(\bar{S}_1,\dots,\bar{S}_n)$ with $H_{C(S_1,\dots,S_n)} \subset$

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[1] A similar result was showed for bimatrix games (see Winkels(1979), Jansen(1981a), (1981b)).

$H_{C(\bar{S}_1,\ldots,\bar{S}_n)}$ then the points in $VH_{C(S_1,\ldots,S_n)}$ are called Extreme Equilibrium Points *[2].

We also define:

$$H(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)} = \bigcap_{i\epsilon N} \tilde{H}_i(S_i) \cap C(\bar{S}_1,\ldots,\bar{S}_n)$$

Proposition 3.4:

$$VH(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)} = \{ x \in \prod_{i=1}^{n} \bar{\Sigma}_i : x \in V(\bigcap_{i\epsilon N} \tilde{H}_i(\bar{S}_i)) \text{ with}$$

$x \in C(\bar{S}_1,\ldots,\bar{S}_n)$ }

(We denote the second set by: $K(\bar{S}_1,\ldots,\bar{S}_n;S_1,\ldots,S_n)$). *[3]

Proof:

Given $x \in K(\bar{S}_1,\ldots,\bar{S}_n;S_1,\ldots,S_n)$ then $x \in V(\bigcap_{i\epsilon N} \tilde{H}_i(S_i)) \subseteq \bigcap_{i\epsilon N} \tilde{H}_i(S_i)$ and $x \in C(\bar{S}_1,\ldots,\bar{S}_n)$; so that $x \in H(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)}$.

Suppose that $x \notin VH(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)}$. That is :

(3.5) $\quad x = \sum_{t=1}^{z} \lambda_t x^t$ with $x^t \in VH(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)}$ and $0 < \lambda_t < 1$

However, as $x \in \bigcap_{i\epsilon N} V(\tilde{H}_i(S_i))$ and $x^t \in \bigcap_{i\epsilon N} \tilde{H}_i(S_i)$ then (3.5) becomes impossible to be fulfilled. Then:

$$K(S_1,\ldots,S_n;\bar{S}_1,\ldots,\bar{S}_n) \subseteq VH(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)}.$$

Suppose that there would be a point x such that:

$$x \in VH(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)} - K(S_1,\ldots,S_n;\bar{S}_1,\ldots,\bar{S}_n) \text{ then}$$

$x \in C(\bar{S}_1,\ldots,\bar{S}_n)$ and $x \in V(\bigcap_{i\epsilon N} \tilde{H}_i(S_i))$

At the same time: $x \in \bigcap_{i\epsilon N} \tilde{H}_i(S_i)$, so that:

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[2] It corresponds to the notion of Extreme Equilibrium Points defined for bimatrix games in terms of Maximal Nash Subsets.

[3] It says that the vertices of $H(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)}$ are those vertices of $\bigcap_{i\epsilon N} \tilde{H}_i(S_i)$ which are also in $C(\bar{S}_1,\ldots,\bar{S}_n)$.

(3.6) $x = \sum_{t=1}^{w} x^t \lambda_t$ with $x^t \epsilon V(\bigcap_{i \epsilon N} \tilde{H}_i(S_i)) \subseteq \bigcap_{i \epsilon N} \tilde{H}_i(S_i)$ and $0 < \lambda_t < 1$.

Then for each $i \epsilon N$ $x_i = \sum_{i=1}^{w} \lambda_t x_i^t$

On the other hand we have : $x \epsilon C(\bar{S}_1,\ldots,\bar{S}_n)$ and by (3.6) we obtain:

$$S(x_i^t) \subseteq S(x_i) \subsetneq \bar{S}_i \quad \text{for each } i \epsilon N. \text{ Then:}$$

(3.7) $x_i^t \epsilon C(\bar{S}_1,\ldots,\bar{S}_n)$

Therefore, from (3.6) and (3.7) :

(3.8) $x^t \epsilon H(S_1,\ldots,S_n)_{C(S_1,\ldots,S_n)}$ and $x \epsilon VH(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)}$

Then (3.6) and (3.8) will be incompatible and we obtain:

$$K(S_1,\ldots,S_n;\bar{S}_1,\ldots,\bar{S}_n) = VH(S_1,\ldots,S_n)_{C(\bar{S}_1,\ldots,\bar{S}_n)} \qquad \text{(Q.E.D)}$$

If for each $i \epsilon N$ $\bar{S}_i = S_i$ we will use the notation:

$K(\bar{S}_1,\ldots,\bar{S}_n;S_1,\ldots,S_n) = K(S_1,\ldots,S_n)$. We remark that:

(3.9) $K(S_1,\ldots,S_n) = VH_{C(S_1,\ldots,S_n)}$

We will define:

$$E(\bigcap_{i \epsilon N} \tilde{H}_i(S_i)) = \{ x \epsilon \prod_{i=1}^{n} \tilde{\Sigma}_i \text{ with } x \text{ being a convex combination of}$$

points $x^t \epsilon K(S_1,\ldots,S_n) \}$

We note that we can rewrite :

$$K(S_1,\ldots,S_n) = \{ x^t \epsilon V(\bigcap_{i \epsilon N} \tilde{H}_i(S_i)) \text{ with } S(x_i) \subsetneq S_i \text{ for each } i \epsilon N\}$$

Proposition 3.10:

a) $K(S_1,\ldots,S_n) \subseteq E(\Gamma)$ b) $E(\bigcap_{i \epsilon N} \tilde{H}_i(S_i)) \subseteq E(\Gamma)$

Proof:

a) If $x^t \epsilon K(S_1,\ldots,S_n)$ then $x^t \epsilon H(S_1,\ldots,S_n)_{C(S_1,\ldots,S_n)}$ .It implies :

$S(x_i^t) \subseteq S_i \subseteq J_i(x^t_{N-\{i\}})$ and in view of (2.6) $x^t \epsilon E(\Gamma)$.

b) Given $x \epsilon E(\bigcap_{i \epsilon N} \tilde{H}_i(S_i))$, x is then a convex combination:

$x = \sum_{t=1}^{z} \lambda_t x^t$ with $x^t \epsilon K(S_1,\ldots,S_n)$; $0 \le \lambda_t \le 1$ and $\sum_{t=1}^{z} \lambda_t = 1$

and we obtain:

(3.11) $S(x_i) \subseteq \bigcap_{t=1}^{z} S(x_i^t) \subseteq S_i$

As x is a convex combination of $x^1,\dots,x^z$, which are points of $\bigcap_{i\epsilon N} \tilde{H}_i(S_i)$ and it is a convex set, then $x \epsilon \bigcap_{i\epsilon N} \tilde{H}_i(S_i)$, so that:

(3.12) $S_i \subseteq J_i(x_{N-\{i\}})$ for each $i\epsilon N$.

(3.11), (3.12) and (2.6) imply : $x \epsilon E(\Gamma)$. (Q.E.D)

We define: $$\tilde{E} = \bigcup_{(S_1,\dots,S_n)\subseteq(\Sigma_1,\dots,\Sigma_n)} E(\bigcap_{i\epsilon N} \tilde{H}_i(S_i))$$

Theorem 3.13:

$\tilde{E} = E(\Gamma)$

Proof:

In view of Proposition 3.11 b) $\tilde{E} \subseteq E(\Gamma)$. Let $x \epsilon E(\Gamma)$ be a Nash equilibrium of the polymatrix game, then by (3.2) and (3.3):

$$x \epsilon H_{C(S_1,\dots,S_n)} \quad \text{with } S_i = J_i(x_{N-\{i\}}) \text{ for each } i\epsilon N.$$

Then $VH_{C(S_1,\dots,S_n)}$ is a non-empty finite set.Moreover, as remarked in (3.3): $x^t \epsilon VH_{C(S_1,\dots,S_n)} \subseteq E(\Gamma)$.

By Proposition 3.4 and (3.9): $x^t \epsilon K(S_1,\dots,S_n)$ and as x is a convex combination of elements of $K(S_1,\dots,S_n)$ then $x \epsilon \bigcap_{i\epsilon N} \tilde{H}_i(S_i)$ and thus we obtain: $x \tilde{\epsilon} E$. (Q.E.D.)

In this way every Equilibrium Point of a Polymatrix game can be obtained as a convex combination of a finite set of Extreme Equilibrium Points.

We conclude this section with some examples:

Example 1: Consider the 3-person polymatrix game Γ, where $\Sigma_i=\{a,b\}$ for i=1,2,3 and the payoff matrices are defined by:

$$a^{ij}(\sigma^i,\sigma^j) \begin{cases} 1 \text{ if } \sigma^i = \sigma^j \\ 0 \text{ otherwise} \end{cases}$$

The total payoff for player i will be the sum of the number of players, other than himself, matching his strategy. The payoff matrices for the normal form of this game are given by:

Player 3 = a:

Player 1 \ Player 2	a	b
a	2,2,2	1,0,1
b	0,1,1	1,1,0

Player 3 = b:

Player 1 \ Player 2	a	b
a	1,1,0	0,1,1
b	1,0,1	2,2,2

(Player 1 chooses the row, player 2 the column and player 3 the matrix).

There are two Nash equilibrium points (in pure strategies): All the players choosing strategy a or all of them choosing b. The payoff at these equilibrium points will be 2 for everyone.

There is another, less appealing, equilibrium point if mixed strategies are allowed: Each player mixing (0.5,0.5) each pure strategy, with an expected payoff of 1.

<u>Example 2:</u> Consider the 3-person polymatrix game defined by:

$$A^{12}=\begin{pmatrix} 0,0,0 \\ 1,0,1 \\ 0,1,0 \end{pmatrix} \quad A^{23}=\begin{pmatrix} 3,0,0 \\ 2,2,0 \\ 0,3,0 \end{pmatrix} \quad A^{31}=\begin{pmatrix} 0,0,0 \\ 1,1,1 \\ 0,0,1 \end{pmatrix} \quad \text{and the remaining } A^{ij}\equiv 0$$

This game belongs to the subclass of cyclic games studied by Marchi and Quintas (1983). The set of equilibrium points is:

$E = \{((0,1,0),(0,0,1),(0,1,0))\} \cup \{((0,0,1),y,(0,0,1))$; where y is any convex combination of (0.5,0.5,0), (0,0.5,0.5) and (0,1,0)$\}$

<u>4.- Characterizations of polymatrix games</u>

Until now, we considered polymatrix games defined by means of the matrices A^{ij} (see Examples 1 and 2). However, if the game is given in the usual normal form it arises the problem of recognizing whether it is a polymatrix game.

In this section we will state necessary and sufficient conditions in order that a given n-person noncooperative normal form game $\Gamma = \{ \Sigma_i, A_i, i \epsilon N \}$ be a polymatrix game. We remark that every polymatrix game is defined by n(n-1) matrices (see (2.1)) and it forms a n-person normal form game. However many sets of matrices (i.e. many polymatrix games)define the same normal form game.*[4]

We will introduce some nomenclature. Given two n-1 tuples : $\sigma_{N-\{i\}} = (\sigma^1,\ldots,\sigma^{i-1},\sigma^{i+1},\ldots,\sigma^n)$ and $\tau_{N-\{i\}} = (\tau^1,\ldots,\tau^{i-1},\tau^{i+1},\ldots,\tau^n)$, for each subset $S \subseteq N-\{i\}$, we denote :

$$(4.1) \qquad w^S_{N-\{i\}} = \begin{cases} w^j = \sigma^j & \forall\ j \ \epsilon\ S \\ w^j = \tau^j & \forall\ j \ \epsilon\ S' \end{cases}$$

where $S' = (N-\{i\}) - S$

We will use the above notation to state the following conditions:

$$(4.2) \qquad \forall\ \sigma^i \ \epsilon\ \Sigma_i,\ \forall\ \text{(n-1) tuples} : \sigma_{N-\{i\}},\ \tau_{N-\{i\}} \text{ and } \forall\ S \subseteq N-\{i\} :$$

$$A_i(\sigma^i,\sigma_{N-\{i\}}) + A_i(\sigma^i,\tau_{N-\{i\}}) = A_i(\sigma^i,w^S_{N-\{i\}}) + A_i(\sigma^i,w^{S'}_{N-\{i\}})$$

In a polymatrix game, for each strategy $\sigma^i \ \epsilon\ \Sigma_i$, each other players strategies (σ^j or τ^j with $j \neq i$) provide a fixed amount ($a^i(\sigma^i,\sigma^j)$ or $a^i(\sigma^i,\tau^j)$ respectively) to the total payoff player i receives ($A_i(\sigma^i,\sigma_{N-\{i\}})$ or $A_i(\sigma^i,\tau_{N-\{i\}})$).Then a polymatrix game will trivially fulfill (4.2). We will prove that, for a normal form game, (4.2) is also a sufficient condition in order to be a polymatrix game.

$$(4.3) \qquad \forall\ \sigma^i \ \epsilon\ \Sigma_i\ ,\ \sigma^j \ \epsilon\ \Sigma_j\ ,\ \tilde{\sigma}^j \ \epsilon\ \Sigma_j\ ,\ \sigma^k \ \epsilon\ \Sigma_k,\ \tilde{\sigma}^k \ \epsilon\ \Sigma_k \text{ with } j,k \neq i$$

and $\forall$ (n-3) tuple $\underset{\sim}{\sigma}_{N-\{i,j,k\}}$: $\quad A_i(\sigma^i,\tilde{\sigma}^j,\tilde{\sigma}^k,\underset{\sim}{\sigma}_{N-\{i,j,k\}})$ -

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[4] The formula (4.10) will allow to generate many sets of matrices A^{ij} defining the same n-person normal form game.

$$A_i(\sigma^i,\sigma^j,\sigma^k,\sigma_{N-\{i,j,k\}}) = A_i(\sigma^i,\sigma^j,\sigma^k,\sigma_{N-\{i,j,k\}}) - A_i(\sigma^i,\sigma^j,\sigma^k,\sigma_{N-\{i,j,k\}})$$

Lemma 4.4:

Conditions (4.2) and (4.3) are equivalent.

Proof:

For each $S \subseteq N-\{i\}$ we denote σ_S the $|S|$-tuple of strategies where each player $j \in S$ plays the strategy $\sigma^j \in \Sigma_j$.

Condition (4.2) can be rewritten as follows:

(4.5) $\forall\ \sigma^i \in \Sigma_i \quad \forall\ S \subseteq N-\{i\}$ and $S'=(N-\{i\})-S$

$$A_i(\sigma^i,\sigma_{N-\{i\}}) + A_i(\sigma^i,\tau_{N-\{i\}}) = A_i(\sigma^i,\sigma_S,\tau_{S'}) + A_i(\sigma^i,\tau_S,\sigma_{S'})$$

When $S=\{k\}$ (i.e. $|S|=1$) we obtain the condition (4.3).

In order to prove that (4.3) implies (4.2); Suppose that:

(4.6) this implication holds for each set $S\hat{}$ with $|S\hat{}|=s-1$.

Then, we will prove it also holds for each set S with $|S|=s$.

For any $k \in S \subseteq N-\{i\}$, we apply (4.6) to the set $S-\{k\}$. We obtain:

(4.7) $A_i(\sigma^i,\sigma_{N-\{i\}}) + A_i(\sigma^i,\tau_{N-\{i\}}) = A_i(\sigma^i,\sigma_S,\tau_{S'}) +$
$A_i(\sigma^i,\tau_{S-\{k\}},\sigma_{(S-\{k\})'})$ and

(4.8) $A_i(\sigma^i,\sigma_{N-\{i\}}) + A_i(\sigma^i,\tau_{N-\{i\}}) = A_i(\sigma^i,\tau_S,\sigma_{S'}) +$
$A_i(\sigma^i,\sigma_{S-\{k\}},\tau_{(S-\{k\})'})$

Summing up (4.8) and (4.7), and using again (4.6) we obtain (4.5) for the set S. (Q.E.D.)

Theorem 4.9:

A noncooperative n-person normal form game $\Gamma = \{\ \Sigma_i,\ A_i,\ i \in N\ \}$ is a polymatrix game if and only if it fulfills (4.3). If it is the case, then the elements of the n.(n-1) matrices A^{ij} defining the polymatrix game can be written as follows:

(4.10) Fixed any n-1 tuple $\bar{\sigma}_{N-\{i\}}$, $\tilde{j}\neq i$ and n-2 arbitrary values C_j :

$$a^{ij}(\sigma^i,\sigma^j) = A_i(\sigma^i,\sigma^j,\bar{\sigma}_{N-\{i\}}) + C_j \text{ for } \sigma^j \in \Sigma_j \text{ (for } j\neq i \text{ and } j\neq\tilde{j}).$$

$$a^{i\tilde{j}}(\sigma^i,\sigma^{\tilde{j}}) = A_i(\sigma^i,\sigma^{\tilde{j}},\bar{\sigma}_{N-\{i\}}) - (n-2)\, A_i(\sigma^i,\bar{\sigma}_{N-\{i\}}) - \sum_{j\neq i,\tilde{j}} C_j$$

Proof:

It is immediate to verify that every polymatrix game fulfills (4.3). On the other hand, we will denote $\Gamma=\Gamma_n$ when $|N|=n$. If Γ_n fulfills (4.3) we are going to show that:

(4.11) $\forall\, \sigma^i \in \Sigma_i \quad \forall\, \sigma_{N-\{i\}} \in \prod_{j\neq i} \Sigma_j$:

$A_i(\sigma^i,\sigma_{N-\{i\}}) = \sum_{j\neq i} a^{ij}(\sigma^i,\sigma^j)$ i.e. (4.10) defines a polymatrix game.

It can be rewritten as follows:

(4.12) $A_i(\sigma^i,\sigma_{N-\{i\}}) + (n-2)\, A_i(\sigma^i,\bar{\sigma}_{N-\{i\}}) = \sum_{j\neq i} A_i(\sigma^i,\sigma^j,\bar{\sigma}_{N-\{i\}})$

For n=3 (4.12) is just (4.3).

For each $\bar{\sigma}_k \in \Sigma_k$ we define the game: $\Gamma_n(\bar{\sigma}^k) = \{\ \Sigma_i(\bar{\sigma}^k),\ A_i,\ i \in N\ \}$ with $\Sigma_i(\bar{\sigma}^k) = \Sigma_i$ for each $i\neq k$ and $\Sigma_k(\bar{\sigma}^k) = \{\bar{\sigma}^k\}$

It is equivalent to consider the n-1 person game :

$\Gamma_{n-1}(\sigma^k) = \{\ \Sigma\,(\bar{\sigma}^k)\ ,\ A_i^k\ ,\ i \in N-\{k\}\ \}$ where $A_i^k(\sigma_{N-\{i,k\}}) = A_i(\bar{\sigma}^k, \sigma_{N-\{i\}})$.

We will assume (4.12) holds for $\Gamma_{n-1}(\sigma^k)$, that is:

(4.13) $A_i(\sigma^i,\bar{\sigma}^k,\sigma_{N-\{i,k\}}) + (n-3)\, A_i(\sigma^i,\bar{\sigma}_{N-\{i\}}) + A_i(\sigma^i,\sigma^k,\bar{\sigma}_{N-\{i,k\}})$

$= \sum_{j\neq i} A_i(\sigma^i,\sigma^j,\bar{\sigma}_{N-\{i\}})$

Using (4.3), the left term in (4.13) becomes:

(4.14) $(n-2)\, A_i(\sigma^i,\bar{\sigma}_{N-\{i\}}) + A_i(\sigma^i,\sigma_{N-\{i\}})$

Then (4.12) holds for Γ_n. (Q.E.D.)

It is immediate that different choices of C_j's and $\bar{\sigma}_{N-\{i\}}$ give origin to different payoff matrices A^{ij} in the polymatrix game, but they define the same n-person normal form game.

Condition (4.3) is useful to recognize a polymatrix game when the matrices A^{ij} are not explicitly given in the description of the game. We illustrate this in the oligopoly model considered in the following example:

Example 3: Consider an industry with a fixed number of firms producing an homogeneous good. Each firm may choose to produce nothing or to produce a positive quantity q_i, from a finite set of possible outcome levels Q_i. Each firm has a cost function $c_i(q_i)$. Let Q be the total output of the industry and p the market price. Let q be the n-tuple of quantities produced for each firm: $q=(q_1,...,q_n)$. The profit function of firm i can be written: $A_i(q)=q_i p(Q)-c_i(q_i)$. We imagine the firms make their decisions (choose each output level q_i) simultaneously. Then, the above situation can be described as a n-person game $\Gamma=\{Q_i, A_i, i\epsilon N\}$

Q_i a finite set for each $i\epsilon N$. $A_i(q_1,...,q_n)=q_i\ p(\sum_{i=1}^{n} q_i) - c_i(q_i)$.

Γ is a polymatrix if and only if:

$\forall\ q_i\epsilon Q_i \quad \forall\ q_{-i}$ and $r_{-i}\ \epsilon\ Q_{-i} = \prod_{j\neq i} Q_j$ and for each partition $\{S,S'\}$ of the set $N-\{i\}$:

$$(*)\qquad \begin{aligned} &p(q_i + \sum_{j\epsilon N-\{i\}} q_j) + p(q_i + \sum_{j\epsilon N-\{i\}} r_j) = \\ &p(q_i + \sum_{j\epsilon S} q_j + \sum_{j\epsilon S'} r_j) + p(q_i + \sum_{j\epsilon S'} q_j + \sum_{j\epsilon S} r_j) \end{aligned}$$

If we assume linear cost functions: $c_i(q_i)=K_i\ q_i$ and also a linear demand function $p(Q)=a-bQ$ with $a,b>0$. Then the profits (payoff functions) are given by:

$A_i(q)=q_i(a-K_i) - bq_i^2 - q_i\ b\ (\sum_{j\neq i} q_i)$ and (*) becomes the following identity:

$$\sum_{j\epsilon N-\{i\}} q_j + \sum_{j\epsilon N-\{i\}} r_j = \sum_{j\epsilon S} q_j + \sum_{j\epsilon S'} r_j + \sum_{j\epsilon S} r_j + \sum_{j\epsilon S'} q_j$$

Then, Γ is a polymatrix game and we can use (4.10) to define explicitly the coefficients of the matrices A^{ij}.

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