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NONLINEAR DYNAMICS APPLIED TO NUMERICAL ANALYSIS AND ECONOMICS

by

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The purpose of this paper is to show how recent advances in nonlinear dynamics allow us to resolve several longstanding, classical issues from numerical analysis and from economics. The kind of problems I describe are the numerical analysis' problem of finding zeros of polynomials and the economics' problem of finding the price equilibrium. Dynamically, both problems are essentially the same; they both reduce to a search for an iterative method that converges to a zero of any function from a specified class. For instance, let $P = \{f \mid f$ is a polynomial with at least one real root$\}$. A classical approach to find a zero of $f \in P$ is Newton's method

1.1 $$x_{n+1} = x_n - f(x_n)/f'(x_n),$$

where $x_j \in (-\infty, \infty)$. Standard results confirm that if the initial iterate is "sufficiently close" to a zero to of f, then Newton's process converges. But, what if the initial iterate isn't sufficiently close. After all, the problem is to find the zero; what if we don't know where to start. Here, the results are not as reassuring. The pioneer for these issues is Barna [1,2,3]. Among other results, Barna showed that if a polynomial has at least three real zeros, then there are Cantor sets of initial iterates where Eq. 1.1 does not converge. Indeed, Barna constructed functions in P where an open set of initial iterates do not allow convergence. Barna's work depended on the properties of polynomials. By use of nonlinear dynamics, it is shown in (Saari - Urenko [9]) that these conclusions depend on the geometry of the graph of the function, so they extend to any continuous function with at least three real zeros. Moreover, it turns out that the associated dynamical behavior is very complicated -- it involves the currently fashionable topic of chaos. (See [9].)

If Newton's method doesn't work, then what does? Can we create a method, at least for the functions in *P*, that almost always leads to convergence? To see how to generalize Newton's method, note that if $M(u,v) = -u/v$, then Newton's method is

1.2 $$x_{n+1} = x_n + M(f(x_n), f'(x_n)).$$

Are there choices of M that overcome the limitations of Newton's method? Does there exist an M so that for $f \in P$, Eq. 1.2 converges for almost all initial iterates? This central, "algorithm design" problem is the numerical analysis question that I'll discuss here. My results are negative, but I'll indicate a direction where positive conclusions may result.

In economics, a standard concern is efficiency. Namely, is there a market decision mechanism to ensure that market demands are satisfied without incurring an unused surplus of goods? A widely discussed approach is the price mechanism. The textbook description starts with a price vector $\mathbf{p} = (p_1, .., p_c)$ where the j^{th} component, p_j, specifies the price for the j^{th} commodity, $j=1,..,c$. The next step is to postulate that with these prices, consumers create a demand, $D_j(\mathbf{p})$, and producers are willing to supply a certain amount, $S_j(\mathbf{p})$, of the j^{th} commodity, $j=1,..,c$. It is reasonable to suspect that by allowing the prices to adjust with freedom, an equilibrium price, $\mathbf{p}^*$, will be attained whereby efficiency is achieved. This means that $D(\mathbf{p}^*) = S(\mathbf{p}^*)$ where $D(p) = (D_1(\mathbf{p}), .., D_c(\mathbf{p}))$, and $S(\mathbf{p}) = (S_1(\mathbf{p}), ..., S_c(\mathbf{p}))$. The standard argument supporting the movement toward efficiency is a simple one. At a given price, consumers maximize their utility; if something is too expensive, they buy a substitute. So, if a component of $f(\mathbf{p}) = D(\mathbf{p}) - S(\mathbf{p})$ is positive, then it is natural to charge more to meet the demand with the insufficient supply. This higher price reduces demand and encourages an increase in supply. Conversely, if a component of $f(\mathbf{p})$ is negative, then the price will fall to get rid of excess stock. With the cheaper good, demand increases and supply decreases. Consequently, it appears that convergence should be achieved through the governing, market pressure dynamics of price adjustment given by

1.3 $$\mathbf{p}_{n+1} = \mathbf{p}_n + f(\mathbf{p}_n).$$

To justify the "supply-demand" story, we need to analyze the dynamics of Eq. 1.3. In turn, this requires us to know the functional form of f. A surprising example, developed by Scarf [11], proves that even in a simple pure exchange model with only three commodities, this supply - demand story need not work. Scarf creates a "text-book" economy where the consumers' utilities satisfy the appropriate neo-classical assumptions, but where the market pressures force the prices to stay a certain distance *away* from the price equilibrium. In fact, work starting with H. Sonnenschein [12, 13] and continuing with R. Mantel [5] and with G. Debreu [4] proves that even with standard, simplified assumptions concerning consumer preferences and production, if $c \geq 2$, then f can be just about any desired function! (There are boundary conditions reflecting that if a desired commodity is free (i.e., $p_j = 0$) then everyone will want it.) From this body of research, it now is commonly accepted that there are serious difficulties with convergence for the standard supply - demand story if $c \geq 3$. Recently, it was shown (Saari [6]) that similar problems occur for Eq. 1.3 whenever $c \geq 2$; indeed, it was shown that the accepted market pressures, even for a pure exchange model, can cause a chaotic price dynamic.

A possible rejoiner to this defect in the price adjustment story is that the model is wrong. The story leading to Eq. 1.3 is very simple, so perhaps it doesn't accurately capture how the market pressures affect the prices. Maybe the real market pressures involve a more subtle interaction of added information. Maybe an accurate, sophisticated modelling of the dynamics -- one that takes predictions, etc. into account -- does prove that market pressures can force the prices to converge. For instance, maybe the actual adjustment mechanism should include changes in the aggregate excess demand, f, as determined by the derivative Df. Thus, the natural issue is to determine whether there are alternative models that justify convergence. Does there exist *any* kind of mechanism of the form

1.4 $$p_{n+1} = p_n + M(f(p_n), Df(p_n))$$

that almost always leads to a price equilibria? This is the issue I'll address here concerning economics. I'll provide both negative and positive assertions.

It is obvious from Eqs. 1.2, 1.4 that both questions are essentially the same. By restricting attention to the special setting of a $c = 2$ commodity model, the similarity becomes stronger. With this assumption and by using standard economic assumptions, what happens for one commodity uniquely determines what happens for the other. Thus the problem is reduced to one on [0,1] where f is any smooth function such that $f(0) > 0$, $f(1) < 0$ and $p \in [0,1]$. So, the main difference between the problems is that the numerical analysis problem is on the real line, while the natural setting for the economic problem is [0,1].

2. Main Results

Our concern is the existence of an M that does what we want it to do -- it allows the dynamics of Eq. 1.2 or 1.4 to converge to a zero of f for almost all choices of an initial iterate. By "almost all", we mean that for a given M, the set of initial iterates that fail to lead to convergence does not contain an open set.

The choice of M's are governed by the following assumptions. With the possible exception of a finite union of smooth varieties or manifolds, V,

2.1 $$M: R^2 \setminus V \longrightarrow R$$

is a continuous mapping that is piecewise smooth. The choice of V can vary with M, and V is introduced to allow for a wider choice of options. For instance, for Newton's method $V = R \times \{0\}$ to correspond to the singularity when $f'(x) = 0$. The flexibility introduced both by V and by the "piecewise smoothness" assertion is meant to allow for branching techniques, etc. For the price mechanism, the image is slightly more complicated because $p+M(f(p),f'(p))$ must remain in the unit interval. This can be handled by imposing a truncation condition. (A natural one is if $p+M(f(p),f'(p)) \geq 1$, then assign the value 1, and if $p+M(f(p),f'(p)) \leq 0$, then assign the value 0.) For both topics -- economics and numerical analysis -- the dynamics must stop at a zero of f. This is the reason for the condition

2.2 $$M(0,-) = 0.$$

The objective is to determine whether any choice of M will allow

2.3 $$x_{n+1} = x_n + M(f(x_n),\ f'(x_n))$$

to converge to a zero of f for almost all initial iterates. For a given M and f, let $E_{M,f}$ = {initial iterate x| Eq. 2.3 does not converge to a zero of f}.

These minimal conditions on the admissible M's defines a wide class from which to select a possible algorithm. Yet, even with this abundance of possible choices, Theorem 1 asserts that none of them do what we want. There does not exist a single method that achieves convergence of all polynomials in *P* for almost all initial iterates. Indeed, the assertion is robust; it holds for open sets in *P* or for open sets of economies. (The topology on *P* is any natural topology determined by the coefficients of the polynomials. The topology on the set of economies is the one inherited by a function topology on the space of aggregate excess demand functions.)

Theorem 1. *a. Let M that satisfies Eqs. 2.1 and 2.2 define a numerical algorithm Eq. 2.3. There exists a non-empty, open set* $U \subset P$ *so that if* $f \in U$*, then* $E_{M,f}$ *contains a non-empty, open set.*

b. Let M that satisfies Eqs. 2.1 and 2.2 define a price adjustment procedure Eq. 1.4. There exists a non-empty, open set of neo-classical economies, U, so that for the excess demand function f for an economy in U, $E_{M,f}$ *contains a non-empty, open set.*

In other words, no matter what is the economic theory leading to a choice of M, what is the numerical technique defining a M, if the iterative process can be modelled with Eqs. 2.3, 1.4, then there are robust situations where the desired goal is not achieved. For instance, even a economic theory of price adjustment that includes speculation, as determined by changes in the aggregate excess demand as indicated by f', need not lead to convergence.

Of course, we could tolerate this negative conclusion and even a very large set $E_{M,f}$ if we know how to avoid it. One way would be if there is a point that never is in $E_{M,f}$ for any choice of f. This doesn't happen.

Theorem 2. *Let M satisfy Eqs. 2.1 and 2.2. Let x be a point in the domain. (The domain is* $(-\infty,\infty)$ *for the numerical analysis problem, [0,1] for the economics problem.) There exists an f in U so that* $x \in E_{M,f}$.

One reason convergence is not achieved is that an f always can be found so that the iterative dynamics creates an attracting cycle. This suggests a possible resolution. For instance, if we notice that Newton's method is caught in a cycle, then we would make the appropriate adjustments by changing some iterate. Can we incorporate such insight into the design of a mechanism M? To avoid a periodic orbit, we would want M to depend upon the past s+1 iterates and, perhaps, their first k derivatives. This requires

2.4. $$M: (R^{k+1})^{s+1}\backslash V \longrightarrow R$$

where V is a finite union of smooth varieties or manifolds. (The choice of V depends upon the choice of M.) Again, we require

2.5 $$M(0,-,-,-,..,-) = 0.$$

The iterative method is

2.6 $$x_{n+1} = x_n + M(f(x_n),f'(x_n),..,f^{(k)}(x_n),...,f(x_{n-s}),..,f^k(x_{n-s}))$$

The definition of $E_{M,f}$ remains the same except that Eq. 2.6 is the dynamics.

Theorem 3. *a. Let M that satisfies Eqs. 2.4 and 2.5 define a numerical algorithm Eq. 2.6. There exists a non-empty, open set $U \subset P$ so that if $f \in U$, then $E_{M,f}$ contains a non-empty open set. Furthermore, for any point x, there is an $f \in U$ so that $x \in E_{M,f}$.*

b. Let M that satisfies Eqs. 2.4 and 2.5 define a price adjustment procedure of the form Eq. 2.6. There exists a non-empty, open subset of neo-classical economies, U, so that the aggregate excess demand function f for any economy in U has an $E_{M,f}$ with an non-empty open set. Let $p \in [0,1]$; there is a $f \in U$ so that $p \in E_{M,f}$.

The implications for numerical analysis are obvious. If we want an algorithm of the form Eq. 2.6 to converge for all $f \in P$, then either *we need a new approach using different kinds of information, or we may need an infinite amount of information.* For economics, this means that *any price adjustment theory based on the aggregate excess demand functions -- such as using trends from the derivatives and the last s iterates -- will not work for all economies.* Examples always can be created to show that convergence does not occur. One reason convergence may not occur is that no matter how large the value of s, a function f or an economy can be found so that the system defines an attractive, periodic orbit with period m where $m > s$. So, no matter how many past iterates we include, there can be an attractive periodic orbit that just exceeds the limitations of M. To require all values of s requires an infinite amount of information.

Are there any positive conclusions? For economics, there are, but, at this stage they must be viewed strictly as "existence of positive conclusions," because a practical statement is yet to be developed. A weaker assertion holds for numerical analysis. The idea is this. Perhaps, the real source of non-convergence is the tacit assumption that we should use only one choice of M. To explain the idea with an analogy, consider the problem of representing a globe with "flat maps." This is impossible because of the topological nature of the sphere; one map can't cover the globe, two or more are required. For similar reasons, the topological barriers to convergence of an algorithm of the form Eq. 2.3, 2.6, require more than one M. Namely, we want a finite set $\{M^j\}$ so that for any $f \in P$, there is a M^j so that $E_{M^j,f}$ does not contain an open set. To be more specific, we say that M *covers f if* $E_{M,f}$ does not contain an open set. We say that $\{M^j\}$ *covers a function class F* if $\forall f \in F$, $\cap\{E_{M^j,f}\}$ does not contain an open set. The objective is to determine for a given function space *F* what is the minimal kind of cover it admits. With such knowledge, then if one algorithm doesn't work, we know what other ones to try. It can be derived (from Saari [7]) that *P* admits a countable cover. Consequently, a finite set of methods can be found for any compact subset of *P*. What remains is to understand whether there is a finite cover for all of *P*, and to characterize which methods apply for which subsets of *P*. (For instance, Newton's method applies for the subset where all zeros are real.)

In economics, we have more to work with because the agents can communicate with each other. Using this fact, we (Saari - Williams [10]) obtain a similar statement that, for a compact set of economies, a finite number of price mechanisms will suffice. Moreover, the algorithm particularly simple and natural for the theory; it is determined by multiplying f(p) with a diagonal matrix. Thus a story that permits the convergence is a simple extension of the standard supply - demand algorithm! What differs is that the matrix changes with the economy. (See Saari - Williams [10] for more details and proofs.)

3. An Outline of the Proofs.

Details of the proofs and additional assertions are in two basic references. The results about numerical analysis are in Saari [7]. The assertions about economics are in Saari [6]; some of the ideas in this last reference are extensions of concepts developed in Saari-Simon [8] for "continuous" price adjustment methods. An argument why we need more than one choice of M, how this helps to solve the problem facing price adjustment in economics, and how the new mechanisms can be found in a "privacy preserving" manner is in Saari-Williams [10].

For the remainder of this concluding section, I will show why we should expect the stated conclusions. I'll do this with a special case of Theorem 1. Let $G_f(x) = x + M(f(x),f'(x))$. This changes the iterative scheme to $x_{n+1} = G_f(x_n)$. To prove that convergence is not achieved, it suffices to show that there is an f and two distinct points x_1 and x_2 such that i) $G_f(x_1) = x_2$, $G_f(x_2) = x_1$, and that ii) $|G_f'(x_1)G_f'(x_2)| < 1$. This is because conditions i) ensures that there is a period two cycle. The term inside the absolute value signs of condition ii) is the chain rule expansion of $\{G_f(G_f(x))\}'$ evaluated at either x_1 or at x_2. The fact that this derivative is less than unity in magnitude forces the periodic orbit to be attractive. Namely, there is an open set of initial conditions containing x_1 so that the subsequent orbit approaches the periodic orbit as the number of iterates increases. Thus, for any of these initial iterates, convergence to a zero is denied.

There are some technical difficulties that need to be handled. First, if $M \equiv 0$, then G_f is the identity map for all choices of f. Here, nothing happens, so the conclusion of the theorem follows immediately. Secondly, if $M > 0$, then the iterates are always forced to the right. So, for any initial iterate to the right of the last zero of f, the system cannot converge. Again, the theorem follows immediately. A similar argument shows that it is impossible to have a successful algorithm for M <

0. Thus, we assume that M does not have these properties.

The first task is to show that i) is satisfied. Here there are several technical difficulties. To illustrate the ideas, I'll consider only a simple setting, then I'll outline what needs to be done in the more general case. Suppose, as true for Newton's method $M(u,v) = -u/v$, that there are v^j, $j = 1,2$, so that for an interval $0< u \leq 2$, sign $(M(u,v^j)) = (-1)^j$. Using the fact that $M(0,-) = 0$ and continuity, this forces the two surfaces in R^3 given by $z = -M(u,v^1)$ and $z = M(y,v^2)$ to have a point of intersection over $(0,2]x(0,2]$. (The coordinates in R^3 are (u,y,z); the values of v^j are treated as parameters.) This intersection follows immediately because the first surface is independent of y, the second is independent of u, and both are positive. Thus, you can see the conclusion by drawing a graph; indeed, it follows that the intersection contains a continuum that contains arbitrarily small values of M.

Let (u^1,y^2) be a point of intersection of these graphs. Let x_1 be any point in the interior of the domain, let $f(x_1) = u^1$, $f'(x_1) = v^1$, $x_2 = x_1 + M(u^1,v^1)$, $f(x_2) = y^2$, and $f'(x_2) = v^2$. Since the intersection points (u^1) and y^2 can be selected to make the value of M sufficiently small, it is clear that if such an f exists, then the points x_1 and x_2 can be in any designated interval. Moreover, since this is imposes only point information on f, it is clear that such an f exists. From this construction, it is obvious that with this f, x_1 and x_2 form a period two cycle.

The general setting is slightly more difficult; it may be that the variety V is complicated enought not to allow an interval bordering on u = o to satisfy the conditions for M. But since there are regions where M is positive and where it is negative, certain number of iterates of the negative values will have the same magnitude as a certain number of iterates of the positive values. Thus, for such an M, the period two orbit is replaced with a periodic orbit of some higher order. The details of such a construction are in [7].

Next, we need to show that condition ii) is satisfied. To do this, note that $G_f'(x) = 1 + M_1(f(x),f'(x))f'(x) + M_2(f(x),f'(x))f''(x)$. The values of M_j (the partial derivative with respect to the j^{th} variable), $f(x)$, $f'(x)$ are all specified at $x = x_1, x_2$, by the above construction. However, at both points the value of $f''(x)$ is free to be chosen. Clearly there are choices for $f''(x_j)$ so that $|G_f'(x_j)| < 1$. This completes the proof. (The minor technical details to handle the situation where $M_2 = 0$ at such a point are easy to supply.)

What we see from this proof that no matter how many iterates we include, how many derivatives we require, if both are a finite number, then, by the chain rule, we have a parameter to vary. This parameter is the value of the "next" derivative of f. By altering this value appropriately, the negative conclusions specified in Section 2 follow.

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