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ON THE NUMBER OF SOLUTIONS TO THE LINEAR COMPLEMENTARITY PROBLEM

by

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ABSTRACT

Given an $n \times n$ matrix A and an n-dimensional vector q let $N(A,q)$ be the cardinality of the set of solutions to the linear complementarity problem defined by A and q. It is shown that if A is nondegenerate then $N(A,q) + N(A,-q) \leq 2^n$, which in turn implies $N(A,q) \leq 2^n-1$ if A is also a Q-matrix.

It is then demonstrated that $\min_{q \neq 0} N(A,q) \leq 2^{n-1}-1$, which concludes that the complementary cones cannot span R^n more than $2^{n-1}-1$ times around. For any n, an example of an $n \times n$ nondegenerate Q-matrix spanning all R^n, but a subset of empty interior, $2^{[\frac{n}{3}]}$ times around is given.

Given a square real matrix A of order n and an n-dimensional vector q, the Linear Complementarity Problem, (LCP), denoted by (q,A), is to find n-dimensional vectors x and y satisfying

$$y = Ax + q,\ x \geq 0,\ y \geq 0 \qquad x'y = 0 \tag{1}$$

(See [3,5,6,7] for the bibliography and history of the LCP.)

Letting N(A,q) be the cardinality of the set of solutions to (q,A) we study the quantities $N^*(A) = \max_{q \in R^n} N(A,q)$ and $N_*(A) = \min_{0 \neq q \in R^n} N(A,q)$ and establish bounds on both. The latter quantity has the following geometrical interpretation. If we consider the 2^n complementary cones generated by the columns of [I,-A], (see [8]), then $N_*(A)$ is the number of times R^n is spanned around by these cones.

This study shows that this (spanning) number is bounded above by $2^{n-1}-1$ if the matrix A is nondegenerate, i.e. all its principal minors are nonzero. For any n, an example of an $n \times n$ nondegenerate matrix spanning all R^n, but a subset of empty interior, $2^{[\frac{n}{3}]}$ times around is given.

It is also demonstrated that the nondegeneracy of A implies $N(A,q) + N(A,-q) \leq 2^n$ for all q. Thus, if A is a Q-matrix (i.e. $N(A,q) \geq 1$ for all q) then $N^*(A) \leq 2^n-1$.

This work was motivated by a conjecture due to Ingleton [4] and Censor [1]. Studying nondegenerate matrices Ingleton showed that if for some q* (q*,A) has a unique solution (x*,y*) and x* + y* is strictly positive, then A is a Q-matrix. (Extensions of this result appear in [5,8,9,10]). Ingleton conjectured that under the above conditions the maximal number of solutions to (q,A) is 2^n-1, provided that $N(A,\overline{q}) > 1$ for some $\overline{q}$.

The conjecture was studied and partially resolved by Censor, [1], who also raised the following general question: Given an $n_x n$ nondegenerate Q-matrix A and supposing that $N(A,\bar{q}) > 1$ for some $\bar{q}$, is the maximal number of solutions to (1) 2^n-1 for all q in R^n?

The following result demonstrates that 2^n-1 is indeed a valid bound. (We later show that the bound is not necessarily attained.)

Lemma 1: Let A be an $n_x n$ matrix such that $N(A,q) \geq 1$ for all $q \leq 0$. Then for any q $N(A,q) < \infty$ implies $N(A,q) \leq 2^n-1$.

Proof:

For any $I \subseteq \{1,2,\ldots,n\}$ consider the (complementary) cone generated by the n columns $-A\cdot i$, $i \in I$ and e_i $i \in \bar{I}$, where $A\cdot i$ is the i^{th} column of A and e_i is the i^{th} unit vector. Note that the correspondence between subsets of $\{1,2,\ldots,n\}$ and the complementary cones is not necessarily one to one.

It is obvious that if there are two different solutions to (2)

$$y = Ax + q, x \geq 0,\ y \geq 0,\ x'y = 0, \qquad \begin{array}{ll} x_i > 0 & i \in I \\ \text{and } x_i = 0 & i \notin I \end{array} \qquad (2)$$

for some $I \subseteq \{1,\ldots,n\}$ then $N(A,q)$ is not finite. (Consider convex combinations of these two solutions.)

Hence, if $N(A,q)$ is finite each set $I \subseteq \{1,2,\ldots,n\}$ contributes at most one solution to (q,A) and $N(A,q) \leq 2^n$. If $q \not\geq 0$, then the cone defined by $I = \emptyset$ does not contribute a solution and $N(A,q) \leq 2^n-1$. Note that $N(A,0)$ is finite only if $N(A,0) = 1$. Consider now a nonzero, nonnegative q. By the lemma's assumption $(-q,A)$ has a solution.

Defining a cone to be nondegenerate if it has a nonempty interior, we observe that the union of all the degenerate complementary cones and the (proper) faces of the nondegenerate cones is closed and no where dense in R^n. Thus there exists a sequence $\{q^k\}$, converging to $-q$ where $\forall k\ q^k \leq 0$ and q^k belongs to a nondegenerate cone. Since there is a finite number of complementary cones we can assume without loss of generality (choose a subsequence if necessary) that $\forall k\ q^k$ belongs to the same nondegenerate cone. Hence $-q$ is in that cone. It is then clear that q is not contained in this nondegenerate cone, since otherwise we would have that the n generators are linearly dependent - a contradiction to the nondegeneracy assumption. Hence $N(A,q) \leq 2^n-1$.

From Lemma 1 $N^*(A) \leq 2^n-1$ for any nondegenerate Q-matrix A. Censor, [1], provided an example of an $n \times n$ nondegerate Q-matrix, (which we denote by A_n), such that $N^*(A_n) = 2^n-1$. Murty, [8], illustrated that the matrix

$$M = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \tag{3}$$

is a nondegenerate Q-matrix and $N^*(M) = 4$. Combining the above two illustrations and using the direct sum operation we show that for each n 2^n-1 is not always a sharp bound.

Recalling that the direct sum of the square matrices A and B, $A \oplus B$, is given by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

we notice that

$$N(A \oplus B,q) = N(A,q^1)\ N(B,q^2) \tag{4}$$

where (q^1,q^2) is the corresponding partition of q.

<u>Lemma 2</u>: Given nonnegative integers n, r, t, such that $3t+r \leq n$ there exists an $n \times n$ nondegenerate Q-matrix A such that $N^*(A) = 4^t \max\{1,2^r-1\}$.

<u>Proof</u>: Define $A = \underbrace{M \oplus \cdots \oplus M}_{t} \oplus A_r \oplus I_{n-3t-r}$, where M is given by (3), A_r is the above mentioned matrix due to Censor [1] and I_{n-3t-r} is the identity of order n-3t-r.

Having established an upper bound on N(A,q), we next turn to the lower bound question and prove that if A is an $n \times n$ nondegenerate Q-matrix then $\min\limits_{q \neq 0} N(A,q) \leq 2^{n-1}-1$, provided $n \geq 2$.

<u>Lemma 3</u>: Let A be an $n \times n$ matrix and $q \in R^n$. If $N(A,q) + N(A,-q) < \infty$, then $N(A,q) + N(A,-q) \leq 2^n$.

<u>Proof</u>:

If $q = 0$, then $N(A,0) < \infty$ implies $N(A,0) = 1$. Thus consider $q \neq 0$. Let $I \subseteq \{1,2,\ldots,n\}$, then we show that the complementary cone corresponding to I does not contain both q and -q. This will prove the lemma. Observe first that if the cone corresponding to I is nondegenerate, i.e. has an interior, then $q \neq 0$ implies that the cone cannot contain both q and -q. Suppose that there exists a cone C containing q and -q. Let B be an $n \times n$ matrix corresponding to the n generators of C, then $Bx^1 = q$ and $Bx^2 = -q$ for some $x^1 \geq 0$ and $x^2 \geq 0$. Hence $\forall t \geq 0$ $B[(1+t)x^1 + tx^2] = q$ and there exists a (complementary) ray of solutions - a contradiction to $N(A,q) < \infty$.

Note that Lemma 3, which has been proved independently of Lemma 1, also implies that $N(A,q) \leq 2^n-1$ for any nondegenerate $n \times n$ Q-matrix. Another consequence is that $\min\limits_{q \neq 0} N(A,q) \leq 2^{n-1}$.

We now refine this bound to $2^{n-1}-1$, applying a result due to Murty [8].

<u>Theorem 4</u>: Let A be an $n \times n$ nondegenerate matrix.

Then $\min\limits_{q \neq 0} N(A,q) \leq 2^{n-1}-1$ when $n \geq 2$.

<u>Proof</u>:

From Lemma 3 it follows that $\min\limits_{q \neq 0} N(A,q) \leq 2^{n-1}$. Suppose that for all $q \neq 0$ $N(A,q) = 2^{n-1}$. We then apply Theorem (7.2) of [8] to have $N(A,q) = 1$ for all q. Hence the proof is complete.

Theorem 4 concludes that the complementary cones cannot span the space R^n more than $2^{n-1}-1$ times around. This bound is trivially attained for 2×2 Q-matrices.

Finally we show that for any $n \geq 3$ there exists an $n \times n$ nondegenerate Q-matrix A such that the complementary cones defined by the columns of $[I,-A]$ span $R^n-R(A)$ $2^{[\frac{n}{3}]}$ times around, where R(A) is a subset of R^n with empty interior.

To define R(A) precisely, we recall that q in R^n is said to be nondegenerate with respect to the $n \times n$ matrix A if for any (x,y) solving (q,A) $x + y$ is a strictly positive vector (i.e. q is neither contained in a complementary cone with no interior nor in a proper face of a nondegenerate cone). q is degenerate if it is not non-degenerate. $R(A)$ is defined as the set of vectors in R^n that are degenerate with respect to A.

We need the following lemma for our discussion.

Lemma 5: Let A be an $n_x n$ nondegenerate matrix and consider q in R^n which is not contained in any face of dimension less than $n-1$ of a complementary cone defined by the columns of $[I,-A]$. Then, $\forall\ \epsilon > 0$ there exist nondegenerate vectors P^1 and P^2 such that

$$\| P^1-q\| < \epsilon,\quad \| P^2-q\| < \epsilon \text{ and } N(A,q) = \tfrac{1}{2}[N(A,P^1) + N(A,P^2)].$$

Proof:

If q itself is nondegenerate choose $P^1 = P^2 = q$.

Suppose that q is contained in the intersection of k differnet $(n-1)$ dimensional faces $C_1,\ldots,C_k$ of complementary cones but not in any face of a lower dimension. Choose $\epsilon > 0$ such that $S(q,\epsilon) = \{x \mid \|x-q\| \leq \epsilon\}$ is contained in any complementary cone having q in its interior, and such that each complementary cone having one of $C_1,\ldots,C_k$ as its $(n-1)$ dimensional face contains the corresponding "half" ball chopped off by that face. ϵ is well defined since each complementary cone is nondegenerate and q is not in any face of dimension less than $n-1$.

Let $u \in R^n$, $\|u\| < \epsilon$, such that $q + u$ and $q - u$ are nondegenerate vectors. We then have that $P^1 = q + u$ and $P^2 = q - u$ are on opposite sides of each C_i, $i=1,\ldots,k$. Also, P^1 and P^2 are contained in any complementary cone having q in its interior.

Consider the i^{th} face C_i, and let t_i be the number of different complementary cones having C_i as their face. Using the assumption that q is not on a face of dimension less than $n - 1$ we may suppose, without loss of generality, that $\bar{t}_i$ cones, $0 \leq \bar{t}_i \leq t_i$ contain P^1 and $t_i - \bar{t}_i$ contain P^2.

Let $\bar{T}_i$ $(T_i-\bar{T}_i)$ be the cardinality of the set of solutions to (P^1,A) $((P^2,A))$ corresponding to the above $\bar{t}_i(t_i-\bar{t}_i)$ cones. (Note that we introduce T_i and $\bar{T}_i$ since the correspondence of subsets of $\{1,2,\ldots,n\}$ to the different complementary cones is not injective.) Note that T_i is even by the nondegeneracy assumption. Thus the t_i cones contribute $\frac{T_i}{2}$ solutions to (q,A), $\bar{T}_i$ solutions to (P^1,A) and $T_i - \bar{T}_i$ solutions to (P^2,A). Noting that P^1 and P^2 are on opposite sides of each C_i, $i=1,\ldots,k$ and denoting by r the number of nondegenerate solutions to (q,A) we have

$$N(A,P^1) = r + \sum_{i=1}^{k} \bar{T}_i, \quad N(A,P^2) = r + \sum_{i=1}^{k} (T_i-\bar{T}_i) \text{ and } N(A,q) = r + \sum_{i=1}^{k} \frac{T_i}{2}$$

Hence $N(A,q) = \frac{1}{2}[N(A,P^1) + N(A,P^2)]$ and the proof is complete.

Dealing with the 3×3 matrix M defined in (3), Murty [8] claims that $N(M,q)$ is positive and even for all nondegenerate q. Lemma 5 and a simple inspection of the six generators of $[I,-M]$ show that $N(M,q) \geq 2$ for all $q \neq 0$. Clearly $N(M,0) = 1$.

Let M be given by (3) and define $M^{(j)} = \underbrace{M \oplus \ldots \oplus M}_{j}$. If q is in R^{3j} then we have $q = (q^1, q^2, \ldots, q^j)$, where q^t, $1 \leq t \leq j$, is in R^3. We then observe that for $q \in R^{3j}$ such that $q^t \neq 0$, $t=1,\ldots,j$ $N(M^{(j)},q) \geq 2^j$, and that $\{q \mid q^t = 0 \text{ for some } 1 \leq t \leq j\}$ is (strictly) contained in $R(M^{(j)})$, the set of degenerate vectors corresponding to $M^{(j)}$. Thus the following theorem is implied.

Theorem 6:

For any $n \geq 3$ there exists an $n \times n$ nondegenerate Q-matrix A such that $N(A,q) \geq 2^{[\frac{n}{3}]}$ for any q in $R^n - B$, where B is a subset of R(A), the set of degenerate vectors corresponding to A.

Finally, while noting that for $j \geq 2$ $N(M^{(j)}, e_1) = 2$ where $e_1 = (1,0,\ldots,0)^1$ we conclude that Lemma 5 does not hold for all $q \neq 0$. In fact, the above illustrates that if q is contained in a face of dimension less than n-1, where n is the order of the matrix A, the existence of nondegenerate vectors P^1 and P^2 such that $N(A,P^1) \leq N(A,q) \leq N(A,P^2)$ is not guaranteed.

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