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DECOMPOSITION AND REPRESENTATION OF COALITIONAL GAMES

by

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Decomposition and Representation of Coalitional Games*

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Abstract

A coalitional game is a real-valued set function ν defined on an algebra $\mathcal{F}$ of subsets of a space X such that $\nu(\varnothing) = 0$. We prove the existence of a one-to-one correspondence between coalitional games bounded with respect to the composition norm and countably additive measures defined on an appropriate space.

1 Introduction

Let $\mathcal{F}$ be an algebra of subsets of a given space X, and V the set of all set functions ν on $\mathcal{F}$ such that $\nu(\varnothing) = 0$. These set functions are called transferable utility coalitional games (games, for short). In Gilboa and Schmeidler (1995) it is proved the existence of a one-to-one correspondence between the games in V that have finite composition

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norm (see next section) and the bounded finitely additive measures defined on an appropriate algebra Ψ of subsets of $2^{\mathcal{F}}$. The algebra Ψ is constructed as follows: for $T \in \mathcal{F}$, define $T^* \subseteq \mathcal{F}$ by $T^* = \{S \in \mathcal{F} : \varnothing \neq S \subseteq T\}$. Denote $\Theta = \{T^* : \varnothing \neq T \in \mathcal{F}\}$. Then Ψ is defined as the algebra of subsets of $2^{\mathcal{F}}$ generated by Θ. On the basis of this representation, Gilboa and Schmeidler (1995) show that every game with finite composition norm can be decomposed in the difference of two totally monotone games (i.e. belief functions). Their work is related to Choquet (1953-54) and Revuz (1955-56), as discussed on p. 211 of their article.

In this paper we first give a direct proof of the mentioned decomposition theorem, a proof based on the well-known Dempster-Shafer-Shapely Representation Theorem for finite games (see e.g. Shapley [1953] and Shafer [1976]). On the basis of such a decomposition we obtain a one-to-one correspondence between the games in V that have finite composition norm and the bounded regular countably additive measures defined on an appropriate Borel σ-algebra. To construct this σ-algebra we proceed as follows. Let U_b be the set of all $\{0, 1\}$-valued convex games (a game ν is convex if $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$ for all $A, B \in \mathcal{F}$). The set U_b can be endowed with a natural topology τ_v as defined in next section. Let $\mathcal{B}(U_b)$ be the Borel σ-algebra on (U_b, τ_v). This is the σ-algebra we use to get the mentioned one-to-one correspondence.

A main advantage of this novel representation theorem is that the space of bounded regular countably additive measures on $\mathcal{B}(U_b)$ has much more structure than the space of finitely additive measures on the algebra Ψ. Besides, unlike finitely additive measures, regular countably additive measures are widely studied in measure theory, and technically more convenient.

For finite algebras, both the Gilboa-Schmeidler representation and the one proved here reduce to the Dempster-Shafer-Shapley Representation Theorem.

Finally, in this work we use a topological approach, while in Gilboa and Schmeidler (1995) an algebraic one is used. This is a secondary contribution of the paper. In particular, with our topological approach it is possible to reobtain also their finitely additive representation.

In sum, the approach taken in this paper leads to novel results, without losing any of the results already proved with the different algebraic approach taken in Gilboa and Schmeidler (1995). This gives a unified perspective on this topic.

The paper is organized as follows. The next section contains some preliminary material. In section 3 some properties of a locally convex topological vector space on V are proved. In section 4 a direct proof of the decomposition theorem is provided. In section 5, which is the heart of the paper, the main result is proved. As a consequence of this result, in section 6 it is proved that every Choquet integral on $\mathcal{F}$ can be represented by a standard additive integral on $\mathcal{B}(U_b)$. In section 7 it is showed how the finitely additive representation result of Gilboa and Schmeidler (1995) can be reobtained in our set-up. Finally, in section 8 some dual spaces of V are studied.

2 Preliminaries

A set function ν on the algebra $\mathcal{F}$ is said to be a game if $\nu(\emptyset)=0$. The symbol V denotes the set of all games defined on $\mathcal{F}$. The space V becomes a vector space if we define addition and multiplication elementwise:

$$(\nu_1+\nu_2)(A)=\nu_1(A)+\nu_2(A) \text{ and } (\alpha\nu)(A)=\alpha\nu(A) \text{ for all } A\in\mathcal{F} \text{ and } \alpha\in\Re.$$

A game ν is monotone if $\nu(A)\leq\nu(B)$ whenever $A\subseteq B$. A game ν is convex if $\nu(A)+\nu(B)\leq\nu(A\cap B)+\nu(A\cup B)$ for all $A,B\in\mathcal{F}$. A game ν is normalized if $\nu(X)=1$. A game ν is totally monotone if it is nonnegative and if for every $n\geq 2$ and $A_1,\ldots,A_n\in\mathcal{F}$ we have:

$$\nu\left(\bigcup_{i=1}^{n}A_i\right)\geq\sum_{\{I:\emptyset\neq I\subseteq\{1,\ldots,n\}\}}(-1)^{|I|+1}\nu\left(\bigcap_{i\in I}A_i\right)$$

For $T\in\mathcal{F}$, the $\{0,1\}$-valued game $u_T\in V$ such that $u_T(A)=1$ if and only if $T\subseteq A$ is called unanimity game. We can now present the Dempster-Shafer-Shapley Representation Theorem, which will play a central role in the sequel. Given a finite algebra $\mathcal{F}=\{T_1,\ldots,T_n\}$, the atoms of $\mathcal{F}$ are the sets of the form $T_1^{i_1}\cap T_2^{i_2}\cap\cdots\cap T_n^{i_n}$ where $i_j\in\{0,1\}$ and $T_j^0=-T_j$, $T_j^1=T_j$ ($-T$ denotes the complement of T). We denote by Ω the set of all atoms of $\mathcal{F}$. It holds $n\leq|\Omega|\leq 2^n$.

Theorem 1 *Suppose $\mathcal{F}$ is finite. Then $\{u_T:\emptyset\neq T\in\mathcal{F}\}$ is a linear basis for V. Given $\nu\in V$, the unique coefficients $\{\alpha_T^\nu:\emptyset\neq T\in\mathcal{F}\}$ satisfying*

$$\nu(A)=\sum_{\emptyset\neq T\in\mathcal{F}}\alpha_T^\nu u_T(A)\quad \textit{for all } A\in\mathcal{F}$$

are given by

$$\alpha_T^\nu = \sum_{S\subseteq T}(-1)^{|T|-|S|}\nu(S) = \nu(T) - \sum_{\{I:\varnothing\neq I\subseteq\{1,\dots,k\}\}}(-1)^{|I|+1}\nu\left(\bigcap_{i\in I}T_i\right)$$

where $T_i = T\backslash\omega_i, T = \bigcup_{i=1}^k \omega_i$. *Moreover,* ν *is totally monotone on* $\mathcal{F}$ *if and only if* $\alpha_T^\nu \geq 0$ *for all nonempty* $T \in \mathcal{F}$.

For a finite algebra $\mathcal{F}$, Gilboa and Schmeidler (1995) define the composition norm of $\nu \in V$ to be $\|\nu\| = \sum_{T\in\mathcal{F}} |\alpha_T^\nu|$. On infinite algebras, Gilboa and Schmeidler (1995) define the composition norm $\|\cdot\|$ of $\nu \in V$ in the following way. Given a subalgebra $\mathcal{F}_0 \subseteq \mathcal{F}$, let $\nu_{|\mathcal{F}_0}$ denote the restriction of ν to $\mathcal{F}_0$. Then define:

$$\|\nu\| = \sup\left\{\left\|\nu_{|\mathcal{F}_0}\right\| : \mathcal{F}_0 \text{ is a finite subalgebra of } \mathcal{F}\right\}.$$

The function $\|\cdot\|$ is a norm, and in what follows $\|\cdot\|$ will always denote the composition norm. V^b will denote the set $\{\nu \in V : \|\nu\| \text{ is finite}\}$. The pair $(V^b, \|\cdot\|)$ is a Banach space (see Gilboa and Schmeidler [1995] p.204).

Another important norm in transferable utility cooperative game theory is the variation norm $\|\cdot\|_v$, introduced in Aumann and Shapley (1974). This norm is defined by

$$\|\nu\|_v = \sup\left\{\sum_{i=0}^n |\nu(A_{i+1}) - \nu(A_i)| : \varnothing = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{n+1} = X\right\}.$$

BV^b will denote the set $\{\nu \in V : \|\nu\|_v \text{ is finite}\}$. If ν is additive, then $\|\cdot\|$ coincides with $\|\cdot\|_v$, and both norms coincide with the standard variation norm for additive set functions (cf. Gilboa and Schmeidler [1995] p.201).

A filter p of $\mathcal{F}$ is a collection of subsets of $\mathcal{F}$ such that

1. $X \in p$,
2. if $A \in p$, $B \in \mathcal{F}$, and $A \subseteq B$, then $B \in p$,
3. if $A \in p$ and $B \in p$, then $A \cap B \in p$.

Let p be a filter of $\mathcal{F}$. Then

1. p is a proper filter if $\varnothing \notin p$. i.e. $p \neq \mathcal{F}$.

2. p is a principal filter if $p = \{B \in \mathcal{F} : A \subseteq B\}$ for some $A \in \mathcal{F}$; p is then the principal filter generated by A

3. p is a free filter if it is not a principal filter.

4. p is an ultrafilter if it is a filter and for every $A \in \mathcal{F}$ either $A \in \mathcal{F}$ or $A \notin \mathcal{F}$.

In a finite algebra all filters are principal. This is no longer true in infinite algebras. For example, let X be an infinite space. A simple example of a free filter in the power set of X is the collection of all cofinite sets $\{A \subseteq X : -A \text{ is finite}\}$.

Every filter p can be directed by the binary relation $\geq$ defined by

$$A \geq B \iff A \subseteq B \quad \text{where } A, B \in p.$$

Let $f : X \to \Re$ be a real-valued function on X. Set

$$f_A = \inf_{x \in A} f(x) \quad \text{for each } A \in p.$$

The pair $(f_A, \geq)$ is a monotone increasing net. Using it we can define $\liminf_p f$ as follows

$$\liminf_p f \equiv \lim_{A \in p} f_A.$$

If p is a principal filter generated by a set $A \in \mathcal{F}$, then $\liminf_p f = \inf_{x \in A} f(x)$. This shows that $\liminf_p f$ is the appropriate generalization of $\inf_{x \in A} f(x)$ needed to take care of free filters.

We denote by F the set of all bounded functions $f : X \to \Re$ such that for every $t \in \Re$ the sets $\{x : f(x) > t\}$ and $\{x : f(x) \geq t\}$ belong to $\mathcal{F}$. For a monotone set function $\nu \in V$ and a function $f \in F$, the Choquet integral is defined as

$$\int f d\nu = \int_0^\infty \nu(\{x : f(x) \geq t\}) dt + \int_{-\infty}^0 [\nu(\{x : f(x) \geq t\}) - \nu(X)]\, dt$$

where the r.h.s. is a Riemann integral.

3 A locally convex topological vector space on V

A natural topology on V has as a local base at $\nu_0 \in V$ the sets of the form

$$B(\nu_0; A_1, \ldots, A_n; \varepsilon) = \{\nu \in V : |\nu_0(A_i) - \nu(A_i)| < \varepsilon \text{ for } 1 \leq i \leq n\}$$

where $A_i \in \mathcal{F}$ for $1 \leq i < n$, and $\varepsilon > 0$. We call this topology the $\mathcal{V}$-topology of V. In the next proposition we claim that under this topology the vector space V becomes a locally convex and Hausdorff topological vector space. The proof is standard, and it is therefore omitted.

Proposition 1 *Under the $\mathcal{V}$-topology the vector space V is a locally convex and Hausdorff topological vector space.*

The next proposition is rather important for the rest of the paper and it is a simple extension of Alaoglu Theorem to this set-up.

Proposition 2 *The set $\{\nu \in V : \|\nu\|_v \leq 1\}$ is $\mathcal{V}$-compact in V.*

Proof. Let MO be the set of all monotone set functions on $\mathcal{F}$. Set $K = \{\nu \in MO : \nu(X) \leq 1\}$. Let $I = \prod_{A \in \mathcal{F}} [0,1]_A$. By Tychonoff Theorem, I is compact w.r.t. the product topology. We can define a map $\tau : K \to I$ by $\tau(\nu) = \prod_{A \in \mathcal{F}} \nu(A)$. It is easy to check that τ is a homeomorphism between K endowed with the relative $\mathcal{V}$-topology, and $\tau(K)$ endowed with the relative product topology. Therefore, to prove that K is $\mathcal{V}$-compact it suffices to show that K is $\mathcal{V}$-closed. Let ν_α be a net in K that $\mathcal{V}$-converges to a game ν. Since $\lim_\alpha \nu_\alpha(A) = \nu(A)$ for all $A \in \mathcal{F}$, it is easy to check that $\nu \in MO$ and $\nu(X) \leq 1$. We conclude that K is $\mathcal{V}$-closed, and, therefore, $\mathcal{V}$-compact. Let $\nu \in BV^b$. On p. 28 of Aumann and Shapley (1974) it is proved that there exists a decomposition ν_1, ν_2 of ν such that $\nu_1, \nu_2 \in MO$ and $\|\nu\|_v = \|\nu_1\|_v + \|\nu_2\|_v$. Therefore,

$$\{\nu \in V : \|\nu\|_v \leq 1\} \subseteq K - K$$

Since V is a locally convex and Hausdorff topological vector space, the set $K - K$ is $\mathcal{V}$-compact (see e.g. Schaefer [1966], I.5.2). Therefore, to prove that $\{\nu \in V : \|\nu\|_v \leq 1\}$ is $\mathcal{V}$-compact it suffices to show that it is $\mathcal{V}$-closed. Let ν_α be a net in $\{\nu \in V : \|\nu\|_v \leq 1\}$

that $\mathcal{V}$-converges to a game ν. For each α there exists a decomposition $\nu_{1,\alpha}, \nu_{2,\alpha}$ such that both $\nu_{1,\alpha}$ and $\nu_{2,\alpha}$ are in K and $\|\nu_\alpha\|_v = \|\nu_{1,\alpha}\|_v + \|\nu_{2,\alpha}\|_v$. Since K is $\mathcal{V}$-compact and ν_α $\mathcal{V}$-converges to ν, there exist two subnets $\nu_{1,\beta}$ and $\nu_{2,\beta}$ that $\mathcal{V}$-converge, respectively, to two games ν_1 and ν_2 such that $\nu_1, \nu_2 \in K$ and $\nu = \nu_1 - \nu_2$. We can write:

$$\begin{aligned} \|\nu\|_v &= \|\nu_1 - \nu_2\|_v \leq \|\nu_1\|_v + \|\nu_2\|_v = \nu_1(X) + \nu_2(X) \\ &= \lim_\beta \{\nu_{1,\beta}(X) + \nu_{2,\beta}(X)\} = \lim_\beta \{\|\nu_{1,\beta}\| + \|\nu_{2,\beta}\|\} = \lim_\beta \|\nu_\beta\|. \end{aligned}$$

Therefore, $\|\nu\|_v \leq 1$, as wanted. □

4 Decomposition

In this section we prove the decomposition result mentioned in the introduction. The proof is rather different than that in Gilboa and Schmeidler (1995), and it is essentially based on the properties of the $\mathcal{V}$-topology. It is worth noting that this result is an extension to set functions in V^b of the well-known Jordan Decomposition Theorem for measures.

Theorem 2 *(i) Let $\nu \in V^b$. Then there exist two totally monotone games ν^+ and ν^- such that $\nu(A) = \nu^+(A) - \nu^-(A)$ for all $A \in \mathcal{F}$ and $\|\nu\| = \|\nu^+\| + \|\nu^-\|$. Moreover, this is the unique decomposition that satisfies the norm equation.*

(ii) The set $U(X) = \{\nu \in V : \|\nu\| \leq 1\}$ is $\mathcal{V}$-compact.

Proof. Let $TM = \{\nu \in V : \nu \text{ is totally monotone}\}$. Let $\nu_0 \in V^b$. Let

$$B(\nu_0; A_1, ..., A_n; \varepsilon)$$

be a neighborhood of ν_0. Let $\mathcal{F}(A_1, ..., A_n)$ be the algebra generated by $\{A_1, ..., A_n\}$. Since $\mathcal{F}(A_1, ..., A_n)$ is finite, Theorem 1 holds. Let $\mathcal{F}_+ = \{\varnothing \neq A \in \mathcal{F}(A_1, ..., A_n) : \alpha_T^\nu \geq 0\}$, and $\mathcal{F}' = \{A \in \mathcal{F}(A_1, ..., A_n) : A \neq \varnothing\}$. Set

$$\nu^+ = \sum_{T \in \mathcal{F}_+} \alpha_T^\nu u_T \quad \text{and} \quad \nu^- = \sum_{T \in \mathcal{F}' \setminus \mathcal{F}_+} (-\alpha_T^\nu)\, u_T.$$

As observed in Gilboa and Schmeidler (1994, p. 56), we have $\nu_0(A) = \nu^+(A) - \nu^-(A)$ for all $\mathcal{F}(A_1, ..., A_n)$ and $\left\|\nu_{0|\mathcal{F}(A_1,...,A_n)}\right\| = \|\nu^+\| + \|\nu^-\|$. Moreover, each unanimity game u_T is totally monotone on the entire algebra $\mathcal{F}$. Set $\nu = \nu^+ - \nu^-$. Clearly, $\nu = (\nu^+ - \nu^-) \in B(\nu_0; A_1, \ldots, A_n; \varepsilon)$.

Since ν^+ is totally monotone, $\|\nu^+\| = \nu^+(X)$. Therefore, $\|\nu^+\| = \left\|\nu^+_{|\mathcal{F}(A_1,\ldots,An)}\right\|$. Similarly, $\|\nu^-\| = \left\|\nu^-_{|\mathcal{F}(A_1,\ldots,An)}\right\|$. We now show that $\|\nu\| = \|\nu^+\| + \|\nu^-\|$. Since $\nu(A) = \nu_0(A)$ for all $A \in \mathcal{F}(A_1, \ldots, A_n)$, we have

$$\left\|\nu_{|\mathcal{F}(A_1,\ldots,A_n)}\right\| = \left\|\nu_{0\,|\mathcal{F}(A_1,\ldots,A_n)}\right\| = \left\|\nu^+_{|\mathcal{F}(A_1,\ldots,An)}\right\| + \left\|\nu^-_{|\mathcal{F}(A_1,\ldots,An)}\right\|.$$

Hence, $\|\nu\| \geq \left\|\nu^+_{|\mathcal{F}(A_1,\ldots,A_n)}\right\| + \left\|\nu^-_{|\mathcal{F}(A_1,\ldots,An)}\right\| = \|\nu^+\| + \|\nu^-\|$. On the other hand, since $\|\cdot\|$ is a norm, $\|\nu\| \leq \|\nu^+\| + \|\nu^-\|$. We conclude $\|\nu\| = \|\nu^+\| + \|\nu^-\|$, as claimed. Using this equality we can write:

$$\|\nu_0\| \geq \left\|\nu_{0|\mathcal{F}(A_1,\ldots,A_n)}\right\| = \left\|\nu^+_{|\mathcal{F}(A_1,\ldots,An)}\right\| + \left\|\nu^-_{|\mathcal{F}(A_1,\ldots,An)}\right\| = \left\|\nu^+\right\| + \left\|\nu^-\right\| \tag{1}$$

By what has just been proved, if we consider the family of all $\mathcal{V}$-neighborhoods of ν_0 as directed by the inclusion $\subseteq$, there exists a net ν_α that $\mathcal{V}$-converges to ν_0, and such that for all α we have:

(i) $\nu_\alpha = \nu^+_\alpha - \nu^-_\alpha$

(ii) $\|\nu_\alpha\| = \|\nu^+_\alpha\| + \|\nu^-_\alpha\|$

(iii) $\|\nu_\alpha\| \leq \|\nu_0\|$.

Set $M = \|\nu_0\|$ and $U^M(X) = \{\nu \in V : \|\nu\| \leq M\}$. If $\nu \in TM$, then

$$\|\nu\| = \|\nu\|_v = \nu(X).$$

Therefore, using Proposition 2, it is easy to check that the set $TM \cap U^M(X)$ is $\mathcal{V}$-compact. Since ν^+_α is a net in $TM \cap U^M(X)$, there exists a subnet ν^+_β that $\mathcal{V}$-converges to a game $\nu^+_0 \in TM \cap U^M(X)$. Since the net ν_α $\mathcal{V}$-converges, this implies that also the subnet ν^-_β (which is equal to $\nu^+_\beta - \nu_\beta$) $\mathcal{V}$-converges to a game $\nu^-_0 \in TM \cap U^M(X)$. Clearly, $\nu_0 = \lim_\beta(\nu^+_\beta - \nu^-_\beta) = \nu^+_0 - \nu^-_0$. Moreover:

$$\begin{aligned} \|\nu_0\| &\geq \lim_\beta \|\nu_\beta\| = \lim_\beta \left\{\left\|\nu^+_\beta\right\| + \left\|\nu^-_\beta\right\|\right\} = \lim_\beta \left\{\nu^+_\beta(X) + \nu^-_\beta(X)\right\} \\ &= \nu^+_0(X) + \nu^-_0(X) = \left\|\nu^+_0\right\| + \left\|\nu^-_0\right\| \end{aligned} \tag{2}$$

where the first inequality follows from expression (1). On the other hand, $\nu_0 = \nu_0^+ - \nu_0^-$ implies $\|\nu_0\| \leq \left\|\nu_0^+\right\| + \left\|\nu_0^-\right\|$. Together with (2), this implies $\|\nu_0\| = \left\|\nu_0^+\right\| + \left\|\nu_0^-\right\|$. This proves the existence of the decomposition. As to uniqueness, to prove it we need the techniques used in the proof of Theorem 3. Consequently, uniqueness is proved in the proof of Theorem 3.

As to part (ii), we first show that $U(X)$ is $\mathcal{V}$-closed. Let ν_α be a net in $U(X)$ that $\mathcal{V}$-converges to an element $\nu \in V^b$. By part (i), there exists a decomposition $\nu_\alpha = \nu_\alpha^+ - \nu_\alpha^-$ with $\nu_\alpha^+, \nu_\alpha^- \in U(X) \cap TM$ and $\|\nu_\alpha\| = \|\nu_\alpha^+\| + \|\nu_\alpha^-\|$. Proceeding as before, we can there exists two subnets ν_β^+ and ν_β^- that $\mathcal{V}$-converge, respectively, to ν^+ and ν^-, where $\nu^+, \nu^- \in TM \cap U(X)$. We have:

$$\begin{aligned} \|\nu\| &= \left\|\nu^+ - \nu^-\right\| \leq \left\|\nu^+\right\| + \left\|\nu^-\right\| = \nu^+(X) + \nu^-(X) \\ &= \lim_\beta \left\{\nu_\beta^+(X) + \nu_\beta^-(X)\right\} = \lim_\beta \left\{\left\|\nu_\beta^+\right\| + \left\|\nu_\beta^-\right\|\right\} = \lim_\beta \|\nu_\beta\| . \end{aligned}$$

Since $\|\nu_\beta\| \leq 1$ for all β, we can conclude $\|\nu\| \leq 1$, so that $\nu \in U(X)$. This proves that $U(X)$ is $\mathcal{V}$-closed. On the other hand, from part (i) it follows that $U(X) \subseteq [TM \cap U(X)] - [TM \cap U(X)]$. Since the set $TM \cap U(X)$ is $\mathcal{V}$-compact, also the set $[TM \cap U(X)] - [TM \cap U(X)]$ is $\mathcal{V}$-compact, and this implies that $U(X)$ is $\mathcal{V}$-compact, as desired. □

5 Countably additive representation of games in $\mathbf{V}^b$

We first introduce a class of simple games that will play a key role in the sequel. As observed in Shafer (1979), these games can already be found in Choquet (1953-54).

Definition 1 *Let p be a proper filter of $\mathcal{F}$. A normalized game $u_p(\cdot)$ on $\mathcal{F}$ is called a filter game if $u_p(A) = 1$ whenever $A \in p$, and $u_p(A) = 0$ whenever $A \notin p$. We denote by U_b the set of all filter games on $\mathcal{F}$.*

Unanimity games are a subclass of filter games. In particular, a filter game u_p is a unanimity game if and only if p is a principal filter. For, if p is a principal filter, i.e. $p = \{B \in \mathcal{F} : A \subseteq B\}$ for some $A \in \mathcal{F}$, then u_p coincides with the unanimity game u_A. In finite algebras all filters are principal, so that all filter games are unanimity games. This is no longer true in infinite algebras, where there are free filters to consider. For example, if $\mathcal{P}(X)$ is the power set of an infinite space, it is known that there are $2^{2^{|X|}}$ filters (see e.g. Balcar and Franek (1982)), and just $2^{|X|}$ of them are principal.

In sum, filter games are the natural generalization of unanimity games to infinite algebras. We now list some very simple properties of filter games.

Proposition 3

(i) A game is $\{0,1\}$-valued and convex if and only if it is a filter game.

(ii) Every filter game is totally monotone.

(iii) The set U_b is $\mathcal{V}$-compact in V.

Remark. Of course, (i) and (ii) together imply that a game is $\{0,1\}$-valued and totally monotone if and only if it is a filter game.

Proof. (i) "only if" part: let ν be a $\{0,1\}$-valued and convex game. Then ν is monotone. Let $p = \{A : \nu(A) = 1\}$. By monotonicity, if $A \in p$, then $B \in p$ whenever $A \subseteq B$. Now, let $A, B \in p$. Then $\nu(A) = \nu(B) = \nu(A \cup B) = 1$. By convexity,

$$1 = \nu(A) + \nu(B) - \nu(A \cup B) \leq \nu(A \cap B)$$

and so $\nu(A \cap B) = 1$. We conclude that p is a filter, and ν a filter game.

"If" part: tedious, but obvious.

(ii) Let u_p be a filter game, and $A_1, \ldots, A_n \in \mathcal{F}$. Let $I_* = \{i : A_i \in p\}$. If I_* is empty, the claim is obvious. Let $I_* \neq \varnothing$, with I_*=k. Let $C_{k,i}$ be a binomial coefficient. Then:

$$\sum_{\{I:\varnothing \neq I \subseteq \{1,\ldots,n\}\}} (-1)^{|I|-1} u_p\left(\bigcap_{i \in I} A_i\right) = \sum_{i=1}^{k} C_{k,i}(-1)^{i-1} = 1.$$

Since $u_p(\bigcup_{i=1}^{n} A_i) = 1$, it follows that u_p is totally monotone.

(iii) Let ν_α be a net in U_b that $\mathcal{V}$-converges to an element $\nu \in V$. By hypothesis, $\nu_\alpha(A) \in \{0,1\}$ for all $A \in \mathcal{F}$. Then $\nu(A) \in \{0,1\}$ for all $A \in \mathcal{F}$. Hence ν is $\{0,1\}$-valued. It is easy to check that ν is also convex. Therefore, by part (i), ν is a filter game, as wanted. □

Let $\mathcal{B}(U_b)$ be the Borel σ-algebra on the space U_b w.r.t. the $\mathcal{V}$-topology, and let $rca(U_b)$ be the set of all regular and bounded Borel measures on $\mathcal{B}(U_b)$. Moreover, set

$rca^+(U_b)$=$\{\mu \in rca(U_b) : \mu(A) \geq 0$ for all $A \in \mathcal{B}(U_b)\}$ and $rca_1(U_b)$=$\{\mu \in rca^+(U_b) : \mu(U_b) = 1\}$. When the space $rca(U_b)$ is endowed with the weak*-topology, we write $(rca(U_b), \tau_w)$. Similarly, (V^b, τ_v) denotes the space V^b endowed with the $\mathcal{V}$-topology. We remind that an isometric isomorphism between two normed spaces is a one-to-one continuous linear map which preserves the norm. As in the previous section, $U(X) = \{\nu \in V : \|\nu\| \leq 1\}$.

Theorem 3 *There is an isometric isomorphism J^* between $(V^b, \|\cdot\|)$ and $(rca(U_b), \|\cdot\|)$ determined by the identity*

$$\nu(A) = \int_{U_b} u_p(A) d\mu \qquad \text{for all } A \in \mathcal{F}. \tag{3}$$

The correspondence J^* is linear and isometric, i.e. $\|\mu\| = \|J^*(\nu)\| = \|\nu\|$. Moreover, ν is totally monotone if and only if the corresponding μ is nonnegative. Finally, the map J^* is a homeomorphism between $(V^b \cap U(X), \tau_v)$ and $(rca(U_b) \cap U(X), \tau_w)$.

In other words, we claim that for each $\nu \in V^b$ there is a unique $\mu \in rca(U_b)$ such that (3) holds; and, conversely, for each $\mu \in rca(U_b)$ there is a unique ν such that (3) holds.

Remark. The following proof is based on Theorem 2. The hard part is uniqueness. Instead, the proof of existence of the measure μ when ν is totally monotone is based on the well-known Dempster-Shafer-Shapley Representation Theorem for games defined on finite algebras (theorem 2.1), and on a simple compactness argument (similar to those used in Choquet Representation Theory (see Phelps [1965])). Other remarks on the existence part can be found after corollary (1) below.

Remark. Let $Fil(\mathcal{F})$ be the set of all filters of the algebra $\mathcal{F}$, and $Ult(\mathcal{F})$ the set of all ultrafilters. Up to a natural isomorphism, equation (3) can be written as $\nu(A) = \mu\{p \in Fil(\mathcal{F}) : A \in p\}$. If we replace $Fil(\mathcal{F})$ with $Ult(\mathcal{F})$, i.e. $\nu(A) = \mu\{p \in Ult(\mathcal{F}) : A \in p\}$, we have a version of the celebrated Stone Representation Theorem (see e.g. Sikorski (1969)). Note that in this case ν is additive. All this suggests that Theorem 3 may be thought as an extension of the Stone Theorem to non-additive set functions.

Proof. Let $TM_1 = \{\nu \in V : \nu \text{ is totally monotone and } ||\nu|| = 1\}$. Let

$$Una = \{u_T : T \in \mathcal{F} \text{ and } T \neq \varnothing\}$$

be the set of all unanimity games on $\mathcal{F}$. Let $\nu \in TM_1$, and let $B(\nu; A_1, ..., A_n; \varepsilon)$ be a neighborhood of ν. Let $\mathcal{F}(A_1, ..., A_n)$ be the algebra generated by $\{A_1, ..., A_n\}$. By Theorem (1) there exist $\alpha_T^\nu \geq 0$ such that

$$\nu(A) = \sum_{\{\varnothing \neq T \in \mathcal{F}(A_1, ..., A_n)\}} \alpha_T^\nu u_T(A)$$

for all $A \in \mathcal{F}(A_1, ..., A_n)$, $\alpha_T^\nu \geq 0$ for all $\varnothing \neq T \in \mathcal{F}(A_1, ..., A_n)$ and $\sum_T \alpha_T^\nu = 1$. Hence $\sum_{\varnothing \neq T \in \mathcal{F}(A_1, ..., A_n)} \alpha_T^\nu u_T(\cdot)$ belongs both to $B(\nu; A_1, ..., A_n; \varepsilon)$ and to $co(Una)$. This implies $TM_1 = cl\{co(Una)\}$. Clearly, $Una \subseteq U_b$. Moreover, we know by Proposition 3(ii) that $U_b \subseteq TM_1$. Therefore, $TM_1 = cl\{co(U_b)\}$. Hence, there exists a net λ_β contained in $co(U_b)$ that $\mathcal{V}$-converges to ν. For $A \in \mathcal{F}$, let $f_A : U_b \rightarrow \Re$ be defined by $f_A(u_p) = u_p(A)$. The map f_A is $\mathcal{V}$-continuous on U_b. By definition,

$$\lambda_\beta(A) = \sum_{j \in I_\beta} \alpha_j u_{p_j}(A)$$

for all $A \in \mathcal{F}$ and some finite index set I_β. Since $\sum_{j \in I_\beta} \alpha_j = 1$ and $\alpha_j \geq 0$, we can write

$$\lambda_\beta(A) = \int_{U_b} f_A d\mu_\beta$$

where $\mu_\beta(u_{p_j}) = \alpha_j$ if $j \in I_\beta$, and $\mu_\beta(u_{p_j}) = 0$ otherwise. Since, by Proposition 1(iii), U_b is $\mathcal{V}$-compact, it is known that $rca_1(U_b)$ is weak*-compact. Then there exists a subnet μ_γ of μ_β that weak*-converges to some $\mu_0 \in rca_1(U_b)$. Since f_A is $\mathcal{V}$-continuous, $\int_{U_b} f_A d\mu_\gamma$ converges to $\int_{U_b} f_A d\mu_0$ for all $A \in \mathcal{F}$. Set $\nu_\gamma(A) = \int_{U_b} f_A d\mu_\gamma$ for all $A \in \mathcal{F}$. The net $\nu_\gamma(A)$ converges to $\nu(A)$ for all $A \in \mathcal{F}$ because λ_β $\mathcal{V}$-converges to ν. Therefore, it follows that

$$\nu(A) = \int_{U_b} u_p(A) d\mu_0$$

for all $A \in \mathcal{F}$, and we conclude that μ_0 is the measure we were looking for.

We have therefore proved the existence of a correspondence J_1 between TM_1 and $rca_1(U_b)$, where J_1 is determined by $\nu(A) = \int_{U_b} u_p(A) d\mu$ for all $A \in \mathcal{F}$. Clearly, $\nu(X) = \int_{U_b} d\mu = \mu(U_b)$ because $X \in p$ for every filter p in $\mathcal{F}$. Hence, if $\mu \in J_1(\nu)$, then $||\nu|| = ||\mu||$.

Let $TM = \{\nu \in V^b : \nu \text{ is totally monotone}\}$. A simple argument now shows that there exists a correspondence J between TM and $rca^+(U_b)$, where J is determined by

$$\nu(A)=\int_{U_b} u_p(A)d\mu$$

for all $A \in \mathcal{F}$. Moreover, if $\mu \in J(\nu)$, then $\|\nu\| = \|\mu\|$.

Let $\nu \in V^b$. By Theorem 2 (existence part), there exist $\nu^+, \nu^- \in TM$ such that $\nu = \nu^+ - \nu^-$ and $\|\nu\| = \|\nu^+\| + \|\nu^-\|$. Let $\mu^+ \in J(\nu^+)$ and $\mu^- \in J(\nu^-)$. Set $\mu = \mu^+ - \mu^-$. Since $\{u_p : A \in p\} \in B(U_b)$, we have:

$$\nu(A) = \int_{U_b} u_p(A)d\mu^+ - \int_{U_b} u_p(A)d\mu^- = \int_{U_b} u_p(A)d\mu$$

for all $A \in \mathcal{F}$. We claim that $\|\nu\| = \|\mu\|$. On one hand, since $\|\nu^+\| = \|\mu^+\|$ and $\|\nu^-\| = \|\mu^-\|$, we have:

$$\|\mu\| \leq \left\|\mu^+\right\| + \left\|\mu^-\right\| = \left\|\nu^+\right\| + \left\|\nu^-\right\| = \|\nu\|.$$

On the other hand, let μ_1 and μ_2 be the Jordan decomposition of the signed measure μ. We have

$$\|\mu\| = \|\mu_1\| + \|\mu_2\| = \|\nu_1\| + \|\nu_2\| \geq \|\nu\|$$

where $\nu_i(A)=\int_{U_b} u_p(A)d\mu_i$ for all $A \in \mathcal{F}$ and $i = 1, 2$. The inequality holds because $\nu=\nu_1 - \nu_2$ by construction. We conclude that $\|\nu\| = \|\mu\|$, as wanted. We now prove that this μ is unique. Indeed, suppose to the contrary that there exist two signed regular Borel measures μ, μ' such that $\|\mu\| = \|\mu'\| = \|\nu\|$ and

$$\nu(A) = \int_{U_b} u_p(A)d\mu = \int_{U_b} u_p(A)d\mu'. \tag{4}$$

for all $A \in \mathcal{F}$. We first observe that $\|\mu\| = \|\mu'\| = \|\nu\| < \infty$ implies $\sup|\mu(B)| < \infty$ and $\sup|\mu'(B)| < \infty$, i.e. μ and μ' are bounded. Next, define a map s from $\mathcal{F}$ into the power set of U_b by $s(A) = \{u_p : A \in p\}$. Let $\Gamma = \{s(A) : A \in \mathcal{F}\}$. Every set $s(A)$ is $\mathcal{V}$-closed, so that $\Gamma \subseteq \mathcal{B}(U_b)$. From (4) it follows that μ and μ' coincide on Γ. The set Γ is a π-class (i.e. it is closed under intersection) because, as it is easy to check, it holds $s(A) \cap s(B) = s(A \cap B)$. Then μ and μ' coincide on $\mathcal{A}(\Gamma)$, the algebra generated by Γ. For, let $\mathcal{L}=\{B \subseteq U_b : \mu(B) = \mu'(B)\}$. We check that $\mathcal{L}$ is a λ-system (see e.g. Billingsley [1985] pp.36-38). Since $X \in p$ for every filter p in $\mathcal{F}$,(4) implies $U_b \in \mathcal{L}$.

Moreover, if $B \in L$, then $B^c \in \mathcal{L}$ because μ and μ' are additive. Finally, suppose $\{B_i\}_{i=1}^{\infty}$ is an infinite sequence of pairwise disjoint subsets of U_b. If $B_i \in \mathcal{L}$ for all $i \geq 1$, then $\bigcup_{i=1}^{\infty} B_i \in \mathcal{L}$ because both μ and μ' are countably additive. We conclude that $\mathcal{L}$ is a λ-system. By the $\pi - \lambda$ theorem (see e.g. Billingsley [1985] p.37), this implies that μ and μ' coincide on $\mathcal{A}(\Gamma)$, as wanted. The algebra $\mathcal{A}(\Gamma)$ is a base for a topology on U_b. Let us denote by τ_s such a topology. Next we prove that τ_s coincide with the relative $\mathcal{V}$-topology τ_v on U_b. Let $B(u_{p_0}; A_1, ..., A_n; \varepsilon)$ be a neighborhood of u_{p_0}. Set

$$I_1 = \{i \in \{1, ..., n\} : u_{p_0}(A_i) = 1\} \text{ and } I_2 = \{i \in \{1, ..., n\} : u_{p_0}(A_i) = 0\}.$$

and

$$G = \left[\bigcap_{i \in I_1} s(A_i)\right] \cap \left[\bigcap_{i \in I_2} (s(A_i))^c\right].$$

Of course, $G \in \mathcal{A}(\Gamma)$. Moreover, $u_{p_0} \in G$. If $p \in G$, then $u_p(A_i) = u_{p_0}(A_i)$ for all $1 \leq i \leq n$, so that $u_p \in B(u_{p_0}; A_1, ..., A_n; \varepsilon)$. Therefore, we conclude

$$u_{p_0} \in G \subseteq B(u_{p_0}; A_1, ..., A_n; \varepsilon)$$

and this implies $\tau_v \subseteq \tau_s$ because the sets of the form $B(u_p; A_1, ..., A_n; \varepsilon)$ are a local base for $\mathcal{V}$. As to the converse, let $G \in \mathcal{A}(\Gamma)$ be a neighborhood of u_{p_0}. W.l.o.g. the set G has the form $[\bigcap_{i \in I_1} s(A_i)] \cap [\bigcap_{i \in I_2} (s(A_i))^c]$. This follows from the usual procedure used for the construction of $\mathcal{A}(\Gamma)$ from Γ and from the fact that $u_{p_0} \in G$. Let us consider $B(u_{p_0}; A_1, ..., A_n; \varepsilon)$. Clearly, $u_{p_0} \in B(u_{p_0}; A_1, ..., A_n; \varepsilon)$. Let $u_p \in B(u_{p_0}; A_1, ..., A_n; \varepsilon)$. Then $u_p(A_i) = u_{p_0}(A_i)$ for all $1 \leq i \leq n$. This implies $A_i \in p$ if $i \in I_1$ and $A_i \notin p$ if $i \in I_2$. Then $u_p \in s(A_i)$ if $i \in I_1$ and $u_p \notin s(A_i)$ if $i \in I_2$. Consequently, $u_p \in G$. Therefore,

$$u_{p_0} \in B(u_{p_0}; A_1, ..., A_n; \varepsilon) \subseteq G$$

and this implies $\tau_s \subseteq \tau_v$. We can conclude $\tau_s = \tau_v$, as desired. Propositions 1 and 3 imply that τ_v is a compact Hausdorff space. From $\tau_s = \tau_v$ it follows that also τ_s is a compact Hausdorff space. Since μ and μ' are regular Borel measures on a compact Hausdorff space, they are τ-additive. For, let μ_1, μ_2 be the Jordan decomposition of μ, and let $\{G_\alpha\}$ be a net of open sets such that $G_\alpha \subseteq G_\beta$ for $\alpha \leq \beta$. Both μ_1 and μ_2 are regular (see Dunford and Schwartz [1957] p.137). Therefore, they are τ-additive (see Gardner [1981] p.47), i.e.

$$\lim_\alpha \mu_i(G_\alpha) = \mu_i(\bigcup_\alpha G_\alpha) \quad \text{for } i = 1, 2.$$

On the other hand, it holds

$$\mu(\bigcup_\alpha G_\alpha) = \mu_1(\bigcup_\alpha G_\alpha) - \mu_2(\bigcup_\alpha G_\alpha)$$
$$= \lim_\alpha \mu_1(G_\alpha) - \lim_\alpha \mu_2(G_\alpha) = \lim_\alpha [\mu_1(G_\alpha) - \mu_2(G_\alpha)] = \lim_\alpha \mu(G_\alpha)$$

and this proves that μ is τ-additive. A similar argument holds for μ'. Now, let G be an open et in τ_v. Since $\mathcal{A}(\Gamma)$ is a base for τ_s, we have $G = \bigcup_{i\in I} G_i$ where $G_i \in \mathcal{A}(\Gamma)$ for all $i \in I$. Let $|I|$ be the cardinal number of I. If $|I| \leq |\mathbb{N}|$ set $G_n^* = \bigcup_{i=1}^n G_i$. Since $\mathcal{A}(\Gamma)$ is an algebra, $G_n^* \in \mathcal{A}(\Gamma)$, so that countable additivity implies

$$\mu(G) = \lim_n \mu(G_n^*) = \lim_n \mu'(G_n^*) = \mu'(G).$$

If $|I|$ is any infinite cardinal, we can again order $\{G_i\}_{i\in I}$ so that $\{G_i\}_{i\in I} = \{G_\alpha : \alpha < |I|\}$ (greek letters denote ordinal numbers). Define G_α^* as follows:

(i) $G_1^* = G_1$;

(ii) if α is not a limit ordinal, then set $G_\alpha^* = G_{\alpha-1}^* \cup G_\alpha$;

(iii) if α is a limit ordinal, then set $G_\alpha^* = \bigcup_{\gamma<\alpha} G_\gamma^*$. To prove that $\mu(G) = \mu'(G)$ we can then use a transfinite induction argument on the increasing net of open sets G_α^*, an argument based on τ-additivity and on the fact that $\alpha < |I|$.

Of course, $\mu(F) = \mu'(F)$ for all closed subsets $F \subseteq U_b$. The class of all closed subsets is a π-class, and $\mathcal{B}(U_b)$ is the σ-algebra generated by the closed sets. We have already proved that $\mathcal{L}$=$\{B \subseteq U_b : \mu(B) = \mu'(B)\}$ is a λ-system. Therefore, by the $\pi - \lambda$ theorem, $\mathcal{B}(U_b) \subseteq L$, as wanted.

This completes the proof that $\mu = \mu'$. This implies that there exists a unique decomposition of ν that satisfies the norm equation. For, suppose there exist two pairs ν_1, ν_2 and ν_1', ν_2' such that

$$\nu(A) = \nu_1(A) - \nu_2(A) = \nu'(A) - \nu(A) \qquad \text{for all } A \in \mathcal{F}$$

and

$$\|\nu\| = \|\nu_1\| + \|\nu_2\| = \|\nu_1'\| + \|\nu_2'\| .$$

Let μ, μ_i and μ_i' be the unique measures on $\mathcal{B}(U_b)$ associated to ν, ν_i and ν_i' for $i = 1, 2$. It is easy to check that

$$\mu(s(A)) = \mu_1(s(A)) - \mu_2(s(A)) = \mu(s(A)) - \mu(s(A)) \qquad \text{for all } A \in \mathcal{F}$$

and

$$\|\mu\| = \|\mu_1\| + \|\mu_2\| = \|\mu_1'\| + \|\mu_2'\|.$$

It is then easy to check that

$$\mu(A) = \mu_1(A) - \mu_2(A) = \mu(A) - \mu(A) \qquad \text{for all } A \in \mathcal{A}(\Gamma).$$

Using transfinite induction as we did before, this equality can be extended to all open sets in U_b, and it is then easy to see that

$$\mu(A) = \mu_1(A) - \mu_2(A) = \mu(A) - \mu(A) \qquad \text{for all } A \in \mathcal{B}(U_b).$$

But, there is only a unique decomposition of μ on $\mathcal{B}(U_b)$ that satisfies the norm equation, i.e. the Jordan decomposition. Therefore, $\mu_i = \mu_i'$ for $i = 1, 2$ and so $\nu_i = \nu_i'$ for $i = 1, 2$ as desired.

We have already defined a correspondence J between TM and $rca^+(U_b)$. By what has been proved, this correspondence is indeed a function, i.e. $J(\nu)$ is a singleton for every $\nu \in TM$. Define a function J^* on V^b by $J^*(\nu) = J(\nu^+) - J(\nu^-)$, where ν^+, ν^- is the unique decomposition of ν that satisfies the norm equation. Clearly $J^*(\nu) \in rca(U_b)$, and J^* is onto. By now we know that $\mu \in J^*(\nu)$ implies $\|\nu\| = \|\mu\|$. Therefore, we conclude that J^* is an isometric isomorphism.

Finally, we show that J^* is an homeomorphism between $(V^b \cap U(X), \tau_v)$ and $(rca(U_b) \cap U(X), \tau_w)$. Since $V^b \cap U(X)$ is $\mathcal{V}$-compact and J^* is a bijection, it suffices to show that J^* is continuous on $V^b \cap U(X)$. Let ν_α be a net in $V^b \cap U(X)$ that $\mathcal{V}$-converges to an element $\nu \in V^b \cap U(X)$. Since $rca(U_b) \cap U(X)$ is weak*-compact, to show that $J^*(\nu_\alpha)$ weak*-converges to $J^*(\nu)$ it suffices to prove that every convergent subnet $J^*(\nu_\beta)$ of $J^*(\nu_\alpha)$ $\mathcal{V}$-converges to $J^*(\nu)$. Suppose $\lim_\beta J^*(\nu_\beta) = J^*(\nu')$. Then

$$\nu(A) = \lim_\beta \int \nu_\beta(A) = \lim_\beta \int_{U_b} u_p(A) dJ^*(\nu_\beta) = \int_{U_b} u_p(A) dJ^*(\nu')$$

Since J^* is bijective, this implies $\nu = \nu'$, as wanted. □

As a simple corollary of Theorem 3, we can obtain the following interesting result, proved in a completely different way in Choquet (1953-54). Let TM be the set of all totally monotone games on $\mathcal{F}$.

Corollary 1 *A game is an extreme point of the convex set* $\{\nu \in TM : \|\nu\| = 1\}$ *if and only if it is a filter game.*

Proof. It is well-known that the Dirac measures are the extreme points of the set of all regular probability measures defined on the Borel σ-algebra of a compact Hausdorff space. Since $\mathcal{B}(U_b)$ satisfies this condition, a simple application of Theorem 3 proves the result. □

As observed in Shafer (1979), using this result of Choquet, the existence part in Theorem 3 for totally monotone set functions can be obtained as a consequence of the celebrated Krein-Milman Theorem. However, we think that the simple proof of existence we have given, based on the well-know Dempster-Shafer-Shapley Representation Theorem for finite algebras, is more attractive in the context of this paper. Indeed, in section 7 it will proved that this technique leads to a new proof of the finitely additive representation of Revuz (1955-56) and Gilboa and Schmeidler (1995).

6 Integral representation

As a consequence of Theorem 3, we have the following representation result for the Choquet integral.

Theorem 4 *Let* ν *be a monotone set function in* V^b *and* $f \in F$. *Then*

$$\int_X f d\nu = \int_{U_b} \left[\liminf_p f\right] d\mu$$

where $\mu = J^*(\nu)$.

Proof. If $f \in F$ is a simple function, it is to check that

$$\int_X f d\nu = \int_{U_b} \left(\int_X f du_p\right) d\mu.$$

If $f \in F$ is not a simple function, then there exists a sequence of simple functions that converges uniformly to f. Since the standard convergence theorems hold for the Choquet integral under uniform convergence, we conclude that the above equality is true for any $f \in F$. To complete the proof we now show that $\int_{U_h} (\int_X f du_p) d\mu = \int_{U_h} [\liminf_p f] d\mu$. The pair $(u_A, \geq)$ is a net, where $\geq$ is the binary relation that directs p. We want to show that $\lim_{A \in p} u_A = u_p$. Let $B(u_p; A_1, ..., A_n; \varepsilon)$ be a neighborhood of u_p. Set $I = \{i \in \{1, ..., n\} : A_i \in p\}$. Suppose first that $I = \varnothing$. Then $u_A(A_i) = u_p(A_i) = 0$ for all $1 \leq i \leq n$ and all $A \in p$. This implies $u_A \in B(u_p; A_1, ..., A_n; \varepsilon)$. Suppose $I \neq \varnothing$. Set $T = \bigcap_{i \in I} A_i$. Let $A \in p$ such that $A \geq T$. Then $u_A(A_i) = u_p(A_i) = 0$ whenever $i \notin I$, and $u_A(A_i) = u_p(A_i) = 1$ whenever $i \in I$. Again, this implies $u_A \in B(u_p; A_1, ..., A_n; \varepsilon)$. All this proves that $\lim_{A \in p} u_A = u_p$. Then

$$\lim_{A \in p} \int_X f du_A = \int_X f du_p.$$

But $\int_X f du_A = \inf_{x \in A} f(x)$. Therefore

$$\begin{aligned} \int_{U_h} \left(\int_X f du_p \right) d\mu &= \int_{U_h} \left(\lim_{A \in p} \int_X f du_A \right) d\mu = \int_{U_h} \left[\lim_{A \in p} \left(\inf_{x \in A} f(x) \right) \right] d\mu \\ &= \int_{U_h} \left[\liminf_p f \right] d\mu \end{aligned}$$

as wanted. □

This result suggests a simple, but useful observation. Let $f : \mathcal{N} \to X$ be a bounded infinite sequence in X. For convenience, set

$$x_n = f(n) \quad \text{for all } n \geq 1.$$

Let us consider the power set $\mathcal{P}(\mathcal{N})$. Let p_c be the free filter of all cofinite subsets of $\mathcal{N}$, and δ_{p_c} the Dirac measure concentrated on p_c. Then

$$\int_X f du_{p_c} = \int_{U_h} \liminf_{p_c} f d\delta_{p_c} = \liminf_n x_n. \tag{5}$$

This shows that the lim inf of an infinite bounded sequence may be seen as a Choquet integral. This is interesting because Choquet integrals have been axiomatized as a decision criterion in the so-called Choquet subjective expected utility (CSEU, for short; see Schmeidler (1989)). As equation (5) shows, the ranking of two infinite payoff streams through their lim inf can then be naturally embedded in CSEU. Of course, here we interpret games as weighting functions over periods and not as beliefs. In repeated games choice criteria based on lim inf have played an important role (see e.g. Myerson (1991) ch.7). One might hope that elaborating on equation (5) a better understanding of the decision-theoretic bases of these criteria may be obtained. This is the subject of future research.

7 Finitely additive representation of games in V^b

In this section we give a proof of the finitely additive representation already proved, with an algebraic approach, in Revuz (1955-56) and in Gilboa and Schmeidler (1995). The proof we give is a modification of the proof we used to prove the countably additive representation of the previous section.

Let $Una = \{u_T \colon T \in \mathcal{F} \text{ and } T \neq \varnothing\}$ be the set of all unanimity games on $\mathcal{F}$. Unlike U_b, the set Una is not $\mathcal{V}$-compact. Consequently, the set of all bounded regular Borel measures on Una is not as well-behaved as it was on U_b. To get a representation theorem based on Una it is therefore natural to look directly at $ba(Una,\Sigma)$, the set of all bounded finitely additive measures on an appropriate algebra $\Sigma \subseteq 2^{Una(X,\mathcal{F})}$. Indeed, the unit ball in $ba(Una,\Sigma)$ is weak*-compact, whatever topological structure on X we have. The problem is now to find out an appropriate algebra Σ. For any $A \in \mathcal{F}$ let $f_A : Una \to \Re$ be defined by $f_A(u_T) = u_T(A)$. Moreover, let Ψ^* be the algebra generated by the sets $\{u_T : T \subseteq A \text{ and } \varnothing \neq T \in F\}$ for $A \in \mathcal{F}$. It turns out that the appropriate algebra Σ is just Ψ^*, which is also the smallest algebra w.r.t. the functions f_A belong to $B(Una,\Psi^*)$, where $B(Una,\Psi^*)$ denotes the closure w.r.t. the supnorm of the set of all simple functions on Ψ^*. Since, as it is easy to check, there is a one-to-one correspondence between Ψ^* and Ψ, where Ψ is the algebra generated by the sets of the form $\{T : T \subseteq A \text{ and } \varnothing \neq T \in \mathcal{F}\}$ for $A \in \mathcal{F}$, we finally have the following result.

Theorem 5 *There is an isometric isomorphism T^* between $(V^b, \|\cdot\|)$ and $(ba\,(Una, \Psi)\,, \|\cdot\|)$ determined by the identity*

$$\nu(A) = \int_{\{T \in \mathcal{F}: T \neq \varnothing\}} u_T(A) d\mu \qquad \text{for all } A \in \mathcal{F}. \tag{6}$$

Thus, for each $\nu \in V^b$ there is a unique $\mu \in ba(Una,\Psi)$ such that (6) holds; conversely, for each $\mu \in ba(Una,\Psi)$ there is a unique ν such that (6) holds. The correspondence T^* is linear and isometric, i.e. $\|\mu\| = \|T^*(\nu)\| = \|\nu\|$. Moreover, ν is totally monotone if and only if the corresponding μ is nonnegative.

In other words, we claim that for each $\nu \in V^b$ there is a unique $\mu \in ba(Una, \Psi)$ such that (6) holds; conversely, for each $\mu \in ba(Una, \Psi)$ there is a unique ν such that (6) holds.

Proof. Let Ψ^{**} be the algebra on Una generated by $\{u_T\}_{T\in Y}$ and Ψ^*. Let $TM_1 = \{\nu \in V : \nu \text{ is totally monotone and } \|\nu\| = 1\}$. From the proof of Theorem 3 we already know that $TM_1 = cl\{co(Una)\}$. Since the unit ball in $ba(Una, \Psi^{**})$ is weak*-compact, an existence argument similar to that used in the proof of theorem 3 proves that there exists a finitely additive probability measure $\mu^{**} \in ba(Una, \Psi^{**})$ such that:

$$\nu(A) = \int_{\{T\in\mathcal{F}:T\neq Z\}} u_T(A)d\mu^{**} \qquad \text{for all } A \in \mathcal{F}. \tag{7}$$

At this stage we have to consider the whole Ψ^{**} because measures on Una with finite support might not be in $ba(Una,\Psi^*)$, while they always are in $ba(Una, \Psi^{**})$. We can rewrite (7) as $\nu(A)= \int_Y f_T(A)d\mu^{**}$ for all $A \in \mathcal{F}$. Let μ^* be the restriction of μ^{**} on Ψ^*. Since $f_T \in B(X, \Psi^*)$, by lemma III.8.1 in Dunford and Schwartz (1957) we have:

$$\nu(A) = \int_{\{T\in\mathcal{F}:T\neq Z\}} u_T(A)d\mu^* \qquad \text{for all } A \in \mathcal{F}. \tag{8}$$

As to uniqueness, suppose there exists a probability measure $\mu' \in ba(Una, \Psi^*)$ such that

$$\nu(A)= \int_{\{T\in\mathcal{F}:T\neq Z\}} u_T(A)d\mu^*= \int_{\{T\in\mathcal{F}:T\neq Z\}} u_T(A)d\mu' \qquad \text{for all } A \in \mathcal{F}.$$

This implies that μ^* and μ' coincide on $\{u_T : T \subseteq A \text{ and } \varnothing \neq T \in \mathcal{F}\}$. Since the sets of the form $\{u_T : T \subseteq A \text{ and } \varnothing \neq T \in \mathcal{F}\}$ are a π-class and Ψ^* is generated by them, it is easy to check that $\mu^*=\mu'$, as wanted.

There is a one-to-one correspondence g between Ψ^* and Ψ such that for $A \in \mathcal{F}$ we have

$$g\left(\{u_T : T \subseteq A \text{ and } \varnothing \neq T \in \mathcal{F}\}\right) = \{T : T \subseteq A \text{ and } \varnothing \neq T \in \mathcal{F}\}.$$

Setting $\mu(g(A)) = \mu^*(A)$ for all $A \in \Psi^*$, we finally get

$$\nu(A) = \int_{\{T\in\mathcal{F}:T\neq Z\}} u_T(A)d\mu \qquad \text{for all } A \in \mathcal{F}. \tag{9}$$

We have therefore proved the existence of a bijection T_1 between TM_1 and the set of all probability measures in $ba(Una, \Psi)$, and T_1 is determined by $\nu(A)= \int_{\{T\in\mathcal{F}:T\neq Z\}} u_T(A)d\mu$ for all $A \in \mathcal{F}$. Clearly, $\nu(X)= \int_Y d\mu = \mu(Una)$ because $T \subseteq X$ for all $T \in \mathcal{F}$. Hence $\|\nu\| = \|T_1(\nu)\|$. This shows that T_1 is isometric.

The rest of the proof, i.e. the construction of the isometric isomorphism T^* that extends T_1 from TM_1 to V^b, can be done through a simple application of the decomposition obtained in theorem 2. This observation completes the proof. □

As a corollary we can obtain also Theorem E in Gilboa and Schmeidler (1995). It is interesting to compare this result with corollary 4. In the next corollary the argument of the Choquet integral is $\inf_{x\in T} f(x)$ because only unanimity games are considered, while in theorem 4 we used the more general $\liminf_p f$ because we integrated over all filter games, included those defined by free filters.

Corollary 2 *Let ν be a monotone set function in V^b and $f \in F$. Then*

$$\int_X f d\nu= \int_{Una} [\inf_{x\in T} f(x)]d\mu$$

where $\mu = T^*(\nu)$.

Remark. This corollary is a bit sharper than Theorem E in Gilboa and Schmeidler (1995). In fact, instead of V^b they use its subset

$$V^\sigma = \left\{\nu \in V^b : \mu = T^*(\nu) \text{ is a } \sigma\text{-additive signed measure}\right\}.$$

Proof. If $f \in F$, we can apply the same argument (based only on finite additivity) used in the first part of the proof of corollary 4 to prove that

$$\int_X f d\nu = \int_{Una} \left(\int_X f du_T\right) d\mu.$$

Since $\int_X f du_T = \inf_{x\in T} f(x)$, we get the desired result. □

8 Dual spaces

On the basis of the representation results proved in sections 5 and 6, in this section we give two different Banach spaces that have duals congruent to V^b, i.e. there exists an isometric isomorphism between V^b and these dual space.

Proposition 4 *The following Banach spaces have their duals congruent with the space* V^b*:*

(i) Let $\mathcal{C}(U_b)$ *be the space of all continuous functions on the space* U_b *endowed with the* $\mathcal{V}$*-topology, and let* $\|\cdot\|_s$ *be the supnorm. Then the dual space of the Banach space* $(\mathcal{C}(U_b), \|\|_s)$ *is congruent to* $(V^b, \|\cdot\|)$.

(ii) Set $H = \{T : T \in \mathcal{F} \text{ and } T \neq \varnothing\}$. *Let* $B(H, \Psi)$ *be the closure w.r.t. the supnorm of the set of all simple functions on* Ψ. *Then the dual space of the Banach space* $(B(H, \Psi), \|\cdot\|_s)$ *is congruent to* $\left(V^b, \|\cdot\|\right)$.

In Ruckle (1982) it is proved that $BV^b([0,1], \mathcal{B})$ is congruent with a dual space. We can obtain his result as a consequence of propositions 1 and 2, together with the following result from Functional Analysis (see e.g. theorem 23.A p.211 in Holmes [1975]).

Proposition 5 *Suppose that there is a Hausdorff locally convex topological space* τ *on a Banach space* X *such that the unit ball* $U(X)$ *is* τ*-compact. Then* X *is congruent to a dual space.*

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