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RANDOM-PLAYER GAMES

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ABSTRACT

This paper introduces games of incomplete information in which the number, as well as the identity, of the participating players is determined by chance. The participation of certain players may not be independent of the participation of others, and hence the very fact that a particular player was chosen to play may give that player a clue as to the number and the identity of the other players chosen. However, players have to choose their strategies before the identity of the other players is fully revealed to them and thus, effectively, before they know whether or not they will take part in the game. Pure-strategy, mix-strategy, and correlated equilibria of random-player games are defined. These definitions extend the corresponding definitions for finite games, Bayesian games with consistent beliefs, and Poisson games—all of which can be seen as special cases of random-player games. Sufficient conditions for the existence of pure- and mixed-strategy equilibria are given.

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1. Introduction

The aim of this paper is to set up a basis for the study of games where the participating players have incomplete information regarding the number and the identity of the other players. In games of incomplete information commonly studied by game theorists and economists the characteristics—or types—of the participating players may be incompletely known, but their number is assumed common knowledge. However, many game-like real life interactions, like auctions, elections, and road jamming, involve a random number of non-identical players, who have to decide on their course of action before even the number of other players is revealed to them.

Even though players may not know the number and the identity of the other players, they may nevertheless have certain beliefs about them. In this paper we assume that these beliefs are derived from a common prior that the potential players share. That is, the selection of actual players is viewed as a random event, having commonly known probabilistic properties. Potential players do not know in advance whether or not they will participate in the game; and once chosen to play, they must do so. The fact that a particular player was chosen to play is a private information of that player that may or may not give him a clue about the identity of the other players. Although initially equally ignorant, after being notified of their participation different players may come to have different beliefs, or posteriors, regarding who else is in the game.

The idea of a random set (of actual players) is captured by the mathematical notion of a simple point process. A simple point process is a mapping from some probability space into the space of finite (or, sometimes, countably infinite) subsets of a topological space (the space of potential players, in the present case). We recall several definitions and results from the theory of point processes on compact metric spaces in Section 2 below.

A formal definition of a random-player game, as well as several examples, is given in Section 3, where we also present the increasingly more general definitions of pure-strategy profile, mixed-strategy profile, and correlated strategy and define correlated equilibrium. Pure- and mixed-strategy equilibria are characterized as special cases of correlated equilibrium in Section 4. Sufficient conditions for the existence of pure- and mixed-strategy equilibria are given in Section 5.

2. Point Processes

The space $\mathfrak{N}(\mathfrak{X})$ of all (finite) integer-valued measures[1] on a compact metric space $\mathfrak{X}$ is topologically complete and separable in the weak topology (Kallenberg, 1983, p. 170). $\mathfrak{N}(\mathfrak{X})$ consists of all measures on $\mathfrak{X}$ that can be written as a finite sum of Dirac measures (Daley and Vere-Jones, 1988, p. 198). Thus, every element of $\mathfrak{N}(\mathfrak{X})$ has a finite support, X, and can be written as $\sum_{x \in X} n_x \delta_x$, where n_x is a positive integer that expresses the size of the atom $\{x\}$. An element of $\mathfrak{N}(\mathfrak{X})$ is called **simple** if all its atoms are of size one. The mapping that sends every such measure to its support is a one-to-one correspondence between the simple measures in $\mathfrak{N}(\mathfrak{X})$ and the finite subsets of $\mathfrak{X}$. A **point process** on $\mathfrak{X}$ is a random element in $\mathfrak{N}(\mathfrak{X})$, that is, a measurable mapping from some probability space into $\mathfrak{N}(\mathfrak{X})$. A point process is called **simple** if it is almost surely simple-valued. The **distribution** of a point process $\boldsymbol{\mu}$ is defined as the probability distribution it induces on $\mathfrak{N}(\mathfrak{X})$. This probability distribution is completely determined by the distribution of the random vectors $(\boldsymbol{\mu}(T_1), \boldsymbol{\mu}(T_2), \ldots, \boldsymbol{\mu}(T_m))$, $\{T_i\}_{1 \le i \le m}$ a finite measurable partition of $\mathfrak{X}$ (Kallenberg, 1983, p. 27). The space of probability distributions on $\mathfrak{N}(\mathfrak{X})$ is topologically complete and separable in the weak topology (Parthasarathy, 1967, Theorems II.6.2 and II.6.5). We will often identify point processes with their distributions.

<u>Examples.</u>

1. Every finite subset of $\mathfrak{X}$ can be seen as a simple point process.

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[1] In this paper, the measurable structure associated with a topological space is always assumed to be the Borel σ-algebra, i.e., the σ-algebra generated by the open sets.

2. Every probability measure ω on $T_1 \times T_2 \times \ldots \times T_n$, where $\{T_i\}_{1 \le i \le n}$ is a finite measurable partition of $\mathfrak{X}$, defines a simple point process on $\mathfrak{X}$ (or, rather, a distribution of such a point process). Exactly one point is selected from each subset T_i.

3. A **Bernoulli** (or sample) process is defined as a sampling (with replacement) of n elements in $\mathfrak{X}$, chosen independently according to a fixed probability measure λ on $\mathfrak{X}$. Formally, a Bernoulli process may be written as $\sum_{i=1}^{n} \delta_{x_i}$, where $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are i.i.d. random elements in $\mathfrak{X}$.

4. If, in the previous example, we take n to be a random variable, rather than a fixed number, then a **mixed** Bernoulli process results. A mixed Bernoulli process is simple if and only if λ is nonatomic or $P(n>1)=0$.

5. A mixed Bernoulli process is called a **Poisson process** if n is a Poisson random variable.

The requirement that the process be simple is often added to the definition of a Poisson process (see, e.g., Kingman, 1993). The importance of the simple Poisson process lies in the fact that this is the only simple point process in which no point is selected with positive probability, and different points are selected or not selected independently of one another (Kallenberg, 1983, p. 58):

Proposition 1. *Suppose that the point process $\boldsymbol{\mu}$ satisfies $\boldsymbol{\mu}(\{x\})=0$ almost surely for every x in $\mathfrak{X}$. Then $\boldsymbol{\mu}$ is a Poisson process if and only if it is simple and $\boldsymbol{\mu}(A)$ and $\boldsymbol{\mu}(A')$ are independent whenever A and A' are disjoint measurable subsets of $\mathfrak{X}$.*

For every point process $\boldsymbol{\mu}$, the set function $A \mapsto E\boldsymbol{\mu}(A)$ is a measure on $\mathfrak{X}$, denoted $E\boldsymbol{\mu}$. We will call it the **mean measure** of $\boldsymbol{\mu}$. If the mean measure of $\boldsymbol{\mu}$ is finite, then there exists a family $\{\boldsymbol{\mu}_x\}$ of point processes, indexed by the elements of $\mathfrak{X}$, such that, for every bounded measurable function $h : \mathfrak{X} \times \mathfrak{N}(\mathfrak{X}) \to \mathbb{R}$,

$$(1) \qquad \int E h(x, \boldsymbol{\mu}_x)\, E\boldsymbol{\mu}(dx) = E \int h(x, \boldsymbol{\mu} - \delta_x)\, \boldsymbol{\mu}(dx)$$

(Kallenberg, 1983, p. 84). The distribution of $\boldsymbol{\mu}_x$ can be interpreted as the conditional distribution of $\boldsymbol{\mu} - \delta_x$, given that $\boldsymbol{\mu}(\{x\}) \ge 1$. The mapping that sends x to the distribution of $\boldsymbol{\mu}_x$ is a measurable function on $\mathfrak{X}$. If $\{\boldsymbol{\mu}'_x\}$ is another family of point processes which satisfy (1) for every bounded measurable function h, then the distribution of $\boldsymbol{\mu}'_x$ is equal to the distribution of $\boldsymbol{\mu}_x$ for $\mathrm{E}\boldsymbol{\mu}$-almost every x (this follows from Theorem V.8.1 of Parthasarathy, 1967).

Example. Let $\boldsymbol{\mu}$ be a mixed Brenoulli process, i.e., n independent draws of elements in $\mathfrak{X}$ according to a fixed probability measure λ, where n is random. If the expected number of draws, $\mathrm{E}\boldsymbol{\mu}(\mathfrak{X})$, is finite, then $\mathrm{E}\boldsymbol{\mu} = \mathrm{E}\boldsymbol{\mu}(\mathfrak{X})\ \lambda$. If, in addition, it is strictly positive, then, for $\mathrm{E}\boldsymbol{\mu}$-almost every x, $\boldsymbol{\mu}_x$ is also a mixed Bernoulli process, namely, n' independent draws of elements in $\mathfrak{X}$ according to λ, and $\mathrm{P}(n'=k-1) = k/\mathrm{E}\boldsymbol{\mu}(\mathfrak{X})\ \mathrm{P}(n=k)$ for k=1,2,.... It follows that, *conditioned on the number of draws*, $\boldsymbol{\mu}_x$ has the same distribution as $\boldsymbol{\mu}$; and n and n' are identically distributed if and only if n has a Poisson distribution.

Furthermore (Kallenberg, 1983, p. 97 and 101),

Proposition 2. *Let $\boldsymbol{\mu}$ be a point process such that* $\mathrm{E}\boldsymbol{\mu}(\mathfrak{X})$ *is finite. Then the distribution of $\boldsymbol{\mu}_x$ is the same for* $\mathrm{E}\boldsymbol{\mu}$*-almost every x if and only if $\boldsymbol{\mu}$ is a mixed Bernoulli process, and in such a case $\boldsymbol{\mu}_x$ is also a mixed Bernoulli process. The distribution of $\boldsymbol{\mu}_x$ is equal to the distribution of $\boldsymbol{\mu}$, for* $\mathrm{E}\boldsymbol{\mu}$*-almost every x, if and only if $\boldsymbol{\mu}$ is a Poisson process.*

3. THE MODEL.

The space of **potential players** is a compact metric space $\mathfrak{X}$.[2] A random finite set of **actual players** is given as a simple point process $\boldsymbol{X}$ on $\mathfrak{X}$ satisfying $E\boldsymbol{X}(\mathfrak{X})<\infty$.[3] Strategy sets are defined by a continuous function ξ from a compact metric space $\mathfrak{S}$ into $\mathfrak{X}$;[4] the **strategy set** of player x is $\xi^{-1}(\{x\})$.[5] The payoff $u(s,S)$ to an actual player who plays s when the plays of the other actual players are S is given by a bounded and measurable **utility function** $u:\mathfrak{S}\times\mathfrak{N}(\mathfrak{S})\rightarrow\mathbb{R}$. For every x and S, the restriction of $u(\cdot,S)$ to $\xi^{-1}(\{x\})$ is assumed continuous. The quintuple $(\mathfrak{X},\boldsymbol{X},\mathfrak{S},\xi,u)$ is called a **random-player game**.

Examples.

1. Every finite game, and more generally every finite-player game with compact strategy sets and bounded and measurable payoff functions, can be viewed as a random-player game; $\boldsymbol{X}$ in this case is a constant set.

2. Every Bayesian game Γ with a finite set of players $N=\{1,2,\ldots,n\}$, compact, disjoint sets of types $\{T_i\}_{i\in N}$, compact sets of actions $\{A_i\}_{i\in N}$, bounded and measurable payoff functions $\{u_i:\times_i T_i\times\times_i A_i\rightarrow\mathbb{R}\}_{i\in N}$, and consistent beliefs derived from a common prior ω can be seen as a random-player game. Potential players in this random-player game correspond to types in Γ. Formally, the player

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[2] The generalization to locally compact separable metric spaces (such as $\mathbb{R}$) is immediate: if $\mathfrak{X}$ is such a space, that $\boldsymbol{X}$ can be defined as a point process on its one-point *compactification*.

[3] Throughout this paper, random elements of a topological space are denoted by bold letters. Sets are denoted by capital letters. Since simple measures are of a dual nature, the realizations of $\boldsymbol{X}$, and of other simple point processes, will sometimes be treated as measures, sometimes as sets.

[4] Note that the inverse relation, ξ^{-1}, is an upper semicontinuous correspondence (multifunction) with domain $\mathfrak{X}$. We could, of course, use that correspondence for defining the strategy sets.

[5] The strategy sets of different players are thus assumed to be disjoint. This harmless assumption makes for a considerable simplification of notation.

space is $\mathfrak{X}=\bigcup_i T_i$. $\boldsymbol{X}$ is defined as in Example 2 in Section 2. ξ is defined on $\bigcup_i (T_i \times A_i)$ as the projection on the first coordinate, and u is defined by $u((t_i,a_i),\{(t_j,a_j)\}_{j\in J}) = u_i(t_1,t_2,\ldots,t_n;a_1,a_1,\ldots,a_n)$ if $t_i \in T_i$, $J=N\setminus\{i\}$, and $t_j \in T_j$ for all $j\in N$, and $= 0$ otherwise.

3. A **Poisson game** (Myerson, 1994) is a random-player game where $\boldsymbol{X}$ is a Poisson process, and there exists a finite partition $\{A_i\}_{1\le i\le m}$ of $\mathfrak{S}$ such that $u(s,S) = u(s',S')$ whenever s and s' are in the same element of $\{A_i\}_{1\le i\le m}$ and $S(A_i)=S'(A_i)$ for every $1\le i\le m$. (The total number of strategies in a Poisson game is thus effectively finite.)

A **pure-strategy profile** is a measurable function from $\mathfrak{X}$ to $\mathfrak{S}$ that assigns to each potential player x an element of $\xi^{-1}(\{x\})$. Since ξ is a continuous function defined on a compact domain, a pure-strategy profile exists (Parthasarathy, 1967, Theorem I.4.2 and Corollary I.3.3). A **mixed-strategy profile** is an assignment of a random element $\boldsymbol{s}_x$ in $\xi^{-1}(\{x\})$ to every potential player x such that, for every measurable set $A\subseteq\mathfrak{S}$, the function $x \mapsto \mathrm{P}(\boldsymbol{s}_x\in A)$ is measurable. A point process $\boldsymbol{S}$ on $\mathfrak{S}$ is a **correlated strategy** if the distribution of $\boldsymbol{S}\circ\xi^{-1}$ is equal to the distribution of $\boldsymbol{X}$.

Every mixed-strategy profile $\{\boldsymbol{s}_x\}_{x\in\mathfrak{X}}$ (and, hence, in particular, every pure-strategy profile) can be identified with a particular correlated strategy. The distribution of this correlated strategy $\boldsymbol{S}$ is given by

(2) $$\mathrm{E}f(\boldsymbol{S}) = \mathrm{E}_{\boldsymbol{X}}\, \mathrm{E}_{\{\boldsymbol{s}_x\}_{x\in\boldsymbol{X}}} f(\{\boldsymbol{s}_x\}_{x\in\boldsymbol{X}}),$$

for every bounded measurable function $f : \mathfrak{N}(\mathfrak{S}) \to \mathbb{R}$. Thus, $\boldsymbol{S}$ results from first choosing a realization X of $\boldsymbol{X}$ and then, independently for each element x in X, choosing a realization of $\boldsymbol{s}_x$. Note that Equation (2) implies

(3) $$\int g(s)\, \mathrm{E}\boldsymbol{S}(ds) = \int \mathrm{E}g(\boldsymbol{s}_x)\, \mathrm{E}\boldsymbol{X}(dx),$$

for every bounded measurable function $g : \mathfrak{S} \to \mathbb{R}$.

A pure-, mixed-, or correlated strategy (profile) $\boldsymbol{S}$ is, respectively, a pure-strategy, mixed-strategy, or correlated **equilibrium** if, for $\mathrm{E}\boldsymbol{S}$-almost every s, $\mathrm{E}u(s,\boldsymbol{S}_s)\ge$

$\mathrm{E}u(s',\boldsymbol{S}_s)$ for every s' in $\xi^{-1}(\{\xi(s)\})$. The point process $\boldsymbol{S}_s$, defined as in Equation (1) above, is interpreted as the posterior of the actual player that plays s on the plays of the other actual players. The equilibrium condition therefore requires s to be a best response for that player against his posterior. This condition can also be stated in terms of the expected aggregate utility (which is the expression on the left-hand side of Equation (4) below):

Proposition 3. *A correlated strategy* $\boldsymbol{S}$ *is an equilibrium if and only if*

$$\mathrm{E} \sum_{s \in \boldsymbol{S}} u(s, \boldsymbol{S} \setminus \{s\}) = \max_{\varphi} \mathrm{E} \sum_{s \in \boldsymbol{S}} u(\varphi(s), \boldsymbol{S} \setminus \{s\}), \tag{4}$$

where the maximum is taken over the set of all measurable functions $\varphi : \mathfrak{S} \to \mathfrak{S}$ *that satisfy* $\xi \circ \varphi = \xi$.

Proof. The restriction of the measurable function $(s, s') \mapsto \mathrm{E}u(s', \boldsymbol{S}_s)$ to the graph of the compact-valued upper semicontinuous correspondence $s \mapsto \xi^{-1}(\{\xi(s)\})$ is continuous in its second argument. Therefore, there exists a measurable selection of that correspondence, $\varphi : \mathfrak{S} \to \mathfrak{S}$, which satisfies $\mathrm{E}u(\varphi(s), \boldsymbol{S}_s) = \max_{s' \in \xi^{-1}(\{\xi(s)\})} \mathrm{E}u(s', \boldsymbol{S}_s)$ for every s (Wagner, 1980, Theorem 3.7). Since, by (1), $\mathrm{E} \sum_{s \in \boldsymbol{S}} u(\varphi(s), \boldsymbol{S} \setminus \{s\}) = \int \mathrm{E}u(\varphi(s), \boldsymbol{S}_s)\, \mathrm{E}\boldsymbol{S}(ds)$, the maximum on the right-hand side of (4) is attained at that function φ. Therefore, (4) is equivalent to $\int \mathrm{E}u(s, \boldsymbol{S}_s)\, \mathrm{E}\boldsymbol{S}(ds) = \int \max_{s' \in \xi^{-1}(\{\xi(s)\})} \mathrm{E}u(s', \boldsymbol{S}_s)\, \mathrm{E}\boldsymbol{S}(ds)$, and this equation is clearly equivalent to $\boldsymbol{S}$ being a correlated equilibrium. □

An immediate corollary of Proposition 3 is that the set of correlated equilibrium *distributions* is closed and convex.

4. Mixed-Strategy Equilibria

Given a mixed-strategy profile, $\boldsymbol{S}$, the posterior of an actual player x on the plays of the other actual players does not depend on that player's action (because the realizations of the mixed strategies of different actual players are chosen independently). This posterior, $\boldsymbol{S}_x$, is only a function of the player's posterior $\boldsymbol{X}_x$ on the identity of the other actual players and of the mixed strategies of these players. More precisely,

Lemma 1. *If $\boldsymbol{S}$ is a mixed-strategy profile, then*

(5) $$\boldsymbol{S}_s = \boldsymbol{S}_{\xi(s)}$$

holds for E$\boldsymbol{S}$*-almost every s, where, for $x \in \mathfrak{X}$, the distribution of $\boldsymbol{S}_x$ is defined by*

(6) $$\mathrm{E}f(\boldsymbol{S}_x) = \mathrm{E}_{\boldsymbol{X}_x}\, \mathrm{E}_{\{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X}_x}} f(\{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X}_x}),$$

for every bounded measurable function $f : \mathfrak{N}(\mathfrak{S}) \to \mathbb{R}$. If $\boldsymbol{X}$ is a mixed Bernoulli process, then so is $\boldsymbol{S}$, and the distribution of $\boldsymbol{S}_x$ is the same for E$\boldsymbol{X}$*-almost every x. The distribution of $\boldsymbol{S}_x$ is equal to the distribution of $\boldsymbol{S}$, and $\boldsymbol{S}$ is a Poisson process, if $\boldsymbol{X}$ is a Poisson process.*

Proof. By (3), (6), (1) and (2), for every bounded measurable function $h: \mathfrak{S} \times \mathfrak{N}(\mathfrak{S}) \to \mathbb{R}$, $\int \mathrm{E}h(s, \boldsymbol{S}_{\xi(s)})\ \mathrm{E}\boldsymbol{S}(ds) = \int \mathrm{E}_{\boldsymbol{s}_x} \mathrm{E}_{\boldsymbol{S}_x} h(\boldsymbol{s}_x, \boldsymbol{S}_x)\ \mathrm{E}\boldsymbol{X}(dx) = \int \mathrm{E}_{\boldsymbol{X}_x} \mathrm{E}_{\{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X}_x}} \mathrm{E}_{\boldsymbol{s}_x} h(\boldsymbol{s}_x, \{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X}_x})$ $\mathrm{E}\boldsymbol{X}(dx) = \mathrm{E}_{\boldsymbol{X}} \sum_{x \in \boldsymbol{X}} \mathrm{E}_{\{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X} \setminus x}} \mathrm{E}_{\boldsymbol{s}_x} h(\boldsymbol{s}_x, \{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X} \setminus \{x\}}) = \mathrm{E} \sum_{s \in \boldsymbol{S}} h(s, \boldsymbol{S} \setminus \{s\})$. Comparing this with Equation (1), we conclude that the distribution of $\boldsymbol{S}_s$ is equal to the distribution of $\boldsymbol{S}_{\xi(s)}$ for E$\boldsymbol{S}$-almost every s. If $\boldsymbol{X}$ is a mixed Bernoulli process, then by Proposition 2 we may assume that the distribution of $\boldsymbol{X}_x$ is independent of x, and is equal to the distribution of $\boldsymbol{X}$ if $\boldsymbol{X}$ is a Poisson process. It follows, by (6) and (2), that the distribution of $\boldsymbol{S}_x$ is independent of x if $\boldsymbol{X}$ is a mixed Bernoulli process, and is equal to the distribution of $\boldsymbol{S}$ if $\boldsymbol{X}$ is a Poisson process. Therefore, by Equation (5) and Proposition 2, the point processes $\boldsymbol{S}$, $\boldsymbol{S}_s$, and $\boldsymbol{S}_x$ are mixed Bernoulli processes if $\boldsymbol{X}$ is a mixed Bernoulli process, and have the same distribution if $\boldsymbol{X}$ is a Poisson process. □

A mixed-strategy equilibrium is characterized by the actual players' mixed strategies being best responses to these players' posteriors:

Proposition 4. *A mixed-strategy profile $\boldsymbol{S}$ is an equilibrium if and only if, for* E$\boldsymbol{X}$*-almost every x,*

(7) $$\mathrm{E}_{\boldsymbol{s}_x, \boldsymbol{S}_x} u(\boldsymbol{s}_x, \boldsymbol{S}_x) = \max_{s \in \xi^{-1}(\{x\})} \mathrm{E}_{\boldsymbol{S}_x} u(s, \boldsymbol{S}_x).$$

Proof. $\boldsymbol{S}$ is an equilibrium if and only if $\int \mathrm{E}u(s,\boldsymbol{S}_s)\ \mathrm{E}\boldsymbol{S}(ds) = \int \max_{s' \in \xi^{-1}(\{\xi(s)\})} \mathrm{E}_{\boldsymbol{S}_s}\ u(s',\boldsymbol{S}_s)\ \mathrm{E}\boldsymbol{S}(ds)$. By (5) and (3), this condition is equivalent to $\int \mathrm{E}_{s_x}\ \mathrm{E}_{\boldsymbol{S}_x}\ u(\boldsymbol{s}_x,\boldsymbol{S}_x)\ \mathrm{E}\boldsymbol{X}(dx) = \int \max_{s \in \xi^{-1}(\{x\})} \mathrm{E}_{\boldsymbol{S}_x} u(s,\boldsymbol{S}_x)\ \mathrm{E}\boldsymbol{X}(dx)$. And this equation clearly holds if and only if Equation (7) holds for $\mathrm{E}\boldsymbol{X}$-almost every x. □

The condition for a mixed-strategy profile to be an equilibrium can also be formulated in terms of the expected aggregate utility. Specifically, as an immediate corollary of Proposition 3 and Equation (2) we have

Proposition 5. *A mixed-strategy profile is an equilibrium if and only if, for every measurable functions $\varphi : \mathfrak{S} \to \mathfrak{S}$ that satisfies $\xi \circ \varphi = \xi$*

(8) $\mathrm{E}_{\boldsymbol{X}} \sum_{x \in \boldsymbol{X}} \mathrm{E}_{\{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X}}} u(\boldsymbol{s}_x, \{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X} \setminus \{x\}}) \geq \mathrm{E}_{\boldsymbol{X}} \sum_{x \in \boldsymbol{X}} \mathrm{E}_{\{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X}}} u(\varphi(\boldsymbol{s}_x), \{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X} \setminus \{x\}})$.

5. Existence of an Equilibrium

Milgrom and Weber (1985) proved an equilibrium existence theorem for Bayesian games in which beliefs are derived from a common prior. We now show that, under conditions rather similar to theirs, a mixed-strategy equilibrium exists in a general random-player game.

Theorem 1. *If the distribution of $\boldsymbol{X}$ is absolutely continuous with respect to the distribution of some mixed Bernoulli process $\boldsymbol{X}'$, and if u is bounded and continuous, then a mixed-strategy equilibrium exists.*

Proof. Let $f : \mathfrak{N}(\mathfrak{X}) \to \mathbb{R}_+$ (a Radon-Nikodym derivative) be a measurable function such that $\mathrm{E}\ f(\boldsymbol{X}')\ h(\boldsymbol{X}') = \mathrm{E}\ h(\boldsymbol{X})$ holds for every bounded measurable function $h : \mathfrak{N}(\mathfrak{X}) \to \mathbb{R}$. The expression on the right-hand side of (8) is equal to

$$\mathrm{E}_{\boldsymbol{X}'} \sum_{x \in \boldsymbol{X}'} f(\boldsymbol{X}')\ \mathrm{E}_{\{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X}'}} u(\varphi(\boldsymbol{s}_x), \{\boldsymbol{s}_{x'}\}_{x' \in \boldsymbol{X}' \setminus \{x\}}),$$

and can also be written as

$$(9) \qquad \sum_{n=1}^{\infty} p_n \int\int\ldots\int n \cdot \mathrm{E}_{s_{x_1}, s_{x_2}, \ldots, s_{x_n}} u(\varphi(\boldsymbol{s}_{x_1}), \{\boldsymbol{s}_{x_2}, \ldots, \boldsymbol{s}_{x_n}\}) f(\{x_1, x_2, \ldots, x_n\}) \prod_{i=1}^{n} \lambda(dx_i),$$

where $p_n := \mathrm{P}(\boldsymbol{X}'(\mathfrak{X}) = n)$ and λ is a probability measure on $\mathfrak{X}$ (see the definition of mixed Bernoulli process in Section 2). Let $\mathfrak{M}_\lambda$ be the set of all probability measures η on $\mathfrak{S}$ that satisfy the equation $\eta \circ \xi^{-1} = \lambda$. If $\xi \circ \varphi = \xi$, then both the measure ν defined by

$$(10) \qquad \int g(s)\, \nu(ds) = \int \mathrm{E} g(\boldsymbol{s}_x)\, \lambda(dx),$$

for every bounded measurable function $g : \mathfrak{S} \to \mathbb{R}$, and the measure $\nu \circ \varphi^{-1}$ are in $\mathfrak{M}_\lambda$, and (9) is equal to $F(\nu \circ \varphi^{-1}, \nu)$, where $F : \mathfrak{M}_\lambda \times \mathfrak{M}_\lambda \to \mathbb{R}$ is defined by

$$F(\eta', \eta) := \sum_{n=1}^{\infty} n \cdot p_n \int\int\ldots\int u(s_1, \{s_2, \ldots, s_n\}) f(\{\xi(s_1), \xi(s_2), \ldots, \xi(s_n)\})\ \eta'(ds_1) \prod_{i=2}^{n} \eta(ds_i).$$

It follows, by Proposition 5, that a sufficient condition for $\{\boldsymbol{s}_x\}_{x\in\mathfrak{X}}$ to be a mixed-strategy equilibrium is that the function $F(\cdot, \nu)$ takes its maximum at ν. Equivalently, $\{\boldsymbol{s}_x\}_{x\in\mathfrak{X}}$ is a mixed-strategy equilibrium if ν is a fixed point of the correspondence ψ defined on $\mathfrak{M}_\lambda$ by $\psi(\eta) = \{\eta' \in \mathfrak{M}_\lambda \mid \eta' \text{ maximizes } F(\cdot, \eta)\}$.

For every measure ν in $\mathfrak{M}_\lambda$, there exists a mixed-strategy profile $\{\boldsymbol{s}_x\}_{x\in\mathfrak{X}}$ (a regular conditional probability distribution; see Parthasarathy, 1967, p. 147) such that (10) holds. To prove that a mixed-strategy equilibrium exists, it therefore suffices to show that ψ has a fixed point. Since $\mathfrak{S}$ is compact, the set $\mathfrak{M}_\lambda$ is relatively compact (Parthasarathy, 1967, p. 45). In fact, this set is easily seen to be a compact convex subset of the locally convex topological linear space of all finite signed measures on $\mathfrak{S}$, topologized by weak convergence. Therefore, ψ has a fixed point if it is upper semicontinuous and $\psi(\eta)$ is a nonempty closed convex subset of $\mathfrak{M}_\lambda$ for every $\eta \in \mathfrak{M}_\lambda$ (Fan, 1952). To prove that ψ satisfies these conditions, it suffices to show that F is continuous.

For every n such that $p_n > 0$, the function $f(\{x_1, x_2, \ldots, x_n\})$ is λ^n-integrable. Therefore, for every $k \geq 1$ there exists a bounded continuous function $f_{nk} : \mathfrak{X}^n \to \mathbb{R}$ whose distance from that function in the $L_1(\lambda^n)$ norm is less than $1/(n \cdot k)$ (Halmos, 1950, p. 242).

Since the mapping that sends two probability measures into their product is continuous (Billingsley, 1968, p. 21.),

$$F_k(\eta',\eta):=\sum_{n=1}^{k} n\cdot p_n \int\int\ldots\int u(s_1,\{s_2,\ldots,s_n\})\, f_{nk}(\xi(s_1),\xi(s_2),\ldots,\xi(s_n))\ \eta'(ds_1)\prod_{i=2}^{n}\eta(ds_i)$$

is a continuous function for every k. The continuity of F now follows from the fact that $F_k \to F$ uniformly. Indeed, if M is a bound on u then, for every η and η' in $\mathfrak{M}_\lambda$ and for every k, $|F(\eta',\eta)-F_k(\eta',\eta)|\leq$

$$\leq\sum_{n=1}^{\infty} Mnp_n \int\int\ldots\int |f(\{\xi(s_1),\xi(s_2),\ldots,\xi(s_n)\})-f_{nk}(\xi(s_1),\xi(s_2),\ldots,\xi(s_n))|\ \eta'(ds_1)\prod_{i=2}^{n}\eta(ds_i)$$

$$=\sum_{n=1}^{\infty} Mnp_n \int\int\ldots\int |f(\{x_1,x_2,\ldots,x_n\})-f_{nk}(x_1,x_2,\ldots,x_n)|\ \prod_{i=1}^{n}\lambda(dx_i)\geq M/k.$$

□

Sufficient conditions for the existence of a pure-strategy equilibrium in Poisson (and other) games are given by the next theorem:

Theorem 2. *Let $\{A_1,A_2,\ldots,A_m\}$ be a partition of $\mathfrak{S}$ into finitely many closed sets. If $\boldsymbol{X}$ is a mixed Bernoulli process, if u is bounded, and if, for every s and S, $u(\cdot,S)$ is continuous and $u(s,\cdot)$ can be expressed as a function of $S(A_1),S(A_2),\ldots,S(A_m)$, then a pure-strategy equilibrium exists.*

Proof. If $(s_n,S_n)\to(s,S)$ is a converging sequence in $\mathfrak{S}\times\mathfrak{N}(\mathfrak{S})$, then $S_n(A_i) = S(A_i)$ holds for every i and every large enough n (recall that A_i is both closed and open), and therefore $u(s_n,S_n)\to u(s,S)$. Thus, u is continuous. By Theorem 1, a mixed-strategy equilibrium, $\boldsymbol{S}$, exists. By Lemma 1, $\boldsymbol{S}$ is a mixed Bernoulli process, and it may be assumed that $\boldsymbol{S}_x$ has the same distribution for every player x. For fixed s, the value of the random variable $u(s,\boldsymbol{S}_x)$ depends only on the value of the random vector $(\boldsymbol{S}_x(A_1),\boldsymbol{S}_x(A_2),\ldots,\boldsymbol{S}_x(A_m))$. Therefore, $\mathrm{E}u(s,\boldsymbol{S}_x)$ depends only on the distribution of this random vector. It follows from the example that precedes Proposition 2 that, conditioned on $\boldsymbol{S}_x(\mathfrak{S})$, this distribution is multinomial with parameters $(n;p_1,p_2,\ldots,p_m)$ $=(\boldsymbol{S}_x(\mathfrak{S});\mathrm{E}\boldsymbol{S}(A_1)/\mathrm{E}\boldsymbol{S}(\mathfrak{S}),\mathrm{E}\boldsymbol{S}(A_2)/\mathrm{E}\boldsymbol{S}(\mathfrak{S}),\ldots,\mathrm{E}\boldsymbol{S}(A_m)/\mathrm{E}\boldsymbol{S}(\mathfrak{S}))$. The distribution of $\boldsymbol{S}_x(\mathfrak{S})$ is completely determined by the distribution of the random variable $\boldsymbol{S}(\mathfrak{S})$—which is the same as the distribution of $X(\mathfrak{X})$. Therefore, $\mathrm{E}u(s,S_x)$ depends on S only through $\mathrm{E}S(A_1),\mathrm{E}S(A_2),\ldots,\mathrm{E}S(A_m)$.

Let f_i denote the nonnegative measurable function $x \mapsto \mathrm{P}(s_x \in A_i)$. By (3), $\int f_i(x)\ \mathrm{E}X(dx) = \mathrm{E}S(A_i)$ for every i. Since X is a simple mixed Bernoulli process, its mean measure $\mathrm{E}X$ is nonatomic, unless $\mathrm{P}(X(\mathfrak{X})>1)=0$. Assuming for the moment that $\mathrm{P}(X(\mathfrak{X})>1)>0$, the measure λ_i defined by $\lambda_i(A) = \mathrm{E}X(A \cap \mathrm{supp}\, f_i)$ (supp f_i is the set of points in which $f_i \neq 0$) is a nonatomic measure on $\mathfrak{X}$, for every i. Since $\sum_i f_i = 1$ identically, there exists a measurable partition $\{T_i\}_{1 \leq i \leq m}$ of $\mathfrak{X}$ such that $\lambda_i(T_i) = \int f_i\,(x)\lambda_i(dx)$ for every i (Dvoretzky et al., 1951), and hence $\mathrm{E}X(T_i) \geq \lambda_i(T_i) = \int f_i(x)\ \mathrm{E}X(dx) = \mathrm{E}S(A_i)$. But since $\{T_i\}$ and $\{A_i\}$ are both partitions, and $\mathrm{E}X(\mathfrak{X}) = \mathrm{E}S(\mathfrak{S})$, equality must in fact hold for every i. It follows that T_i can be chosen as a subset of supp f_i, and $\mathrm{P}(s_x \in A_i)>0$ can thus be assumed to hold for every $x \in T_i$. Since S is an equilibrium, this implies that the set $\psi_i(x) = \{s \in \xi^{-1}(\{x\}) \cap A_i \mid s$ maximizes $\mathrm{E}u(\cdot,S_x)$ in $\xi^{-1}(\{x\})\}$ may be assumed to be nonempty for every i and every x in T_i. The correspondence ψ, defined on $\mathfrak{X}$, is closed-valued, and has a measurable graph. Therefore, is admits a measurable selection, $\sigma : \mathfrak{X} \to \mathfrak{S}$ (Wagner, 1980, Theorem 12.1). For every x, $\mathrm{E}u(\sigma(x),S_x) = \max_{s \in \xi^{-1}(\{x\})} \mathrm{E}u(s,S_x)$. By definition, σ maps the elements of T_i, and only them, into elements of A_i. Therefore, if S' is the representation of σ as a correlated strategy, $\mathrm{E}S'(A_i) = \int \chi_{A_i}(\sigma(x))\ \mathrm{E}X(dx) = \mathrm{E}X(T_i) = \mathrm{E}S(A_i)$ for every i. As shown above, these equalities imply $\mathrm{E}u(\cdot,S'_x) = \mathrm{E}u(\cdot,S_x)$, for $\mathrm{E}X$-almost every x. It follows that S' is a pure-strategy equilibrium. If $\mathrm{P}(X(\mathfrak{X})>1)=0$, then any measurable selection of the correspondence ψ' defined by $\psi'(x) = \{s \in \xi^{-1}(\{x\}) \mid s$ maximizes $\mathrm{E}u(\cdot,\varnothing)$ in $\xi^{-1}(\{x\})\}$ is a pure-strategy equilibrium. □

Remark. Theorem 2 is the only place in this paper in which the assumption that X is a simple point process is used. All the other results in Sections 3 through 5 are true also when X is not simple, that is, when the same potential player (now better interpreted as a potential player *characteristic*) may be chosen more than once. Thus, for example, if we write Equation (4) as $\mathrm{E}\int u(s,S\setminus\{s\})\,S(ds) = \max_{\varphi} \mathrm{E}\int u(\varphi(s),S\setminus\{s\})\,S(ds)$, then Proposition 3 is true regardless of whether or not X is simple.

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