Controlling price volatility through financial innovation[*]

Alessandro Citanna[†]
Groupe HEC
Jouy-en-Josas, 78351 France

Karl Schmedders
Kellogg School of Management
Evanston, IL 60208, U.S.A.

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Abstract

In a three-period finite competitive exchange economy with incomplete financial markets and retrading, we study the possibility of controlling asset price volatility through financial innovation. We first give sufficient conditions on preferences and endowments implying that whatever is the innovation which completes markets, it also reduces volatility, typically in this class of economies. We also numerically examine some interesting examples. Then we show the generic existence, even outside this class, of financial innovation which decreases equilibrium price volatility. The existence is obtained under conditions of sufficient market incompleteness. The financial innovation may consist of an asset which is only traded at time zero, or retraded, and with payoffs only at the terminal date. The existence is shown to be robust in the asset payoff space.

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[†]Corresponding author: Alessandro Citanna, Dept. Finance et Economie, Bat. W2, Salle 12, Groupe HEC- Paris, 78351, Jouy-en-Josas, France. Phone: (33 1) 3967 7292. Email: citanna@hec.fr

1. Introduction

In this paper we find sufficient conditions for the robust control of asset price volatility through financial innovation. An intimately related question, often asked in the literature (see our discussion of the literature below), is whether the degree of market incompleteness affects asset price volatility.

We take advantage of a very simple, standard model of discrete-time dynamic trading of assets and multiple goods in a finite economy. In our setup, financial innovation is not the result of optimizing behavior, and asset markets are incomplete. We do not seek to explain why markets are incomplete or why new assets are introduced, but we take this as given and concentrate on the equilibrium effects of having different financial structures. We impose time and state separability of preferences, and use von Neumann-Morgenstern expected utilities. This is done for the sake of exposition, and because of the wide use of the separable case in the finance and macroeconomic literature. Any weaker version of separability (such as time nonseparability, or a habit formation assumption) also gives rise to the same results as the ones presented in this paper. The model we use covers any finite time horizon trading economy, even though here we focus on the three-period case. Although a new financial asset generally plays a role as a hedging device and as an information vehicle, the results in this paper do not address differential information economies. Rather, we concentrate on the spanning role of financial assets.

With these maintained general specifications, we first elaborate on a well-known result (see Oh (1996), e.g.) showing that with quadratic (i.e., mean - variance) preferences price volatility is unaffected by the degree of market incompleteness, and financial innovation has no effect on volatility.

Second, we find a class of economies where market-completing financial innovation never increases price volatility, and almost always reduces it. These are economies with no aggregate risk. Here asset prices show zero volatility with complete markets, provided the stationarity of payoff expectations is assumed. In Proposition 3.2, we prove that any financial innovation is volatility-reducing, generically in endowments, existing asset payoffs and preferences, but within this class, if: i) we go from one asset to complete markets; ii) there is one commodity, and; iii) conditions of effective incompleteness are added to this environment, i.e., idiosyncratic risks cannot be perfectly hedged when markets are incomplete. We also give numerical examples, suggesting that the volatility-reducing effects of innovation extend in this class of economies to a general comparison between incomplete and complete markets. Our examples also suggest that no qualitative role is played by the type of risk aversion. We obtain price volatility with incomplete markets and with CRRA or CARA utilities.

Finally, in Theorem 5.2 and its Corollary we find conditions on the number of states, assets and types of households such that, in an open and dense subset of preferences and endowments, financial innovation allows the *control* of asset price volatility. These conditions imply that, with sufficient market incompleteness to start with, we can find a new asset which reduces price volatility. Moreover, our theorem

allows the comparison between incomplete and (dynamically) complete markets, with one initial asset and a binomial-tree structure for uncertainty, if the new asset is retraded.

The last result of the paper shows that it is *easier* to reduce volatility through financial innovation when traders *cannot rebalance* their holdings of the new asset. More precisely, in the case of *impossibility* of retrading (Theorem 5.2) the condition linking the number of states, households and assets is *weaker* than in the case of retrading (Theorem 5.6). Suppose that over-the-counter financial contracts can be designed in a more customized way and retrading of these contracts is more difficult than on standardized contracts exchanged on organized markets. Then it can be argued that in the absence of greater levels of market incompleteness (higher dimensionality of the state space), it may be convenient to introduce hedging instruments over the counter as opposed to widely retraded assets, when the intent is to control price volatility of existing financial assets (and in the absence of other effects, such as informational).

More formally, Theorem 5.2 states that there is some financial innovation leading to a volatility decrease, while some other leads to an increase in volatility. The intuition here is, along the lines of what is known in the constrained suboptimality literature (see Geanakoplos and Polemarchakis (1986), Citanna, Kajii and Villanacci (1998), e.g.), that price effects induced by payoff changes may allow modifying asset prices in such a way to control volatilities at pleasure. To get the result, we have to impose some extra conditions on the economy. In particular, the degree of incompleteness must be larger than the sum of the number of (types of) households in the economy and of the number of preexisting assets. This condition is generally sufficient, and also tight within our differential framework of analysis, in the sense that this condition is required to obtain robust (open sets of new assets have the same effect on open sets of economies, for almost all initial economies) and predictable (locally one-to-one) effects of financial innovation. The exact meaning of this will become apparent after Lemma 5.1.

The remainder of the paper is organized as follows. After a brief discussion of the related literature, in Section 2 we present the model, and a CAPM economy. In Section 3 we examine the no aggregate risk economies, and study examples illustrating the role of risk aversion. Section 4 introduces a general framework for the comparative statics of financial innovation, generalizing Cass and Citanna (1998). This section can be skipped by the reader familiar with that paper. Section 5 contains the general controllability results without or with retrading, and further examples. The Appendix contains the technical proofs.

1.1. Related literature

Despite the fact that ultimately financial innovation should be judged on the basis of its effects on welfare, its impact on asset price volatility has spurred a considerable debate over the years, also at the academic level. Market volatility has been the focus

of empirical tests of asset pricing models and business cycle theories (the literature is huge and cannot be summarized here; an example is Shiller (1981)). These studies mainly found excess volatility of stock returns, or deviations of asset price volatility from the one determined by the "fundamentals". Traditionally the benchmark volatility is derived from a complete-market, infinite-horizon model of consumption and investment. Market incompleteness, as a form of market imperfection, has been conjectured to be one of the causes of excess volatility (see Shiller (1989)). Computationally, examples of incomplete market models have shown no quantitatively significant amount of this excess volatility (see Telmer (1993), or Rios-Rull (1994)). On the other hand, Geanakoplos (1997) in a multi-commodity world with bankruptcy, and Calvet (2001) in an infinite horizon CARA - normal framework with no aggregate uncertainty have recently confirmed the conjecture.

In its weaker form, the conjecture compares volatility with complete and incomplete markets, asserting that the former is lower than the latter (see Calvet (2001), e.g.[1]). The basic idea behind this hypothesis is that when financial markets are incomplete, risk averse individuals cannot perfectly smooth consumption across time and states, and this causes fluctuations in aggregate endogenous variables across states of the world. In particular, asset prices show volatility in excess of what found when markets are complete. Hence, under the weaker hypothesis, financial innovation which completes markets always has beneficial effects for volatility, no matter what new assets are introduced. Shiller (1993) has also proposed the introduction of index trading as an effective way to reduce market incompleteness and financial market volatility.[2]

Our result in economies with no aggregate risk, and quadratic preferences, are in line with those of Calvet (2001), as we have no stochastic volatility in that case. Our numerical examples with no aggregate risk point to the fact that the absence of stochastic volatility is the effect of mean-variance preferences, and not of constant absolute risk aversion. Moreover, we show that incomplete markets cause an increase in volatility for general preferences and asset payoffs, enlarging Calvet's result on volatility with incomplete markets and no aggregate risk beyond the CARA - normal setup.

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[1]In fact, Calvet (2001) asserts that "More asset markets should not only help improve welfare and reduce social inequalities through better risk sharing; they should also dampen volatility in existing financial markets and the real economy", going past the weaker form of the hypothesis. It should also be noticed that in Calvet (2001), volatility means deterministic fluctuations, while in this paper we focus on stochastic fluctuations.

[2]Using a different modeling framework, Detemple and Selden (1991) studied the effects of option contracts on price volatility, and Detemple (1996), Zapatero (1998) examined general financial innovation with asymmetric information and heterogeneous beliefs, respectively. Parametric restrictions were imposed on preferences (e.g., logarithmic or CRRA) or on asset payoffs, that is, on dividend processes in continuous time, and on prices (diffusion processes). Zapatero (1998) finds that market incompleteness is a source of increased volatility. Our paper addresses situations studied by Zapatero (1998), where traders differ in their beliefs. It does not directly compare to the results obtained in Detemple (1996) because we do not have an asymmetric information economy. In the paper, we choose to expose the case of heterogeneous preferences and homogeneous beliefs to simplify the exposition, although it is immediate to see that the gist of our theorem holds even in the presence of heterogeneous beliefs.

By looking at economies where asset prices show zero volatility with complete markets, we focus on situations where market incompleteness has unequivocal effects on volatility, abstracting from other factors which may entail volatility even in the presence of complete markets. Pure sunspot economies also display this excess volatility phenomenon. However, in our parameter space these economies are a large class only in the nominal asset case (see Cass 1992). With real or numéraire assets, sunspot equilibria are known to occur: generically, only with restrictions on asset payoffs, multiple commodities and a number of states no greater than the number of commodities, and no asset retrading, as in Gottardi and Kajii (1999); or just robustly, as in Hens (2000). By looking at no aggregate risk economies, we attempt at gaining degrees of freedom in the parameters and show that there is endogenous aggregate (i.e., asset price) uncertainty due to market incompleteness even with one commodity. Our economies differ from those of Pietra (1992, 2001), and the perturbation technique developed there cannot be straightforwardly applied. They also differ from those studied in Siconolfi and Villanacci (1991) because here assets are real.

Theorem 5.2 shows that nothing so general can be deduced as "the more financial markets are incomplete, the higher the volatility". Without controlling for asset specifications, more incompleteness may be associated to lower volatility. However, financial innovation can be used as an instrument of volatility control. This generalizes Shiller's proposal to reduce incompleteness in order to reduce price volatility, with the caveat that the innovation must be precisely selected in order to achieve the desired effect. Moreover, in our framework, an option is characterized by a fixed functional form and by only one extra parameter, the strike price. This seems not sufficient to guarantee robustness of volatility-reducing effects.[3] This paper of course does not address the informational requirements needed for the implementation of this policy instrument, a topic linked to the recoverability literature in incomplete markets. While we cannot say anything about calibrated models, or time series, our results and numerical examples complement the quantitative analysis of Telmer (1993) and Rios-Rull (1994) for this finite horizon setup. Changes in volatility may rise to about 10% of their initial value, depending on the level of market incompleteness.

Technically, Theorems 5.2 and 5.6 extend to a multi-period setting the differential framework developed in Cass and Citanna (1998) to study financial innovation in incomplete markets, itself a ramification of the long-debated issue of constrained suboptimality (see Citanna, Kajii and Villanacci (1998)). As we explained, we believe that the study of the effects of financial innovation cannot be reduced to welfare comparisons, already addressed in the literature (see, for example, Cass and Citanna (1998) or Elul (1995)). As a difference with respect to that literature, we study the effects of innovation on price volatility, which cannot be defined in the standard two-period exchange economy. Moreover, the restrictions that naturally arise on the payoff matrix representing financial markets with dynamic trading are not encompassed by the previous theorems, and provide the structural motivation to this work. Finally,

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[3]In this sense, our results do not directly address the robustness of the work by Detemple and Selden (1991) on option contracts.

the analysis of equilibrium volatility is meant to be illustrative of more general issues whose study can be easily embedded in this framework, provided they can be represented by a smooth function defined over the equilibrium set.[4]

2. The model

We consider a standard model of an intertemporal, competitive, pure-exchange economy with incomplete financial markets. Let t denote the time period, with $t = 0, 1, \ldots, T$, where $t = 0$ is today, and $t = T$ is the terminal date. Uncertainty is represented by $1 < \overline{S} < \infty$ states of the world in each period $t > 0$ and at each spot, or realization of previous uncertainty, indexed by $\overline{s}$.[5] The following tree structure represents uncertainty in this economy,
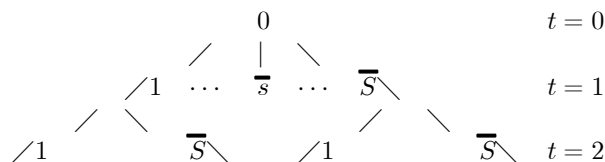
The total number of states in the economy is therefore given by $\sum_{t=0}^{T} \overline{S}^t$. In this paper, we assume that all the information in the economy is publicly available. We will assume that J financial instruments are tradable today and that $\overline{S} > J$, so financial markets are incomplete, even dynamically. These instruments are long-term securities, since they can be held until the terminal date T. Nevertheless, they can be retraded in any period $t < T$. It is notationally convenient also to represent the retraded instruments as independent assets i, where $i = 1, \ldots, I$, and $I = J \sum_{t=0}^{T-1} \overline{S}^t$. We will also index states in different periods all together as spots s, and will write $S + 1 = \sum_{t=0}^{T} \overline{S}^t$. Although the formalization encompasses any finite-horizon economy, we will focus on the three-period case, i.e., $T = 2$.

There are $H \geq 2$ households (also referred to as 'traders') indexed by h. At each date and state, there are C commodities or goods indexed by c, with $C \geq 1$[6]. The commodity (and endowment) space is taken to be $\mathbb{R}_{++}^G$, where $G = C(S + 1)$. A typical household's preferences are represented by the utility function $u_h : \mathbb{R}_{++}^G \to \mathbb{R}$, which is assumed to be smooth, differentially strictly increasing and differentially strictly concave, and to have the closure of indifference surfaces contained in $\mathbb{R}_{++}^G$.

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[4]An example of which is the study of the robustness of the differences in the price level of one asset depending on markets being complete or not, also known as the 'precautionary savings' effect. See Elul (1997), whose robustness conditions can be simplified using our framework.

[5]Although this is not strictly necessary, to simplify the notation we take the number of states to be constant over time and at each spot.

[6]Contrary to Cass and Citanna (1998), robustness can be shown here also in the case when $C = 1$, but the equations considered are slightly different, and we do not give the computational details in this paper.

Moreover, the utility will be assumed of the form

$$u_h(x_h) = \sum_{s=0}^{S} \pi^s v_h(x_h^s),\tag{2.1}$$

with $\pi^s > 0$ and $\sum_{s>0} \pi^s = 2$. That is, we consider von Neumann-Morgenstern preferences, with objective probabilities and time separable utility. The case of non-separability (time and state) is easier to deal with, and follows from the proofs given below. We choose to present the results with this specification because it is the most widely used, although maybe not the most economically plausible. As for using objective probabilities, again this choice derives from the need of comparison between our statements and those made in the related literature, and to simplify computations already burdensome.[7] Note that one can interpret π^s as derived from a (stationary) probability measure $\overline{\pi}$ on the states $s > 0$ in the following standard way:

$$\pi^s = \begin{cases} \overline{\pi}(s) & \text{if } 0 < s \quad \overline{S} \\ \overline{\pi}(\overline{s}|\overline{s}')\overline{\pi}(\overline{s}') & s = \overline{s}'\overline{S} + \overline{s} \end{cases}$$

where $\overline{\pi}(\overline{s}|\overline{s}')$ is the conditional probability of state $\overline{s}$ in period $t = 2$ after state $\overline{s}'$ occurs in period $t = 1$, and $\overline{s}, \overline{s}' \in \{1, \dots, \overline{S}\}$. π^0 is interpreted as a simple intertemporal preference parameter, not as a probability.

The space of households' endowments is $E = (\mathbb{R}_{++}^G)^H$. The space of households' utility functions is $\mathcal{U} = U^H$, where U is a subset of the $\mathcal{C}^3(\mathbb{R}_{++}^G, \mathbb{R})$ mappings[8], endowed with the subspace topology induced by the compact-open topology assigned to the whole space. With $x_h \in \mathbb{R}_{++}^G$, $b_h \in \mathbb{R}^I$, $p \in \mathbb{R}^G$ and $q \in \mathbb{R}^I$ we denote the consumption bundle and the asset portfolio for household h, the commodity price vector and the asset price vector, respectively. It will be convenient to take quantity vectors as columns, and price vectors as rows.

The financial structure is represented by an $(S + 1) \times I$-dimensional matrix of prices and payoffs R expressed in terms of a numéraire commodity, which we take to be the last at each spot s, i.e., $c = C$. It is apparent that we are dealing here with a special case of the usual standard incomplete market model, where the matrix R assumes the following form

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[7]The conditions of Theorem 5.2 may be slightly altered by a subjective probability specification (due to the need of keeping track of volatility as perceived by each household, and therefore requiring a higher degree of market incompleteness for the theorem to hold), but the general framework of analysis does not change.

[8]More precisely, we need the functions to be three times continuously differentiable locally around an equilibrium, although we consider them in the C^2 topology for our genericity statements.

$$R = \Psi^C \begin{bmatrix} -q^0 & 0 & 0 & 0 & 0 \\ y^1(0)+q^1 & -q^1 & 0 & 0 & 0 \\ y^2(0)+q^2 & 0 & -q^2 & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ y^{\overline{S}}(0)+q^{\overline{S}} & 0 & 0 & 0 & -q^{\overline{S}} \\ 0 & Y(1) & & & \\ 0 & & Y(2) & & \\ 0 & & & \ddots & \\ 0 & 0 & & & Y(\overline{S}) \end{bmatrix}$$

where

$$\Psi^C = \begin{bmatrix} p^{0,C} & 0 & \ldots \\ 0 & \ddots & 0 \\ \ldots & 0 & p^{S,C} \end{bmatrix}$$

is an $S+1$-dimensional square matrix of prices of the numéraire commodity, and $Y(s)$, with $s = 0, 1, \ldots, \overline{S}$ is an $\overline{S} \times J$ matrix of payoffs for the traded security. We will assume that each $Y(s)$ be in general position, in that every submatrix of row or column dimension $m \quad J$ has full rank[9]. The argument of this paper adapts to the one-asset case $(J = 1)$ with this condition becoming $Y(s) = Y \gg 0$, which is consistent with intertemporal models of stock and bond trading. Denote by $\Gamma \subset \mathbb{R}^{\overline{S}J(\overline{S}+1)}$ the space of such matrices.

Note that an asset $j \in J$ in this economy promises to deliver $y^{\overline{s},j}(0)$ units of the numéraire good in state $\overline{s}$ in period $t = 1$, and $y^{\overline{s},j}(\overline{s})$ units of the numéraire in state $\overline{s}'$ in period $t = 2$ after $\overline{s}$ occurred in period $t = 1$, with $\overline{s}, \overline{s}' \in \{1, \ldots, \overline{S}\}$. Hence $y^{\overline{s}'}(s) = (y^{\overline{s},j}(s))_{j \in J}$, is the $\overline{s}'$-th row of $Y(s)$, for $s = 0, 1, \ldots, \overline{S}$.

Here q^s is a $J \times 1$ vector of prices, which are endogenously determined in equilibrium, and which correspond to 'different' assets as a function of the state. So, each asset $j \in J$ is exchanged at price $q^{0,j}$ at $t = 0$, and at price $q^{\overline{s},j}$ in state $\overline{s}$ and period $t = 1$, and $q^s = (q^{s,j})_{j \in J}$, for $s = 0, 1, \ldots, \overline{S}$.

Finally, we denote by $b_h^{0,j}$ trader h's holdings of the j-th asset at time zero, and by $b_h^{\overline{s},j}$ trader h's holdings of the same asset at time $t = 1$ in state $\overline{s}$. However, in what follows the natural identification of asset i with a pair (s, j) for $s = 0, 1, \ldots, \overline{S}$ will be used, and each asset j will generate $\overline{S} + 1$ 'different' assets, where b_h^i denotes the holdings of asset i for trader h.[10]

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[9]This is essentially the definition as in Mas-Colell (1985, p.13).

[10]This framework of analysis of dynamic trading in financial markets is common, apart from slight differences in the timing of trading or payoff payments, to several treatments of sequential trading in rational expectations models, and in particular to a model by Duffie and Shafer (1986).

We will parametrize each economy as an element of the pair $E \times \mathcal{U}$, endowed with the product topology, with securities $\left(Y(s)_{s=0}^{\overline{S}}\right) \in \Gamma$. We will later use the notation

$$y(s) = (y^1(s), \ldots, y^{\overline{S}}(s)), \text{ for } s = 0, 1, \ldots, \overline{S}, \text{ and}$$

$$y = [y(0), y(1), \ldots, y(\overline{S})]'$$

for the payoff vector of a newly introduced asset, an element of $\mathbb{R}^S$.

A *financial equilibrium* is a vector $((x_h, b_h)_{h=1}^H, p, q)$ such that:

(H) given p, q, households optimize, that is, for every h, (x_h, b_h) solves the problem

$$\text{maximize}_{x_h, b_h} u_h(x_h)$$
$$\text{subject to } \Psi(x_h - e_h) = Rb_h,$$

where

$$\Psi = \begin{bmatrix} p^0 & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & p^S \end{bmatrix}$$

is an $(S+1) \times C$ price matrix, $e_h \in \mathbb{R}_{++}^G$ is the household's endowment, and

(M) markets clear, that is,

$$\sum_h (x_h - e_h) = 0$$
$$\text{and } \sum_h b_h = 0.$$

An equilibrium is represented in extended form by the system of equations[11] consisting of both the households' Kuhn-Tucker conditions and the market-clearing conditions,

$$\vdots$$
$$Du_h(x_h) - \lambda_h \Psi = 0$$
$$\lambda_h R = 0$$
$$-\Psi z_h + Rb_h = 0 \qquad\qquad (2.2)$$
$$\vdots$$
$$\sum_h z_h = 0$$
$$\sum_h b_h = 0,$$

It suffices to note at this stage that the economy is a dynamic one in the usual sense that trading occurs sequentially, but plans are made once at time $t = 0$. Indeed, in the first period ($t = 0$), households maximize utility given rational expectations about future prices, and the existing financial structure, making plans for trading commodities and assets today and tomorrow. Then, today's trades are carried through, and households consume and hold portfolios to transfer wealth in the future. Tomorrow, given the state of the world, households fulfill their financial obligations, and then again trade commodities and financial instruments, and consume. At $T = 2$, again households fulfill their financial obligations, trade commodities and consume.

[11]The analysis in terms of extended systems was first exploited by Smale (1974) for pure walrasian economies.

where $\lambda_h \in \mathbb{R}_{++}^{S+1}$ is the household's vector of Lagrange multipliers (i.e., marginal utilities of wealth), $z_h = x_h - e_h$ is the household's vector of excess demands, and $z_h^{\backslash}$ is z_h deprived of the element corresponding to commodity C, for all s, by Walras' law. Throughout, we will use the standard normalization $p^{s,C} = 1$, all s.

We define volatility of the $j - th$ financial instrument as

$$\phi^j = \sigma_0^2(q^j) = \sum_{\overline{s}} \pi^{\overline{s}} (q^{\overline{s},j} - E_0(q^j))^2$$

where $E_0(q^j) = \sum_{\overline{s}} \pi^{\overline{s}} q^{\overline{s},j}$. In other words, we look at price volatility, as opposed to return volatility. Return volatility can be studied with a slight modification of the definition given above. The coefficient of variation, defined as $CV^j = \frac{\sigma_0(q^j)}{E_0(q^j)} \cdot 100$, enables a comparison of the price volatility modulo changes in the price level, and will be used as a measure in some numerical examples.

Existence of equilibrium for this model has essentially been analyzed by Duffie and Shafer (1986). Retrading can potentially lead to a matrix R whose rank is less than I. To see this, let Q be the $S \times J$ matrix of security prices at time $t = 1$. The problem of loss of rank, i.e., redundancy, arises because $Y(0) + Q$ may not have full rank. This will hold generically in endowments and security payoffs. Note that for $J = 1$ the rank result follows immediately from the assumption that $Y \gg 0$, in which case the existence proof follows from Geanakoplos and Polemarchakis (1986).

To state the existence result for $J > 1$, we parametrize economies by endowments and securities only, fixing preferences once and for all.[12] Let $\Theta \equiv E \times \Gamma$ denote the parameter space.

Proposition 2.1. *(Duffie and Shafer, 1986) For an open and full-measure subset Θ^* of Θ, a financial equilibrium exists. Moreover, if $\theta \in \Theta^*$, all the equilibria are such that rank $R = I$, and $Y(0) + Q$ is in general position.*

The proof of this Proposition is in essence identical to the one in Duffie and Shafer, hence the reader is referred to that paper for the details. An argument already adapted to this paper's notation is available from the corresponding author upon request.[13]

2.1. An example with no control through innovation: the CAPM

Controlling volatility through financial innovation is not an obvious task. We elaborate on a well-known example for one-period trading models (see Geanakoplos and Shubik (1990), Magill and Quinzii (1996), and Oh (1996)) of an economy with incomplete markets and linear-quadratic utility functions, a CAPM economy. The example

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[12]So, for the time being, we keep $u \in \mathcal{U}$ as fixed; parameterization by utility functions will appear in the next section.

[13]The proposition holds even in the case of the restrictions we impose in the next section, as it can be easily checked. However, for the sake of compactness the details are omitted.

shows that fixing specific preferences leads to no changes in asset prices due to financial innovation, therefore to no changes in volatility. The example also shows that utility perturbations are needed in all the density arguments that will follow.

To this purpose, assume that u_h takes the form

$$u_h(x_h) = \pi_h^0 v(x_h^0) + \sum_{s>0} \pi^s f_h(v(x_h^s)), \tag{2.3}$$

where $v : \mathbb{R}_{++}^C \to \mathbb{R}$ is a smooth, differentially strictly increasing and concave, homogeneous of degree one function, and

$$f_h(y) = -(1/2)(\alpha_h - y)^2$$

Then preferences are homothetic and spot-separable, and spot commodity equilibrium prices are independent of the income distribution across agents and across states. Then the portfolio choice only affects the level of consumption in each spot, but not the commodity prices. Let $w^s = \Sigma_h e_h^s$ be the level of aggregate endowments in spot s, and consider the spot price normalization $p^s w^s = 1$, for all s. From $Dv(w^s)w^s = v(w^s)$, all s, we get

$$p^s = Dv(w^s)/v(w^s)$$

for all s, which shows that p^s only depends on aggregate resources, not on income distribution (hence, and a fortiori, not on financial innovation). Therefore one can reduce the trader's multi-commodity maximization problem to a one-commodity maximization: after defining $m_h^s = p^s e_h^s + r^s b_h$, and noting that the optimal consumption vector in spot s is given by $x_h^s = m_h^s w^s$, we transform (H) into

$$\max_{b_h} \pi_h^0 \tilde{x}_h^0 + \sum_{s>0} \pi^s f_h(\tilde{x}_h^s)$$
$$\text{s. t. } \tilde{x}_h = \omega_h + \tilde{R} b_h$$

with $\tilde{x}_h^s = v(w^s)m_h^s$, $\omega_h^s = v(w^s)p^s e_h^s$ and the $s-$th row in $\tilde{R}$ is $\tilde{r}^s = v(w^s)r^s$. Define $\omega_h^1 = \left(\omega_h^1, \ldots \omega_h^s, \ldots, \omega_h^{\overline{S}}\right)$ and $\omega_h^2(s) = \left(\omega_h^{(s-1)\overline{S}+1}, \ldots, \omega_h^{(s-1)\overline{S}+\overline{S}}\right)$, for $1 \quad s \quad \overline{S}$. Also define $\pi^0 = \Sigma_h \pi_h^0$, $\alpha = \Sigma_h \alpha_h$, $\omega^1 = \Sigma_h \omega_h^1$ and $\omega^2(s) = \Sigma_h \omega_h^2(s)$. At this point, the linear-quadratic assumption on u_h leads to the following equilibrium asset prices

$$q^0 = \frac{1}{\pi^0}\left(\alpha \mathbf{1} - \omega^1\right)' \Pi(0)\left[Y(0) + Q\right]$$

and, for $s = 1, 2, \ldots, \overline{S}$

$$q^s = \frac{1}{\alpha - \omega^s}\left(\alpha \mathbf{1} - \omega^2(s)\right)' \Pi(s) Y(s)$$

where $\Pi(s)$ is the $\overline{S}-$dimensional square, diagonal matrix of conditional probabilities given state s has occurred, $s = 0, 1, \ldots, \overline{S}$. Neither of these expressions changes as we add a new asset. So in particular, moving from complete to incomplete markets does not affect asset prices (see Oh (1996) for the two-period economy result), and price volatility is unchanged.

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3. Price volatility with incomplete markets

In this section, we provide a set of circumstances where market incompleteness brings about price volatility in excess of the uncertainty on fundamentals. In order to focus on the lack-of-smoothing effect due to market incompleteness, we look at the case where asset prices show zero volatility with complete markets.

When financial markets are complete, equilibria are Pareto efficient and the multipliers λ_h are colinear across households, i.e., $\lambda_h = \mu_h \widehat{p}^C$, with $\mu_h > 0$ and $\widehat{p}^C \in \mathbb{R}_{++}^{1+S}$, a vector of numéraire commodity prices in the equivalent Debreu economy. From $D_{.,C} u_h - \lambda_h = 0$ we have $\pi^s D_{.,C} v_h(x_h^s) - \mu_h \widehat{p}^{s,C} = 0$, all s. Hence another equivalent way of writing the no arbitrage conditions is

$$ q^{\overline{s}} = \frac{1}{\widehat{p}^{\overline{s},C}} \sum_{s=\overline{sS}+1}^{\overline{sS+S}} \widehat{p}^{s,C} Y(\overline{s}), \text{ for } \overline{s} = 1, \dots, \overline{S} $$

Looking at the previous expression, we have immediately that if the conditional expected asset payoffs are state - independent, i.e.,

$$ \sum_{\overline{s}'=1}^{\overline{S}} \overline{\pi}(\overline{s}'|\overline{s}) Y(\overline{s}) \text{ does not depend on } \overline{s}, $$

and $\widehat{p}^C$ is proportional to π, then $q^{\overline{s}} = q$ for all $\overline{s}$, all j, that is, zero price volatility.

As a special case, if $Y(s) = Y$ for all $s = 1, \dots, \overline{S}$, and there is a uniform distribution of beliefs, or $\pi^s = \pi$, again Debreu prices proportional to probabilities imply zero volatility.

We now relate this property to the primitives of the economy. Let $\widehat{p} \in \mathbb{R}_{++}^G$ be the whole Debreu price vector, including all state-contingent commodities, and let $\widehat{\widehat{p}} \in \mathbb{R}_{++}^C$. Let $r \in \mathbb{R}_{++}^C$. With a strengthening of the proportionality requirement to all commodity prices, it is immediate to see that, once the state independence of conditional expected payoffs is assumed and utilities are von Neumann - Morgenstern, the following assumption on the primitives is both necessary and sufficient for the desired result.

Lemma 3.1. *The Debreu prices are proportional to π, i.e., $\widehat{p} = (\pi^0 \widehat{\widehat{p}}, \dots, \pi^S \widehat{\widehat{p}})$ if and only if there is no aggregate risk, i.e., $\sum_h e_h^s = r$, all s. Hence, if conditional expected payoffs are state-independent, there is zero volatility if there is no aggregate risk.*

Proof. See the Appendix. ∎

Note that when $C = 1$, the proportionality requirement is necessary to get zero volatility, and if conditional expected payoffs are state-independent, there is zero volatility if *and only if* there is no aggregate risk. Hence, combining this observation with the CAPM example, we obtain that if there is no aggregate uncertainty, asset

prices are deterministic in a mean-variance setup, although they may change over time (from q^0 to $q = q^s$ for all s) as in Calvet (2001). Moreover, volatility cannot go up when moving from incomplete to complete markets.

It turns out that in the case of the economies without aggregate risk, stationary conditional expected payoffs and von Neumann - Morgenstern utilities, incomplete markets economies typically display positive price volatility.

We prove the result for $J = C = 1$. In order to do so, we introduce the condition: for all h, all $\alpha \in \mathbb{R}_+$ and all $\overline{s} \in \{1, \ldots, \overline{S}\}$, there is no b_h such that $\alpha - e_h^{\overline{s}'} = y^{\overline{s}'}(\overline{s}) b_h$ for all $\overline{s}' \in \{1, \ldots, \overline{S}\}$. We call this condition *effective market incompleteness* for obvious reasons. Note that under effective market incompleteness, $e_h^{\overline{s}'} = y^{\overline{s}'}(\overline{s}) b_h$ for no b_h, no h. If not, and if it happened that $e_h = R(q) b_h$, for some b_h and all h, when $\lambda_h = \pi$, no aggregate risk would imply that this individual can synthesize the riskfree asset and construct a fully hedged consumption profile, as with complete markets. Hence we can now state the following result.

Proposition 3.2. *Assume $J = C = 1$. (i) Under the maintained assumptions on utilities, endowments, probabilities and asset payoffs, there is an open and dense subset of utilities such that incomplete markets equilibria are regular; (ii) $q_{(1,s)} \neq q_{(1,s')}$ for all s, s' with $s \neq s'$ in an open and dense subset of utilities, so that there is positive volatility when markets are incomplete.*

Proof. See the Appendix. ∎

Next we present some numerical examples illustrating the result of Proposition 3.2. We also provide an example of economies that does not satisfy the assumptions of the proposition, but still exhibits the positive price volatility result.

3.1. Numerical examples

The first set of examples illustrates and extends the statement of Proposition 3.2. Consider an economy with $\overline{S} = 2$ states at $t = 1$ and so $\bar{S}^2 = 4$ states at $t = 2$. There are $H = 2$ traders who have CRRA utility functions $v_h(x) = \frac{x^{1-\gamma_h}}{1-\gamma_h}$. Traders have identical uniform beliefs, that is, $\pi^s = 0.5$ for $s = 1, 2$, and $\pi^s = 0.25$ for $s = 3, \ldots, 6$. There is a single consumption good.

Example 1a. Traders have the endowments

$$e_1 = (1, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}) \quad \text{and} \quad e_2 = (1, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}).$$

There is a single riskless asset with a payoff of 1 in each state at $t = 1$ and $t = 2$ (a bond). We compute an equilibrium for a variety of different values of the risk-aversion coefficients (for information on the algorithm for computing equilibria and its implementation see the appendix).

γ_1	γ_2	q^1	q^2	$E_0(q)$	$\sigma_0^2(q)$	CV
0.5	0.5	1.04660	1.04660	1.04660	0	0
0.5	1.0	1.06923	1.07712	1.07318	0.00002	0.36737
0.5	2.0	1.10034	1.12303	1.11168	0.00013	1.02042
0.5	3.0	1.12282	1.15924	1.14103	0.00033	1.59607
0.5	5.0	1.15306	1.21377	1.18342	0.00092	2.56492
1.0	2.0	1.20292	1.22856	1.21574	0.00016	1.05437
1.0	3.0	1.25921	1.31406	1.28663	0.00075	2.13141
1.0	5.0	1.33729	1.44784	1.39256	0.00306	3.96958
2.0	3.0	1.56703	1.61647	1.59175	0.00061	1.55293
2.0	5.0	1.80683	1.97654	1.89168	0.00720	4.48570

TABLE 1A: Price volatility for a riskless asset.

Table 1a shows the two possible prices of the asset at $t = 1$, their expected value, variance, and coefficient of variation. When the traders have identical levels of risk aversion then the price of the riskless asset is identical in both states at time $t = 1$. There is zero price volatility. Whenever the traders have heterogeneous levels of risk aversion (a utility perturbation), then there is positive price volatility. The volatility appears to increase both when the levels of risk aversion increase and when the difference between the two levels increases, a fact not explained by Proposition 3.2, though. Similarly, Table 1a shows that CV has no clear relation with an increase in risk aversion.

Example 1b. Traders have the same endowments as in Example 1a. The single asset has now payoffs that exhibit a moderate amount of risk (like 'equity'), namely, $y = (1.1, 0.9, 1.1, 0.9, 1.1, 0.9)$. (Note that these economies are not perfectly symmetric, whenever trader 1 is in his high endowment state, then the asset has a high payoff.)

γ_1	γ_2	q^1	q^2	$E_0(q)$	$\sigma_0^2(q)$	CV
0.5	0.5	1.04924	1.04182	1.04553	0.13774E-04	0.35497
1.0	1.0	1.13465	1.11538	1.12501	0.92855E-04	0.85653
2.0	2.0	1.43223	1.37569	1.40396	0.79902E-03	2.01337
3.0	3.0	1.94693	1.83322	1.89008	0.32330E-02	3.00831
5.0	5.0	4.02786	3.74067	3.88426	0.20620E-01	3.69686
6.0	6.0	5.93438	5.53927	5.73683	0.39028E-01	3.44362
7.0	7.0	8.79662	8.28713	8.54188	0.64896E-01	2.98232

TABLE 1B: Price volatility for a risky asset.

When the asset is risky, then even in an economy with traders who have identical utility functions there is positive price volatility.

Example 1c. In this example the values of endowments and dividends are

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the same as in the previous example. But now traders have CARA utility functions $v_h(x) = -\frac{1}{a_h}e^{-a_h x}$.

a_1	a_2	q^1	q^2	$\mathrm{E}_0(q)$	$\sigma_0^2(q)$	CV
0.5	0.5	1.01497	1.01226	1.01361	0.18285E-05	0.13341
0.5	1.0	1.02917	1.02398	1.02657	0.67373E-05	0.25284
0.5	2.0	1.05431	1.04506	1.04968	0.21370E-04	0.44039
0.5	3.0	1.07459	1.06255	1.06857	0.36226E-04	0.56325
0.5	5.0	1.10261	1.08767	1.09514	0.55769E-04	0.68191
1.0	1.0	1.06024	1.04948	1.05486	0.28950E-04	0.51007
1.0	2.0	1.11446	1.09479	1.10462	0.96796E-04	0.89067
1.0	3.0	1.15991	1.13372	1.14681	0.17156E-03	1.14213
1.0	5.0	1.22565	1.19203	1.20884	0.28256E-03	1.39056
2.0	2.0	1.24708	1.20496	1.22602	0.44356E-03	1.71782
2.0	3.0	1.35812	1.29957	1.32885	0.85697E-03	2.20296
2.0	5.0	1.53510	1.45492	1.49501	0.16074E-02	2.68174
3.0	3.0	1.57926	1.48792	1.53359	0.20858E-02	2.97803
3.0	5.0	1.92089	1.78839	1.85464	0.43890E-02	3.57209
5.0	5.0	2.83208	2.60550	2.71879	0.12834E-01	4.16689

TABLE 1c: Price volatility with CARA utilities.

There is positive, albeit small, volatility in all economies. As in Example 1b, the volatility appears to increase both when the levels of risk aversion increase and when the two agents become more heterogeneous.

Note that contrary to the economies with CRRA utilities in Example 1a, there is no price volatility in a model with CARA utilities and a riskless asset, even when the agents have heterogenous levels of risk aversion. With the symmetric endowment parameterization of the economies in the present example the asset must be risky to create any price volatility.

Example 1d. Traders have the endowments

$$e_1 = (1, \frac{4}{3}, \frac{2}{3}, \frac{3.75}{3}, \frac{2.25}{3}, \frac{4.25}{3}, \frac{1.75}{3}) \quad \text{and} \quad e_2 = (1, \frac{2}{3}, \frac{4}{3}, \frac{2.25}{3}, \frac{3.75}{3}, \frac{1.75}{3}, \frac{4.25}{3}).$$

The traders endowment risk for the last period changes at $t = 1$. In state 1 it increases, in state 2 it decreases. Aggregate endowments are unaffected and remain constant. There is a single riskless asset with a payoff of 1 in each state at $t = 1$ and $t = 2$.

γ_1	γ_2	q^1	q^2	$E_0(q)$	$\sigma_0^2(q)$	CV
0.5	0.5	1.02526	1.07660	1.05093	0.00066	2.44261
1.0	1.0	1.06840	1.21404	1.14122	0.00530	6.38107
2.0	2.0	1.21297	1.72485	1.46891	0.06551	17.42388
3.0	3.0	1.44697	2.70037	2.07367	0.39275	30.22176
4.0	4.0	1.79134	4.44576	3.11855	1.76150	42.55869
5.0	5.0	2.27659	7.49191	4.88425	6.79989	53.38918
6.0	6.0	2.94504	12.75275	7.84890	24.04781	62.47830
7.0	7.0	3.85402	21.79858	12.82630	80.50180	69.95220

TABLE 1C: Price volatility with changing endowment risk.

The change in the endowment risk leads to a substantial increase in the price volatility for the riskless asset in comparison to Example 1a, even though agents have identical levels of risk aversion.

Therefore, these examples suggest that the asymmetry between traders is the cause of the increase in volatility. Usually, when sunspot equilibria are present, these asymmetries are endogenous, or related to such endogenous variables as prices. Here the asymmetries are on the exogenous and individual parameters, as opposed to the aggregate ones (total resources been identical across states).

The next and last example shows that there is positive volatility in economies that are not covered by Proposition 3.2. In particular, we show that there is positive volatility in economies with more than a single asset, that is, when $J > 1$. The gist is again the asymmetry among traders.

Example 2. Consider an economy with $\overline{S} = 4$ states at $t = 1$ and so $\overline{S}^2 = 16$ states at $t = 2$. There are $H = 2$ traders who have CRRA utility functions. Traders have identical uniform beliefs of $\pi^s = 0.25$ for $s = 1, \ldots, 4$ and $\pi^s = 0.0625$ for $s = 5, \ldots, 20$. There is a single consumption good. Traders have an endowment of $e_h^0 = 1, h = 1, 2$, at $t = 0$. For each set of $\overline{S} = 4$ states they have endowments

$$(\frac{4.25}{3}, \frac{3.5}{3}, \frac{2.5}{3}, \frac{1.75}{3}) \quad \text{and} \quad (\frac{1.75}{3}, \frac{2.5}{3}, \frac{3.5}{3}, \frac{4.25}{3}),$$

respectively. There are $J = 2$ assets. The first asset is risky and has payoffs for a set of $\overline{S}$ states of $(1.0, 2.0, 1.5, 1.0)$. The second asset has a safe payoff of 1.

γ_1	γ_2	$E_0(q_1)$	$\sigma_0^2(q_1)$	CV_1	$E_0(q_2)$	$\sigma_0^2(q_2)$	CV_2
0.5	0.5	1.42055	0.0000038	0.13716	1.04296	0.0000025	0.15263
1.0	1.0	1.50188	0.0000266	0.34313	1.11976	0.0000161	0.35842
2.0	2.0	1.79998	0.00020	0.78091	1.40236	0.00010	0.72638
3.0	3.0	2.35545	0.00057	1.01438	1.93273	0.00028	0.85865
4.0	4.0	3.32340	0.00103	0.96361	2.86455	0.00047	0.76011
5.0	5.0	4.97334	0.00134	0.73633	4.46553	0.00061	0.55246
6.0	6.0	7.76769	0.00139	0.47934	7.19599	0.00062	0.34735
7.0	7.0	12.49799	0.00120	0.27755	11.84498	0.00054	0.19629

The prices of both assets show positive price volatility.

We now turn to adjusting a framework formerly developed in Cass and Citanna for comparing equilibria before and after the introduction of new assets, allowing the comparison of price volatility in general economies, and for different degrees of market incompleteness.

4. Introducing a new asset into the economy

For the model described in Section 2, the basic idea of the analysis of the impact of financial innovation on volatility can be easily reconduced to the framework for the study of the welfare impact of financial innovation as found in Cass and Citanna (1998). Although it will become apparent that the logic of the analysis follows very closely that paper, we stress that a straightforward application of the theorems provided there is not possible in this multi-period setup with separable utilities. To reiterate, the steps of the analysis are identical, but the proofs differ because of the special structure of the payoff matrix R. Proofs of Lemmas 4.2, 5.1, of Theorem 5.2, case b), and of the lemmas and theorem in Section 5.3 are similar but not encompassed by statements contained in Cass and Citanna. For instance, in order to derive the condition on multipliers, Cass and Citanna assumed that either the last I rows of the matrix had full rank (general nonseparable case), or that the payoff matrix was in general position (additively separable utility, treated in their Appendix). Here we tied our hands as for the specification of the R matrix, given the dynamic structure of the model, and R is no longer in general position.

Obviously, our proofs in this paper show that the general position of R is not necessary to obtain results in this model[14]. Because of the similarities in the analyses of the two problems (welfare impact and volatility impact), we will show how to proceed with the general logic, we will leave the unchanged proofs to the reader, but will provide the differing proofs in the Appendix.

It is convenient to introduce some general notation for representing equations (2.2). We first normalize the numéraire commodity price at each spot, given that households' budget constraints are homogeneous of degree zero in $p^{s,C}$, all s. Hence, $p^{s,C} = 1$, all s (for simplicity, hereafter we will redefine $q/p^{0,C}$ as q). Moreover, we drop $S + 1$ commodity market clearing conditions, say, those corresponding to the numéraire commodity at each spot, by utilizing the analogue of Walras' law. Let $z_h^{\backslash}$

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[14]Since we can prove the lemmas and theorems essentially without changing conditions on S, I and H, *ex post* this difference turns out not to substantially matter. Indeed, and to anticipate the presentation of our results, in the Cass and Citanna paper the controllability is obtained if $S + 1 - I \geq H + H$, while here if $S + 1 - I \geq H + J$ (Theorem 5.2). The first H conditions account for the new no arbitrage equations, and the last conditions (H there, J here) account for the number of objectives to control.

be z_h less the numéraire commodity at each spot. Let $p^\backslash$ be the commodity price vector without the normalized components.

Let $F_1 : \Xi^{n_1} \times \Theta^* \to \mathbb{R}^{n_1}$ be the mapping representing the left-hand side of (2.2) after the above changes, where Ξ^{n_1} is the n_1-dimensional space of endogenous variables ξ, with

$$\xi = ((x_h, b_h, \lambda_h)_{h=1}^H, p^\backslash, q)$$

and the economy is again parametrized by endowments and securities only. An equilibrium in the original economy is then represented by the equation $F_1(\xi, \theta) = 0$.

We will consider a (smooth) function, defined over the equilibrium manifold of a (now multi-period and) fictitious economy, i.e. the economy where the new asset is chosen so that at the original values of the endogenous variables it is redundant, but traded. For the original economy and *given a fixed* $\theta \in \Theta^*$, we can describe the equilibria by the zeros of a mapping

$$F : \Xi^{n_1} \times \mathcal{T} \to \mathbb{R}^n$$

with $n > n_1$, such that

$$F(\xi, \tau) = (F_1'(\xi, \tau; \theta), F_2(\xi, \tau)),$$

where F_1' is going to be the mapping into $\mathbb{R}^{n_1}$ describing the same equations as those of the original equilibrium, but modified according to the designer's intervention, while F_2 is going to be the mapping into $\mathbb{R}^{n-n_1}$ describing the arbitrage-pricing and market-clearing conditions for the new assets. $\mathcal{T}$ is the space of instrumental variables, including both direct policy variables (new asset payoffs) and related market variables (new asset prices and holdings).

If we can show that there is a subvector of $n - n_1$ instrumental variables τ'' for which a regularity-like result can be established, that is,

$$\left. \text{rank } D_{\xi, \tau''} F(\xi, \tau) \right|_{\substack{F(\xi, \tau) = 0 \\ \tau' = \bar{\tau}'}} = n. \tag{4.1}$$

where τ' is another subvector of τ fixed at the value $\bar{\tau}'$ so to obtain the original equilibrium, then the set

$$M = \{(\xi, \tau) \in \Xi^{n_1} \times \mathcal{T}_{\bar{\tau}'} : F(\xi, \tau) = 0\}$$

where $\mathcal{T}_{\bar{\tau}'}$ is an open neighborhood of $\bar{\tau}'$ in $\mathcal{T}$, is a smooth, finite-dimensional manifold, by a straightforward application of the preimage theorem (see Guillemin and Pollack (1974, p. 21), for instance).

Our standard reference to ξ as "endogenous variables" is justified by the following lemma, which is a well-known regularity result, and is therefore stated here without proof.

Lemma 4.1. $\mathrm{rank} D_\xi F_1 \big|_{F_1(\xi,\theta)=0} = n_1$ *on a generic subset of* Θ^*.

Generic here means in an open, full-measure subset, although in the next section, when introducing volatilities in the analysis, we will use the term in a topological sense only. One property of the Lagrange multipliers and two of equilibrium commodity prices and asset price volatilities are summarized in the following lemma, and will be quite useful in the ensuing analysis. Their demonstration basically involves routine applications of a transversality theorem, hence the proof will be omitted. Let $\sigma(s)$ be a permutation function of $\{0, 1, \ldots, S\}$ into itself.

Lemma 4.2. *Consider the solutions to* $F_1(\xi, \theta) = 0$. *Then,*

$$(i)\ \mathrm{rank}[\lambda_h^{\sigma(s)}, 1 \quad h \quad H, 0 \quad \sigma(s) \quad H - 1] = H \text{ if } H \quad S + 1 - I \text{ and}$$

$$(ii)\ D_{q^{\bar{s},j}} \phi^j = 2\pi^{\bar{s}}(q^{\bar{s},j} - E_0(q^j)) \neq 0$$

$$(iii)\ p^{s'} \text{ is not colinear with } p^{s''}, s' \neq s'' > 0,$$

on a generic subset of Θ^*.

Notice that if $J = 1$, and it is assumed that $Y \gg 0$, we can always extract H spots out of the last $\overline{S}^2$, since the first $\overline{S} + 1 = I$ rows of R form the required full-ranked matrix.[15]

It will be convenient to let

$$\Theta_u = \{\theta \in \Theta^* \mid \theta \text{ satisfies Lemmas 4.1, 4.2}\},$$

which is therefore also a generic subset of Θ^*. From now on, $\theta \in \Theta_u$.

Now let $G : M \to \mathbb{R}^k$ be such that $m \mapsto G(m)$ is a *general* function defined over the equilibrium manifold. Note that if $m \in M$ and $\tau' = \bar{\tau}'$, then $G(m)$ is precisely the value of the function at an equilibrium before innovation takes place. One such function could be the utility vector, with $k = H$ as in Cass and Citanna, or the price level of traded securities or, as hereafter, the price volatility, with $k = J$. So it is clear from their analysis that a (local) sufficient condition to obtain a decrease (or increase) of volatility due to financial innovation is that G be a submersion at every $m \in M$ with $\tau' = \bar{\tau}'$, that is, that

$$dG_m : T_m(M) \to \mathbb{R}^J$$

is onto for all such points. We can restate this condition in terms of the rank of a suitable matrix, which can be expressed using the complementary condition in terms

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[15] 'Always extract H spots' means that the permutation σ can be chosen for all economies so that it maps spot $s = 1 + S + h$ to spot $\sigma(s) = h - 1$, for $h = 1, \ldots, H$.

of the system of equations

$$a' \left.\begin{array}{c} DF \\ DG \end{array}\right]_{\substack{F(\xi,\tau)=0 \\ \tau'=\bar{\tau}'}} = 0 \tag{4.2}$$
$$\text{and } a'a - 1 = 0,$$

where a is an $(n+J)$-dimensional vector.

Financial innovation generated by altering the yields from redundant assets can bring about an increase in volatility if the system of equations (4.2) has *no* solution.[16] We will establish that this property obtains at every equilibrium for an open and dense set of economies. Having accomplished this, we will show robustness in the space of financial innovations, that is openness of the set of volatility-reducing innovations. It will simply follow from establishing that, for an open and dense set of economies, some altered equilibrium (after volatility-reducing innovation) is regular, just as was the original equilibrium.

5. Effects on market volatility

The purpose of this section is to develop conditions on the number of states, assets and households such that, in an open and dense subset of preferences and endowments, financial innovation allows the control of asset price volatility. These conditions imply that, with sufficient market incompleteness to start with, we can find both a new asset which reduces price volatility and a new asset which increases price volatility. We first examine economies in which the new asset cannot be retraded in the middle period. After illustrating our results with some numerical examples we examine the introduction of an asset that can be traded at time $t = 1$.

5.1. Innovation without retrading

When an asset is redundant, it has no effect on the market allocation. If such a new asset is introduced into the economy, then (2.2) becomes

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[16]Note that a necessary condition to have no solution to (4.2) is that $\dim \mathcal{T} \geq n - n_1 + k$, and this will be verified in our case.

$$\vdots$$
$$Du_h(x_h) - \lambda_h \Psi = 0$$
$$\lambda_h R = 0$$
$$-\Psi z_h + [R\; r]\begin{pmatrix} b_h \\ \hat{b}_h \end{pmatrix} = 0$$
$$\vdots$$
$$\sum_h z_h^{\backslash} = 0$$
$$\sum_h b_h = 0$$
$$\vdots$$
$$\lambda_h r = 0$$
$$\vdots$$
$$\sum_h \hat{b}_h = 0,$$

(5.1)

where $\hat{b}_h$, all h, $\hat{q}$ and y are the new asset holdings, price and yields, and $r = \begin{pmatrix} -\hat{q} \\ y \end{pmatrix}$. Note that here it is implicitly assumed that the asset cannot be retraded. For retrading, see Section 5.3 below.

The left-hand side of the equations (5.1) corresponds to our function F, when $(\hat{b}, \hat{q}, y) = ((\hat{b}_h, \text{ all } h), \hat{q}, y)$ is identified with τ. A designer can introduce a new asset by choosing $\hat{b}$, $\hat{q}$ and y, with the constraint that $\hat{b}$ and $\hat{q}$ are equilibrium asset holdings (so they satisfy market clearing) and equilibrium price for given yields y (so they satisfy a no-arbitrage condition). That is, the constraints are

$$\vdots$$
$$\lambda_h r = 0$$
$$\vdots$$
$$\text{and } \sum_h \hat{b}_h = 0,$$

(5.2)

which would have to be appended to equations (2.2), while simultaneously modifying the households' budget constraints accordingly. The full set of constraints facing the designer is then described by equations (5.1). The dimension of the range of F just equals the number of equations defining an equilibrium with $I + 1$ assets. Note that the designer uses $H + 1 + S$ instruments (so that $\mathcal{T} = \mathbb{R}^{H+1+S}$), of which (as many as) $H + 1$ cannot be chosen independently, given equations (5.2).[17]

If the completely redundant asset $y = 0$ is introduced, arbitrage-pricing requires that $\hat{q} = 0$ as well, so that market-clearing is the only effective restriction on $\hat{b}$ (and the planner is free to choose all but two of the remaining "policy" instruments, $\hat{q}$ and

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[17]By this we mean that equations (5.2) restrict the choice of τ, so that in order for these equations to be satisfied, $H + 1$ elements of τ must be endogenously determined, once the others are fixed. The elements of τ which are unrestricted are said to be 'independent'.

$\hat{b}_h$, some h). Thus, in the fictitious equilibrium, a natural choice for the subvector of instrumental variables τ' is $\tau' = ((\hat{b}_h, h > 1), y)$, and for their particular values $\bar{\tau}'$, say,[18]

$$\bar{\tau}' = ((\bar{b}_h, h > 1), \bar{y}) = (1, 0).$$

With these choices, and given Lemma 4.2 (i), we will now prove condition (4.1) by selecting τ'' as

$$\tau'' = (\hat{b}_1, r^{\sigma(s)}, 0 \quad \sigma(s) \quad H - 1)),$$

For this and later purposes it is very convenient to partition

$$r = (r'', r''') = ((r^{\sigma(s)}, 0 \quad \sigma(s) \quad H - 1), (r^{\sigma(s)}, H \quad \sigma(s))),$$

to conform with τ''.

Lemma 5.1. *For every $\theta \in \Theta_u$,*

$$\operatorname{rank} D_{\xi, \hat{b}, r} F \Big|_{\substack{F(\xi, \hat{b}, r) = 0 \\ \hat{b}_h = \bar{b}_h, \ h > 1 \\ y = \bar{y}}} = \operatorname{rank} D_{\xi, \hat{b}_1, r''} F \Big|_{\substack{F(\xi, \hat{b}, r) = 0 \\ \hat{b}_h = \bar{b}_h, \ h > 1 \\ y = \bar{y}}} = n,$$

provided $S + 1 - I = \overline{S}(\overline{S} - J) + \overline{S} + 1 - J \geq H$.

Proof. See the Appendix. ∎

It is not difficult to see that, equipped with Lemma 5.1, we can apply the general methodology outlined in the previous section to this case, where $G(\xi) = \phi(\xi) = (\phi^1, .., \phi^J)$. Therefore, we establish that $\phi(\xi)$ is a submersion at (ξ, τ) with $\tau' = \bar{\tau}'$ for which $F(\xi, \tau)\big|_{\tau' = \bar{\tau}'} = 0$ if and only if the system of equations (4.2), now appearing as

$$\left. \begin{array}{cc} a' & DF \\ & D\phi \end{array} \right]_{\substack{F(\xi, \hat{b}, r) = 0 \\ \hat{b}_h = \bar{b}_h, h > 1 \\ y = \bar{y}}} = 0 \tag{5.3}$$
$$\text{and } a'a - 1 = 0,$$

has no solution. This last is the result that we now verify for an open and dense set of economies parameterized by both endowments and utility functions.

Intuitively, we are using $H + 1 + S$ instruments to achieve J objectives, the changes in volatility. The introduction of a new asset carries additional constraints in the form of H arbitrage pricing equations and one asset market clearing condition. Note that $\dim \mathcal{T} = H + 1 + S \geq H + 1 + J$, where $H + 1 = n - n_1$, which is equivalent to $S \geq J$, obviously true in our context. However, the previous lemma indicates that this is not enough. H instruments (the new asset holdings) are indeed useless, because they only control market clearing. The remaining $S + 1$, the new asset price and yields,

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[18]The argument, and only in Theorem 5.2, requires that $\bar{b}_h \neq 0$, all h.

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are also constrained to spread their effect through elements of the orthogonal space of R (through the λ's), losing I dimensions. Hence $S + 1 - I$ of these instruments must be used to accomplish the control task, but they need to satisfy the H no arbitrage equations as well. Therefore, $S + 1 - I - H$ are really the independent instruments. The following theorem shows that condition $S + 1 - I - H \geq J$ itself is necessary (in our framework) and sufficient in order to show that the equilibrium volatility function is locally onto.

Theorem 5.2. *On an open and dense set $\hat{\Theta} \subset \Theta \times \mathcal{U}$, at any original equilibrium, G is a submersion, so that there are new assets y' and y'' whose introduction decreases and increases volatility, respectively, provided $S + 1 - I \geq H + J$ (that is, again, $\overline{S}(\overline{S} - J) + \overline{S} + 1 - J \geq H + J$).*

The proof of Theorem 5.2 is rather technical and is deferred to the Appendix. From an economic viewpoint, only two aspects of the proof are worth mentioning here. First, the proof shows that it would be possible to choose the new asset payoffs out of the last spots, if these were in sufficient number. That is, if a stronger condition holds, and $\overline{S}^2 - J(\overline{S} + 1) \geq H + J$, then the new asset payoffs can be chosen only at the terminal date. When $J = 1$, no extra condition has to be explicitly imposed in order to establish the dependence of the payoff specification on the terminal date, because the condition occurs automatically. Second, in proving Theorem 5.2 we show that it is always possible to take as "independent" the subvector of instrumental variables

$$\tau''' = (r^{\sigma(s)}, H \quad \sigma(s) \quad H + J - 1).$$

and to fix $r^{\sigma(s)}$, for $\sigma(s) \geq H + J$, and $\widehat{b}_h$, for $h > 1$.[19] Taking advantage of this last observation, we can state and prove a corollary which shows the robustness of the existence of volatility-reducing innovation. It will be convenient hereafter to simply use τ in place of $(\hat{b}, r)$.

Corollary 5.3. *On an open and dense set $\hat{\Theta}^* \subset \hat{\Theta}$, at any original equilibrium, there is some altered equilibrium which is (i) volatility-reducing and (ii) regular, as well as some altered equilibrium which is (i) volatility-increasing and (ii) regular, so that there are open sets of new assets $\mathcal{Y}'$ and $\mathcal{Y}''$ such that the introduction of $y' \in \mathcal{Y}'$ or $y'' \in \mathcal{Y}''$ can decrease or increase market volatility, respectively, provided $\overline{S}(\overline{S} - J) + \overline{S} + 1 - J \geq H + J$.*

Proof. See the Appendix. ■

It should be noticed that, when there is retrading on all initial assets, the minimal degree of incompleteness achievable is $1 + \overline{S}$, when only one asset per period is missing, i.e., $J = \overline{S} - 1$. Therefore a comparison between incomplete and complete markets is

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[19]This is what "dropping the equations corresponding $\cdots$" means, as used in the course of the proof (see the Appendix).

never possible when the new asset is not retraded: the degree of incompleteness goes down to $\overline{S}$ only. Furthermore, note that $H \geq 2$ because otherwise the equilibrium is no trade, and no change in the asset structure can change the price volatility.

5.2. Numerical examples

We describe two examples that illustrate the workings of Theorem 5.2. In the Appendix we explain in detail how we compute volatility-decreasing and volatility-increasing assets by applying the proof of Theorem 5.2 and its Corollary to the economy in the examples.

Other than illustrating the workings of the theorem, the examples also show that even in the case of no aggregate risk financial innovation must be carefully designed when it does not result in complete spanning (Example 3a). No result on unambiguous effects of innovation can be expected for general incomplete versus less incomplete markets even under the assumptions of Section 3, unless the asset effectively completes the span, as at the end of Example 3a.

The analysis underlying Theorem 5.2 is local in nature; as a result, the assets computed in the two examples have very small payoffs and their introduction results in fairly small changes in the equilibrium. In particular, they result in modest volatility changes of the prices of the other assets. Even so, as the examination of Examples 3a and 3b will convincingly show, volatility changes due to a new asset can be significant. The introduction of the riskless asset in Example 3a leads to zero volatility, while the introduction of the ("uncertain Arrow-like") security in Example 3b leads to a drastic increase in volatility. We also see that the presence of a riskless asset may not automatically lead to very small asset price volatility.

We use the same basic framework for both Examples 3a and 3b. Consider an economy with $\overline{S} = 4$ states at $t = 1$. There are $H = 2$ traders who have CRRA utility functions. Traders 1 and 2 have degrees of risk-aversion of $\gamma_1 = 1$ and $\gamma_2 = 4$, respectively. Traders have identical uniform beliefs of $\pi^s = 0.25$ for $s = 1, \ldots, 4$ and $\pi^s = 0.0625$ for $s = 5, \ldots, 20$. There is a single consumption good.

Example 3a. Traders have an endowment of $e_h^0 = 1, h = 1, 2$, at $t = 0$. For each set of $\overline{S} = 4$ states at $t = 1, 2$ they have endowments

$$(1.5, 1.0, 1.0, 0.5) \quad \text{and} \quad (0.5, 1.0, 1.0, 1.5),$$

respectively. There is no aggregate endowment risk. There are $J = 2$ risky assets. For each set of $\overline{S} = 4$ states the two assets have payoffs

$$(2.0, 1.0, 2.0, 1.0) \quad \text{and} \quad (2.0, 1.5, 1.0, 0.5),$$

respectively.

First, we compute an equilibrium for this two-asset economy. The equilibrium asset prices at time $t = 1$ are

$$q_1 = (1.836560, 1.752614, 1.814881, 1.729699)$$

for the first asset and

$$q_2 = (1.552650, 1.466383, 1.529431, 1.444429)$$

for the second asset. The resulting volatility equals

$$\sigma_0^2(q_1) = 0.043728 \quad \text{and} \quad \sigma_0^2(q_2) = 0.044283.$$

One volatility-reducing asset has a payoff vector r with the following properties. It has nonzero payoffs in the four states 5, 10, 15, and 20. (So, from the 16 states at $t = 2$ these are the states 1, 6, 11, and 16.) The computed payoffs in these four states are

$$y(5) = -0.014512, \ y(10) = +0.015973, \ y(15) = +0.04210542, \ y(20) = -0.02143766.$$

The price of this new asset and all other payoffs are set to 0. Furthermore the holdings of this new asset for the two traders are $\hat{b}_1 = 1$ and $\hat{b}_2 = -1$.

The equilibrium asset prices at time $t = 1$ in the altered economy with the additional volatility-reducing asset are

$$\hat{q}_1 = (1.828769, 1.746204, 1.819525, 1.745595)$$

for the first asset and

$$\hat{q}_2 = (1.542404, 1.457999, 1.524497, 1.453454)$$

for the second asset. The resulting volatility is

$$\sigma_0^2(\hat{q}_1) = 0.039261 \quad \text{and} \quad \sigma_0^2(\hat{q}_2) = 0.039407.$$

The price volatility is reduced by 10.2% and 11.0%, respectively.

When we repeat the process to find the payoffs of a volatility-increasing asset we obtain

$$y(5) = +0.007256, y(10) = -0.0079865, y(15) = -0.02371974, y(20) = +0.01280912.$$

The equilibrium asset prices at time $t = 1$ in this altered economy with the additional volatility-enhancing asset are

$$\hat{q}_1 = (1.840544, 1.75591, 1.812808, 1.720833)$$

for the first asset and

$$\hat{q}_2 = (1.557893, 1.470688, 1.532603, 1.439363)$$

for the second asset. The resulting volatility is

$$\sigma_0^2(\hat{q}_1) = 0.046898, \quad \text{and} \quad \sigma_0^2(\hat{q}_2) = 0.047304.$$

The price volatility is increased by 7.25% and 6.82%, respectively.

In the equilibria of both three-asset economies the holdings of the new third asset are $\hat{b}_1 = 1$ and $\hat{b}_2 = -1$ and its asset price at $t = 0$ is 0.[20]

Note that the example has been set up in such a fashion that the introduction of a riskless asset paying 1 in every state at $t = 1$ and $t = 2$ but which is not retraded, results in the spanning of the endowments and thus zero price volatility at $t = 1$. The price of the first asset is 1.5 in all four states, the price of the second asset equals 1.25 in all four states.

Example 3b. Traders have an endowment of $e_h^0 = 1, h = 1, 2$, at $t = 0$. For each set of $\overline{S} = 4$ states at $t = 1, 2$ they have endowments

$$(5.0, 4.0, 3.0, 2.0) \quad \text{and} \quad (2.0, 3.0, 4.0, 5.0),$$

respectively, so that there is aggregate risk. There are $J = 2$ assets, one is risky and one is riskless. For each set of $\overline{S} = 4$ states the two assets have payoffs

$$(1.0, 4.0, 4.0, 1.0) \quad \text{and} \quad (1.0, 1.0, 1.0, 1.0),$$

respectively.

Equilibrium asset prices at time $t = 1$ are

$$q_1 = (2.844958, 2.848529, 2.864825, 2.915806)$$

for the first asset and

$$q_2 = (1.255473, 1.260945, 1.280724, 1.331791)$$

for the second asset. The resulting volatility equals

$$\sigma_0^2(q_1) = 0.028304 \quad \text{and} \quad \sigma_0^2(q_2) = 0.030115.$$

We find that the payoffs of a volatility-decreasing asset are

$$y(5) = 0.005398, \ y(10) = -0.007714, \ y(15) = -0.06571669, \ y(20) = +0.02836388.$$

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[20]The changes in volatility show the same qualitative feature if we don't use the variance or standard deviation of asset prices, but the coefficient of variation. In the two-asset economy the coefficients are $CV_1 = 2.4519$ and $CV_2 = 2.9557$. The corresponding values in the three-asset economy with the new volatility-reducing asset are $CV_1 = 2.1994$ and $CV_2 = 2.6367$, respectively. The corresponding values in the economy with the new volatility-increasing asset are $CV_1 = 2.6310$ and $CV_2 = 3.1533$, respectively.

The equilibrium asset prices at time $t = 1$ in the altered economy with the additional volatility-reducing asset are

$$\hat{q}_1 = (2.845877, 2.846881, 2.879166, 2.908362)$$

for the first asset and

$$\hat{q}_2 = (1.256404, 1.260504, 1.284301, 1.32433)$$

for the second asset. The resulting volatility is

$$\sigma_0^2(\hat{q}_1) = 0.025846 \quad \text{and} \quad \sigma_0^2(\hat{q}_2) = 0.026985.$$

The price volatility is reduced by 8.68% and 10.4%, respectively. For a similar volatility-increasing asset, the volatility is increased by 12.4% and 9.42%, respectively.

Now we examine the introduction of an asset with

$$y(5) = y(10) = y(15) = y(20) = 1.0$$

When the economy is in the middle period, this asset is effectively an Arrow security. At $t = 0$, however, it is unknown for which state this asset is such a security; this uncertainty is only resolved at $t = 1$. The equilibrium asset prices at time $t = 1$ in this altered economy with the additional volatility-increasing asset are

$$\hat{q}_1 = (2.753361, 2.747394, 3.046386, 3.130496)$$

for the first asset and

$$\hat{q}_2 = (1.163808, 1.237002, 1.323859, 1.54719)$$

for the second asset. The resulting volatility is

$$\sigma_0^2(\hat{q}_1) = 0.17164, \quad \text{and} \quad \sigma_0^2(\hat{q}_2) = 0.14396.$$

The price volatility is increased by factors of more than 6 and 4.75, respectively.

5.3. Innovation with retrading

In this section we consider the effects on volatility of the introduction of a new security that can be retraded at time $t = 1$. In the previous section we dealt with innovative instruments that could not be retraded between today, time of the innovation, and the terminal date. Although some hedging instruments which are traded over the counter present this one-time trading feature, most newly traded securities are marketed in exchanges where retrading is possible. It turns out that the general framework of Section 4 is applicable to this case in a fruitful manner, and almost the same analysis developed in Section 5 carries through. In particular Lemma 5.1 and Theorem 5.2 can be recovered when the new asset can be retraded, provided we change slightly

Lemma 4.2 (i). Since the proofs of the results are similar to those presented in the previous sections, the details are omitted. The bottom line of this section is that retrading makes controllability more difficult, in the sense that the condition used to obtain volatility-reducing (or volatility-increasing) innovation is stronger than with no retrading. Introducing an asset with retrading corresponds to introducing $\overline{S}+1$ new markets. Although we have more assets to use, the number of payoffs we control is unchanged (totalling S), while the number of constraints increases, because retrading requires more no arbitrage equations for the new asset.

Let $\hat{R}$ be the matrix representing the payoffs and prices of the new financial instrument. $\hat{R}$ is just a copy of R, that is if we were to assume that $J = 1$, with arbitrary payoffs Y, taken in general to be different across time and states. Let $r^s = (-q^s, y(s))$, for $s = 0, 1, \ldots, \overline{S}$. We need to append $\overline{S}+1$ market clearing equations and $(\overline{S}+1) H$ pricing equations to the original equilibrium system. As before, we will call F the equilibrium function with these appended equations.

First, we need to strengthen Lemma 4.2. Let σ' be a permutation of the states $\{s, s\overline{S}+1, s\overline{S}+2, \ldots, s\overline{S}+\overline{S}\}$, for each $s = 0, 1, \ldots, S$, with generic element $\sigma'(s)$, that is, a permutation of the set including the direct successors of a state s and this state.

Lemma 5.4. *Consider the solutions to $F_1(\xi, \theta) = 0$. Generically in Θ^*, given any set of the $\overline{S}$ direct successors of the state $s = 0, 1, \ldots, \overline{S}$, and a permutation σ',*

$$(ii) \operatorname{rank}[\lambda_h^{\sigma'(s)}, 1 \quad h \quad H, 0 \quad \sigma'(s) \quad H-1] = H \ ,$$

if $1 + \overline{S} \geq H + J$.

Then we can prove the analogue of Lemma 5.1. Let $r^s = (r^{s''}, r^{s'''}) = ((r^{\sigma'(s)}, 0 \quad \sigma'(s) \quad H-1), (r^{\sigma'(s)}, H \quad \sigma'(s))$ be a partition of each vector r^s, for $s = 0, 1 \ldots, \overline{S}$. Let $\hat{b}_1 = \left(\hat{b}_1^0, \hat{b}_1^1, \ldots, \hat{b}_1^{\overline{S}}\right)$, the vector of the new security holdings for trader $h = 1$. After appropriately redefining the set Θ_u in the obvious way, we have the following lemma.

Lemma 5.5. *For every $\theta \in \Theta_u$,*

$$\operatorname{rank} D_{\xi, \hat{b}, r} F \Big|_{\substack{F(\xi, \hat{b}, r) = 0 \\ \hat{b}_h = \bar{b}_h, \ h > 1 \\ y = \bar{y}}} = \operatorname{rank} D_{\xi, \hat{b}_1, r''} F \Big|_{\substack{F(\xi, \hat{b}, r) = 0 \\ \hat{b}_h = \bar{b}_h, \ h > 1 \\ y = \bar{y}}} = n,$$

provided $1 + \overline{S} \geq H + J$.

The proof is based on the use of Lemma 5.4 and the fact that we can extract H states following and including each state $s = 0, 1, \ldots, \overline{S}$ for which the multiplier matrix has rank H.

Finally, Theorem 5.2 can be modified accordingly. We state here only one side of the result, the volatility-reducing part, although the theorem shows that both directions for volatility are possible.

Theorem 5.6. *On an open and dense set $\tilde{\Theta} \subset \Theta \times \mathcal{U}$, at any original equilibrium, G is a submersion, so that in particular there is a new asset with retrading y, whose introduction decreases volatility provided $1 + \overline{S} - J \geq H + J$.*

The proof of this theorem would show that only the terminal payoffs need to be chosen appropriately. Moreover, it is immediate to see that the condition $1 + \overline{S} - J \geq H + J$ needs to hold only for those many states following one state in period $t = 1$. In other words, if the number of states varied from node to node, a weaker condition than the one used in the theorem could be adopted.

When the minimal degree of incompleteness achievable is $1 + \overline{S}$, a comparison between incomplete and complete markets is possible, as the new retradable asset allows markets to be dynamically complete. In this case, as asset markets can be completed by the introduction of the new asset, $(1 + \overline{S})(H - 1)$ no arbitrage equations become redundant: with (dynamically) complete markets, multipliers are colinear across households. Hence, with the minimal degree of incompleteness, only one no arbitrage equation can be appended to system (2.2) as a constraint, say $\lambda_1 \hat{R} = 0$. Then, the condition in Theorem 5.6 becomes $1 + \overline{S} - J \geq 1 + J$. In other words, our Theorem 5.6 allows the comparison between incomplete and (dynamically) complete markets when there is only one asset to start with and two states of the world, i.e., $J = 1$ and $\overline{S} = 2$, and an arbitrary number of households and commodities.

Of course, the theorem fails to apply if one restricts economies to the no aggregate risk class analyzed in Section 3. In fact, it is Lemma 5.4 that needs to be modified to include the additional restrictions on the stationarity of conditional expected payoffs, and this reduces the degrees of freedom making the degree of (dynamic) incompleteness covered by Theorem 5.6 larger than one.

6. Appendix

Proof of Lemma 3.1

Without loss of generality, assume that $\widehat{p}^{s,C} = \pi^s$. Then, $Dv_h(x_h^s) = Du_h(x_h) \cdot x_h/(\widehat{p} \cdot e_h)$ $\widehat{p}$ is state independent, or $x_h^s = \overline{x}_h$, all s. Therefore, for each household, $\overline{x}_h$ solves $\max v_h(\overline{x}_h)$ s.t. $\widehat{\overline{p}}(\overline{x}_h - \overline{e}_h) = 0$, where $\overline{e}_h = \sum_s \pi^s e_h^s$. Substituting the solution $\overline{x}_h(\widehat{\overline{p}}, \overline{e}_h)$ into market clearing, we have

$$\sum_h \overline{x}_h(\widehat{\overline{p}}, \overline{e}_h) = \sum_h e_h^s \tag{6.1}$$

for all s, which implies $\sum_h e_h^s = \sum_h e_h^{s'} = r$ for all s. On the other hand, suppose that there is no aggregate risk. From the previous reasoning we know that $\widehat{p}^C = \pi$ is a complete market equilibrium. To see that it is the unique equilibrium, suppose not. Then consumptions are different across states, while any Pareto efficient allocation has constant consumption vectors across states and any complete market equilibrium is efficient, a contradiction. ∎

Proof of Proposition 3.2

(i) To show the result, we need to prove that the derivative matrix $D_\xi F_1'$ of the equilibrium system (2.2) with respect to all the endogenous variables has full rank. For density, divide the proof in two cases.

Case A. For all h, $x_h^s \neq x_h^{s'}$ for all s, s' with $s \neq s'$. We proceed by group of equations in (2.2), denoted with a number in parenthesis.

(1): use ΔDu^h (Note: perturbed utilities are still state independent), as in Kajii (1991) or Pietra (1992).

(2): for the first, use $\Delta \lambda_h^0$ (since $q^0 > 0$); for the others, use $\Delta \lambda_h^{\overline{s}}$;

(3): use $\Delta x_h^{\cdot,C}$;

(4): this equation has dropped out as $C = 1$;

(5): use Δb_h;

Case B. For some h, $x_h^s = x_h^{s'}$ for all s, s' with $s \neq s'$. Notice that under effective market incompleteness, this cannot occur for more than $\overline{S}$ states. We then perturb equations (1) using ΔDu_h^s for all date-events other than the ones with equal consumption, and including one of them (we always perturb this way the equation corresponding to state $s = 0$). For the remaining date-events, we perturb equations (1) using the corresponding λ_h^s. We substitute the perturbation of equations (2) with a remaining free λ_h^s. Everything else is unchanged.

These perturbations show that the row rank of $[D_\xi F_1' | B]$ is full, where B is the matrix of derivatives of the equilibrium system with respect to utilities. We then apply Sard's theorem to the natural projection of the equilibrium set and obtain by transversality that there is a dense subset of utilities for which the equilibrium is regular, i.e., rank of $D_\xi F_1'$ is full.

As for openness, simply note that $\det D_\xi F_1' = 0$, where $\det D_\xi F_1'$ is the Jacobian of the equilibrium system, is the equation which we add to the equilibrium system when regularity fails. Since det is a continuous function, the set where $\det D_\xi F_1' = 0$ is a closed subset of the equilibrium set. Since the natural projection when restricted to the equilibrium set is proper, the image of this closed subset in the parameter space is also closed. Therefore, its complement is open.

(ii) Fixing arbitrary states s, s', we append the equation $q_{(1,s)} = q_{(1,s')}$ to the equilibrium system. It is immediate to see that the new system can be perturbed by using $q_{(1,s)}$ for this last equation, say, and as in the previous lemma for the rest. Then, by transversality, the row rank of its derivative matrix must be full. However, the number of rows being greater than the number of columns implies that the system with the additional equation has no solution, i.e., that in a dense subset of utilities $q_{(1,s)} \neq q_{(1,s')}$. Openness follows as in the previous proposition. Since there is a finite combination of equalities to be appended, taking the intersection among these finitely many open and dense sets we obtain the desired result. ∎

Proof of Lemma 5.1

The derivative matrix of F at a point as defined above is

eq. / var.	$\cdots$	$\mathbf{x}_h$	$\mathbf{b}_h$	$\boldsymbol{\lambda}_h$	$\cdots$	$\mathbf{p}$	$\mathbf{q}$	$\cdots$	$\hat{\mathbf{b}}_h$	$\cdots$	$\mathbf{r}$
$\vdots$											
$\mathbf{Du}_h - \boldsymbol{\lambda}_h \boldsymbol{\Psi}$		$D^2 u_h$	0	$-\Psi'$		$-\Lambda_h^{\backslash}$	0		0		0
$\boldsymbol{\lambda}_h \mathbf{R}$		0	0	R'		0	Q_h^1		0		0
$-\boldsymbol{\Psi}\mathbf{z}_h + [\mathbf{R}\ \mathbf{r}]\begin{pmatrix}\mathbf{b}_h\\\hat{\mathbf{b}}_h\end{pmatrix}$		$-\Psi$	R	0		$-Z_h^{\backslash}$	Q_h^2		0		$\bar{b}_h I$
$\vdots$											
$\sum_h \mathbf{z}_h^{\backslash}$		$I^{\backslash}$	0	0		0	0		0		0
$\sum_h \mathbf{b}_h$		0	I	0		0	0		0		0
$\vdots$											
$\boldsymbol{\lambda}_h \mathbf{r}$		0	0	0		0	0		0		λ_h
$\vdots$											
$\sum_h \hat{\mathbf{b}}_h$		0	0	0		0	0		1		0

where

$$I^{\backslash} = \begin{bmatrix} \ddots & & \\ & [I\ 0] & \\ & & \ddots \end{bmatrix},$$

$$\Lambda_h^{\backslash} = \begin{bmatrix} \ddots & & & \\ & \lambda_h^s & I & \\ & & 0 & \\ & & & \ddots \end{bmatrix}$$

and

$$Z_h^{\backslash} = \begin{bmatrix} \ddots & & \\ & (z_h^{s\backslash})' & \\ & & \ddots \end{bmatrix},$$

$$Q_h^1 = \begin{bmatrix} -\lambda_h^0 I & \lambda_h^1 I & \cdots & \lambda_h^{\overline{S}} I \\ 0 & -\lambda_h^1 I & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & -\lambda_h^{\overline{S}} I \end{bmatrix}$$

and

$$Q_h^2 = \begin{bmatrix} -b_h^0 & 0 & \cdots & 0 \\ 0 & -b_h^1 + b_h^0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -b_h^{\overline{S}} + b_h^0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lemma 4.1 implies that the block in the upper left corner (corresponding to $D_\xi F_1'$) has full rank. But Lemma 4.2 (i) implies that, when $S + 1 - I \geq H$, the block in the lower right corner (corresponding to $D_{\hat{b},r} F_2$) also has full rank; simply consider the columns corresponding to the particular variables $\hat{b}_1$ and r'' (that is, corresponding to $D_{\hat{b}_1, r''} F_2$). Hence, since the block in the lower left corner (corresponding to $D_\xi F_2$) is 0, the matrix $D_{\xi, \hat{b}_1, r''} F$, and, a fortiori, the matrix $D_{\xi, \hat{b}, r} F$ must have full rank. ∎

Proof of Theorem 5.2

The proof is carried out in two steps.

Step 1 - Openness.

All we need to show is that the projection $\pi : \Xi^{n_1} \times \mathcal{T} \times \Theta \times \mathcal{U} \to \Theta \times \mathcal{U}$ is proper when restricted to the subset of the domain where

$$\begin{aligned} F(\xi, \hat{b}, r, e, Y(s)_{s=0}^{\overline{S}}, u) &= 0 \\ \hat{b}_h &= \bar{b}_h, h > 1 \\ \text{and } y &= \bar{y}. \end{aligned} \tag{6.2}$$

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This follows from the fact that the set of solutions to equations (5.3) yields a closed subset of the solutions to (6.2). Thus, given properness, the complement of its projection into the parameter space is open. But properness can be established through a well-known argument (see Citanna, Kajii and Villanacci, Lemma 1, e.g.), whose details we therefore omit.

Step 2 - Density.
Without loss of generality, we will assume that a (θ, u) is chosen in an open and dense subset of $\Theta \times \mathcal{U}$ such that $\theta \in \Theta_u$, with all the stated properties. Moreover, we will assume that at (θ, u) there is only one equilibrium. To establish density in $\Theta \times \mathcal{U}$, we show that the system (5.3) almost never has a solution for an open and full-measure subset of $\{\theta\} \times \mathcal{A}$, where $\theta \in \Theta_u$ and $\mathcal{A} = \cdots \times \mathcal{A}_h \times \cdots \subset (\mathbb{R}^{G^2})^H$ is the space representing a finite-dimensional parametrization of the households' utility functions around u (again, see Cass and Citanna, e.g.).

The reader should keep in mind that we need to be able to perturb utility spot-by-spot, at the same time keeping the functional form constant across spots. This could be done if we had

$$x_h^{s'} \neq x_h^{s''}, s' \neq s'' > 0, \text{ all } h.$$

But as we proved in Lemma 4.2 (iii), for differentially strictly concave (in fact, quasi-concave) utility functions and on a generic subset of Θ,

$$p^{s'} \text{ is not colinear with } p^{s''}, s' \neq s'' > 0,$$

from which the required diversity in households' consumption follows directly.

Consider the system given by (5.3) in extensive form, that is, (6.2) and

$$
\left.
\begin{array}{lll}
 & \vdots & \\
HG & \alpha_h D^2 u_h - \gamma_h \Psi + \delta I^{\backslash} = 0 & (1) \\
HI & \gamma_h R + \epsilon = 0 & (2) \\
H(S+1) & -\alpha_h \Psi' + \beta_h R' = 0 & (3) \\
 & \vdots & \\
(C-1)(S+1) & \sum_h (\alpha_h \Lambda_h^{\backslash} + \gamma_h Z_h^{\backslash}) = 0 & (4) \\
I & \sum_h (\beta_h Q_h^1 + \gamma_h Q_h^2) + \mu D_q \phi = 0 & (5) \\
 & \vdots & \\
H \ [1] & \hat{\epsilon} = 0 & (6) \\
 & \vdots & \\
S+1 \ [H+J] & \sum_h (\bar{b}_h \gamma_h + \hat{\beta}_h \lambda_h) \ [\sum_h (\gamma_h^{\#} \bar{b}_h + \hat{\beta}_h \lambda_h^{\#})] = 0 & (7) \\
1 & \text{and } \mu' \mu - 1 = 0, & (8)
\end{array}
\right\} \quad (6.3)
$$

where $a' \equiv (\cdots, (\alpha_h, \beta_h, \gamma_h), \cdots, \delta, \epsilon, \cdots, \hat{\beta}_h, \cdots, \hat{\epsilon}, \cdots, \mu_i, \cdots)$, $\gamma_h = (\gamma_h^{\#}, \gamma_h^{\#\#}) = ((\gamma_h^s, 0 \quad \sigma(s) \quad H + J - 1), (\gamma_h^s, \sigma(s) \geq H + J))$ and $\lambda_h = (\lambda_h^{\#}, \lambda_h^{\#\#})$ split accordingly (notation which is only used for the remainder of this argument) and on

the far left side we have displayed the number of equations. Equation (6.3.8) replaces $a'a - 1 = 0$ without loss of generality due to Lemma 5.1. Since they are all identical, hence redundant, we drop $H - 1$ equations corresponding, in particular, to all but the first of (6.3.6). Given that $S + 1 - I \geq H + J$, and consequently that $S + 1 - H - J \geq 0$, it follows that the equations (6.3) still outnumber the additional variables a by this difference. So we can drop all but $H + J$ of (6.3.7) – as indicated in square brackets – and still have one more equation than variables. Now observe that the restriction on the domain in (6.3), that is, (6.2), is equivalent to

$$F_1(\xi, \theta, u) = 0,$$
$$\hat{b} = \bar{b} = (-(H-1), 1, \cdots, 1)$$
$$\text{and } r = \bar{r} = 0.$$

But since $\theta \in \Theta_u$, rank $D_\xi F_1 = n_1$, while by construction, $D_{a,A} F_1 \big|_{F_1(\xi,\theta,u)=0} = 0$. It remains to show that the Jacobian matrix of the truncated subsystem in (6.3) (with respect to (a, A)) has full rank, in order to apply the transversality theorem, and to conclude that, generically in parameterized utility functions, the full system (5.3) has no solution. We need to consider few cases, since the matrix of derivatives of equation (6.3.1) with respect to the elements of the symmetric matrix A_h has full rank if $\alpha_h^s \neq 0$, all s, but, for example, will have less-than-full rank if $\alpha_h^s = 0$, for some s, h. Here is where the general position of R, which we do not have, was heavily used in Cass and Citanna.

Case a - $\alpha_h^s \neq 0$, all s, all h.
In this case it is straightforward to verify that equations (6.3.1), all h, can be perturbed independently by using the utility parameters A. Since μ only appears in equations (6.3.5) and (6.3.8), while, in light of equation (6.3.8), μ can never be equal to zero, this last can then be perturbed independently using μ. Similarly, since α_h only appears in equations (6.3.1), (6.3.8), (6.3.3) and (6.3.4), the last two, all h, can be perturbed independently using α: Use $\alpha_h^{s,C}$, all s, in (6.3.3), and $\alpha_h^{s,c}$, all s, all $c \neq C$, some h, in (6.3.4). Continuing in the same manner, equations (6.3.5) can be perturbed independently using β_h, some h, while, obviously, equation (6.3.6) can be perturbed independently using $\hat{\varepsilon}$. Finally, equations (6.3.2), all h, and (6.3.7) can be perturbed independently using $\gamma_h^{\#\#}$, all h, and $\hat{\beta}_h$, all h, $\gamma_h^{\#,s}$, for J spots s and some h; here we appeal to the assumptions that: a) the first I rows of R (including the relabelled spots $S + 1 \geq \sigma(s) \geq H + J$ and therefore part $\#\#$ of the vectors) form an I^2−dimensional, full rank matrix ; b) by assumption, we have at least J elements $\gamma_h^\#$, and the $\lambda_h^\#$'s selected include the ones forming a matrix of rank H; c) $\bar{b}_h \neq 0$, all h.

Case b - $\alpha_h^s = 0$, some s, some h.
First, note that $\alpha_h = 0$, some h cannot occur. In this case it is straightforward to verify that $\alpha_h = 0$, $\beta_h = 0$ and $\gamma_h = 0$. Then $\delta = 0$ and $\epsilon = 0$, which implies $(\alpha_h, \beta_h, \gamma_h) = 0$ from demand regularity for all other h, and this implies from (6.3.5) and Lemma 4.2.ii) that $\mu = 0$, which contradicts equation (6.3.8).

Let $S_h^0 = \{s \in \{0, 1, \ldots, S\} : \alpha_h^s = 0\}$, with $S_h^0 \neq \emptyset$, some h, and

$$\overline{S} = \cup_h S_h^0$$
$$\underline{S} = \cap_h S_h^0.$$

We need to look at system (6.3) more closely. We rewrite its equations below, state by state:

$$
\left.
\begin{array}{rcl}
s \notin \overline{S} & \alpha_h^s D_{s,s}^2 u_h - \gamma_h^s p^s + \delta^s\,[I0] = 0 & (1a) \\
s \notin S_h^0, s \in \overline{S} & \alpha_h^s D_{s,s}^2 u_h - \gamma_h^s p^s = 0 & (1b) \\
& \gamma_h R + \epsilon = 0 & (2) \\
s \notin S_h^0 & -\alpha_h^s \left(p^s\right)' + \beta_h \left(r^s\right)' = 0 & (3a) \\
s \in S_h^0 & \beta_h \left(r^s\right)' = 0 & (3b) \\
s \notin \underline{S} & \sum_{h:s \notin S_h^0}\left(\alpha_h^{s\backslash}\lambda_h^s + \gamma_h^s z_h^{s\backslash}\right) = 0 & (4) \\
& \sum_h \left(\beta_h Q_h^1 + \gamma_h Q_h^2\right) + \mu D_q \phi = 0 & (5) \\
& \hat{\epsilon} = 0 & (6) \\
s \notin \underline{S} & \sum_h \left(\gamma_h^s \bar{b}_h + \hat{\beta}_h \lambda_h^s\right)] = 0 & (7a) \\
s \in \underline{S} & \sum_h \hat{\beta}_h \lambda_h^s = 0 & (7b) \\
& \text{and } \mu'\mu - 1 = 0. & (8)
\end{array}
\right\} \quad (6.4)
$$

We have substituted for $\alpha_h^s = 0$, discovering that this implies $\gamma_h^s = 0$ and $\delta^s = 0$ for these $s \in S_h^0$. This means that the corresponding equations (6.3.1) will drop for these spots $s \in S_h^0$. Note that also some equations among (6.3.4) have (possibly, for $s \in \underline{S}$) dropped out of the system. We have at least $\sum_h S_h^0$ extra equations. If we decide not to perturb the zeroed variables, we can drop some other equations, since now the equations still outnumber the unknowns by more than one. Notice also that equations (6.4.1) can be perturbed using the utility parameters A_h, equations (6.4.3a) and (6.4.4) can be perturbed using the α's as before, and equations (6.4.6) using $\hat{\epsilon}$. We are left with perturbing equations (6.4.2), (6.4.3b), (6.4.5), (6.4.7) and (6.4.8).

Let $R_h^{\backslash 0}$ be the submatrix of R with rows corresponding to those spots $s \notin S_h^0$, and R_h^0 be the matrix with rows corresponding to spots $s \in S_h^0$; equations (6.4.3b) then read $\beta_h R_h^{0'} = 0$. At this junction, the choice of the equations to be dropped depends on the rank of these two matrices. Let rank $R_h^{\backslash 0} = I_h^* = \min\{I, 1 + S - S_h^0\}$ and rank $R_h^0 = I_h^{**} = \min\{I, S_h^0\}$, using the general position of Y. Of equations (6.4.3b), $S_h^0 - I_h^{**}$ are redundant and must be thrown away. Therefore we have a surplus of at least $\sum_h I_h^{**}$ (i.e., fewer) equations.

If $S_h^0 = \emptyset$, some h, then his equations (6.4.3b) disappear, and equations (6.4.3), (6.4.2) for this h and (6.4.7) are perturbed using this trader α_h, γ_h and $\hat{\beta}$, as in Case a). Then we can use this trader's β_h to perturb equations (6.4.5). The remaining equations (6.4.2) and (6.4.3b) are perturbed as follows. Since there are $\sum_h I_h^{**} + \sum_{s \in \overline{S}}(C - 1)$ extra equations, if $I_h^{**} = I$, we throw away equations (6.4.2) and use β_h to perturb $\beta_h = 0$, implied by (6.4.3b). If $I_h^{**} < I$, then $I_h^{**} = S_h^0$, and we can use β_h to perturb equations (6.4.3b), and perturb (6.4.2) using γ_h possibly (if $I_h^* < I$) after throwing away S_h^0 of these equations. Equation (6.4.8) is perturbed using μ.

So consider the case when $S_h^0 \neq \emptyset$, all h.

1) If $I_h^{**} = I$, all h, equations (6.4.3b) imply $\beta_h = 0$. Then the system of equations for each household is

$$
\begin{array}{rcl}
s \notin \overline{S} & \alpha_h^s D_{s,s}^2 u_h - \gamma_h^s p^s + \delta^s \, [I0] = 0 & (1a) \\
s \in \overline{S}, s \notin S_h^0 & \alpha_h^s D_{s,s}^2 u_h - \gamma_h^s p^s = 0 & (1b) \\
& \gamma_h R + \epsilon = 0 & (2) \\
s \notin \overline{S} & -\alpha_h^s \, (p^s)' = 0 & (3a)
\end{array}
$$

First, we throw away J equations (6.4.7), perturbing the remaining H using $\hat\beta$. Equation (6.4.8) is perturbed using μ. Note that equations (6.4.5) are now $\sum_h \gamma_h Q_h^2 + \mu D_q \phi = 0$. They can be rewritten as

$$
\begin{bmatrix} \gamma_1^0 & \cdots & \gamma_H^0 \end{bmatrix}
\begin{bmatrix} -b_1^0 \\ \vdots \\ -b_H^0 \end{bmatrix}_{H \times J}
= 0
$$

and

$$
\begin{bmatrix} \gamma_1^s & \cdots & \gamma_H^s \end{bmatrix}
\begin{bmatrix} -b_1^s + b_1^0 \\ \vdots \\ -b_H^s + b_H^0 \end{bmatrix}_{H \times J}
+ \mu(D_{q^s} \phi) = 0
$$

for $s = 1, \ldots, \overline{S}$. For each s, let $J^*(s)$ be the rank of these matrices after deleting the rows corresponding to $\gamma_h^s = 0$. In order to perturb equations (6.4.5) using γ_h^s, we need to delete $\sum_{s=0}^{\overline{S}} [J - J^*(s)]$ of them. We observe that it can never be that $\gamma_h^s = 0$ for all h, for $s > 0$, since then $\mu = 0$, contradicting Lemma 5.1. Hence $J^*(s) \geq 1$, for $s > 0$ (this uses the fact that $b_h^s \neq 0$ or $-b_h^s + b_h^0 \neq 0$, all h, s, generically in Θ^*, through an argument similar to the ones in the proof of Lemma 4.2). This implies $I > J + \sum_{s=0}^{\overline{S}} [J - J^*(s)]$. Take $h = 1$ and leave $J + (J - J^*)(\overline{S} + 1)$ of equations (6.4.2), and perturb them using ϵ^i. Then we can throw away the I equations (6.4.2) for $h > 1$, completing this subcase.

2) If $I_h^{**} < I$, for some, but not all h: When $I_{h'}^{**} = I$ for some h', for one of them we keep equations (6.4.2), which we perturb using ϵ. Equations (6.4.3b) are eliminated, thereby freeing the vector $\beta_{h'}$ for use in equations (6.4.5). For all other h', we can throw away the $I_{h'}^{**}$ equations (6.4.2), and use the vector $\beta_{h'}$ in equations (6.4.3b). For h such that $I_h^{**} < I$, we throw away $I_h^{**} = S_h^0$ equations (6.4.2), obtaining a submatrix of $R_h^{\backslash 0}$ with $1 + S - S_h^0$ rows and $I - S_h^0$ columns, and since rows are more than columns, the rank of this submatrix is $I_h^* = 1 + S - S_h^0$ using the general position of Y; we use γ_h to perturb the remaining equations (6.4.2). Equation (6.4.3b) is perturbed using β_h, and equation (6.4.8) using μ. For equations (6.4.7), we can perturb the J (6.4.7a) using γ_h^s, for some h such that $s \notin S_h^0$, and perturb the rest with $\hat\beta$. Note that such a γ_h^s is free because $s \notin \underline{S}$ if s is part of these equations, and because either γ_h^s is not

used to perturb (6.4.2), for h such that $I_{h'}^{**} = I$, or we can always avoid using one such element if h is such that $I_h^{**} < I$, since $S + 1 - S_h^0 > I - S_h^0$, completing the subcase.

3) If $I_h^{**} < I$, all h : Then, as in the previous subcase, for $h > 1$ we throw away $I_h^{**} = S_h^0$ equations (6.4.2), obtaining a submatrix of $R_h^{\backslash 0}$ with more rows than columns, and we use γ_h to perturb the remaining equations (6.4.2). For equations (6.4.3b), we use β_h. For $h = 1$, we keep equations (6.4.2) and perturb them with ϵ. Then we throw away equations (6.4.3b), we use β_1 to perturb equations (6.4.5). As for equations (6.4.7a), again we use γ_h^s for h such that $s \notin \underline{S}$, which we can always do as explained in the previous subcase, and for equation (6.4.8) we use μ.

This ends the proof of case b).

In this way, we have established density. It now follows that, in the (weakly) generic set of economies for which G is a submersion, we can construct a new asset $y' = \Delta y' \neq 0$ so that volatility increases in equilibrium, as well as a new asset $y'' = \Delta y'' \neq 0$ so that volatility decreases. ∎

Proof of the Corollary to Theorem 5.2.

We will focus on decrease of volatility, as the symmetric argument requires merely reversing an inequality. Fix $\hat{b}_h = \bar{b}_h = 1, h > 1$, and $y^{\sigma(s)} = \bar{y}^{\sigma(s)} = 0, \sigma(s) \geq H + J$. Then, for $\hat{\theta} \in \hat{\Theta}$, consider the system of equations and inequalities describing pairs of fictitious and altered equilibria where the latter are both volatility-reducing and regular, that is,

$$
\begin{aligned}
F(\bar{\xi}, \bar{\tau}, \hat{\theta}) = 0 \text{ or } F_1(\bar{\xi}, \hat{\theta}) = 0 \\
F(\xi, \tau, \hat{\theta})\Big|_{\substack{\hat{b}_h = \bar{b}_h, h > 1 \\ y^{\sigma(s)} = \bar{y}^{\sigma(s)}, \sigma(s) \geq H + J}} = 0 \\
\phi(\xi) \ll \phi(\bar{\xi}) \\
\text{and } \det D_{\xi, \hat{b}, \hat{q}} F(\xi, \tau, \hat{\theta})\Big|_{\substack{\hat{b}_h = \bar{b}_h, h > 1 \\ y^{\sigma(s)} = \bar{y}^{\sigma(s)}, \sigma(s) \geq H}} \neq 0.
\end{aligned}
\tag{6.5}
$$

We will show that the projection of the solutions to (6.5) onto $\hat{\Theta}$ is open and dense. The proof is carried out in two steps.

First, it is straightforward to verify that, by virtue of the particular choice for $\bar{b}_h, h > 1$, the projection $\pi : \Xi^{n_1} \times \Xi^{n_1} \times \mathcal{T} \times \hat{\Theta} \to \hat{\Theta}$ restricted to the set defined by the first pair of equations in (6.5) is a proper mapping. Openness then follows directly from the fact that the denial of either of the second pair of inequalities in (6.5) is a closed property in the same set.

Now let $N_{\Delta\xi, \Delta\tau''}$ be an open neighborhood of 0 in $\mathbb{R}^{n_1 + (H+1)}$, $N_{\Delta\tau'''}$ be an open neighborhood of 0 in $\mathbb{R}^J$ and $N_{\Delta v}$ be an open neighborhood of 0 in $\mathbb{R}^J$, intersected with $\mathbb{R}_{++}^J$. Then, for $\hat{\theta} \in \hat{\Theta}$ with $\theta \in \Theta_u$, consider the system of $k' + 1 = (n_1 + J + 2n) + 1$ equations (representing differential volatility decrease with respect to a

critical equilibrium)

$$
\begin{array}{cc}
\overline{\xi} & F_1(\overline{\xi}, \widehat{\theta}) = 0, \\
\Delta v & -D\phi(\overline{\xi}, \overline{\tau})D_{\xi,\tau''}F(\overline{\xi}, \overline{\tau}, \widehat{\theta})^{-1}D_{\tau'''}F(\overline{\xi}, \overline{\tau}, \widehat{\theta})\Delta\tau''' = \Delta\phi, \\
\Delta\xi, \Delta\tau'' & F(\overline{\xi} + \Delta\xi, \overline{\tau} + \Delta\tau, \widehat{\theta}) = 0, \\
a, A & a'D_{\xi, \hat{b}, \hat{q}}F(\overline{\xi} + \Delta\xi, \overline{\tau} + \Delta\tau, \widehat{\theta}) = 0 \\
& \text{and } a'a - 1 = 0
\end{array}
\tag{6.6}
$$

in the k' "variables" $(\overline{\xi}, \Delta\xi, \Delta\tau, a) \in \Xi^{n_1} \times N_{\Delta\xi, \Delta\tau''} \times N_{\Delta\tau'''} \times \{0\} \times \mathbb{R}^n$ and $\ell = J + G^2HJ$ "parameters" $(\Delta v, A) \in N_{\Delta u} \times \mathcal{A}$. It is only tedious to show that, by virtue of the particular choice for $\bar{y}^{\sigma(s)}, \sigma(s) \geq H + J$, for $N_{\Delta\xi, \Delta\tau''}, N_{\Delta\tau''}$ and $N_{\Delta v}$ (as well as $\mathcal{A}$ and $\mathcal{O}$) sufficiently small, the Jacobian matrix of this system has full rank (by perturbing each equation using the variables listed alongside). Hence we can once again apply the transversality theorem, and conclude that, generically in the "parameters", (6.6) has no solution. Density then follows from the fact that $\widehat{\Theta}$ is open. The remainder of the argument is based on the implicit function theorem. Since it is always possible to restrict the analysis to the subset of utilities which have a compact domain containing the total resources of the given economy, and this subset is a Banach space, then (locally) the altered equilibrium depends smoothly on the yields from the volatility-reducing new asset. ∎

On the numerical computations in Section 5.2

We describe how we compute the volatility-changing assets in Section 5.2 by applying the proof of Theorem 5.2 and its Corollary.

The introduction of the new asset adds the bottom three lines to the matrix in the proof of Lemma 5.1; there are 2 no-arbitrage equations and one market-clearing equation. In addition, there two more rows corresponding to the volatility equations. Hence, in order to maintain a square matrix we need to take derivatives with respect to 5 more variables. One variable is the asset holding of one agent, and 4 variables are elements of the payoff vector r of the new asset. We (arbitrarily) pick some of these payoffs; in the examples in Section 5.2 these are the elements $y(5), y(10), y(15)$, and $y(20)$. The price of this new asset and all other payoffs are set to 0. Furthermore we set the holdings of this new asset for the two traders arbitrarily to $\hat{b}_1 = 1$ and $\hat{b}_2 = -1$. Now, by solving system (5.3), without the equation $a'a - 1 = 0$, and normalizing the solution we obtain values for the four nonzero elements of r.

We keep the first two values because there are 2 no-arbitrage conditions, so 2 elements of r are dependent instruments. We now solve system (5.1) where we impose that the equilibrium price at time $t = 0$ must be $q_3^0 = 0$ and the agents hold $\hat{b}_1 = 1$ and $\hat{b}_2 = -1$. We endogenously compute the third and fourth value of r. Before doing so we normalize the first two elements in order to stay within the neighborhood where the Implicit Function Theorem applies.[21] Once we have determined all 4 nonzero val-

---

[21]Note that at this point we could try to estimate the size of the neighborhood in the direction

ues of r the process is complete. We can check our results by simply computing an equilibrium for the resulting three-asset economy.

The only change in the described process for finding the payoffs of a volatility-increasing asset (instead of those of a volatility-reducing asset) is to change the two volatility equations.

Computational methods and implementation

We computed the equilibria for the examples in Sections 3.1 and 5.2 by numerically solving the nonlinear system of equations consisting of both the households' Kuhn-Tucker conditions and the market-clearing conditions. We solved these systems using a variation of the homotopy algorithms of Schmedders (1999) and Kubler and Schmedders (2000). We implemented the algorithm on a 450 MHz Pentium PC using the software package HOMPACK, which is a collection of FORTRAN 77 subroutines for solving systems of nonlinear equations using homotopy methods. Maximal running times for our examples were around four seconds, maximal relative numerical errors were below 10^{-12}.

The calculation of the new asset in Section 5.2 requires us to solve a linear system of equations (in the examples, 119 linear equations and unknowns). This system can be solved in less than two seconds using the subroutine DLSARG from the IMSL library of FORTRAN subroutines and functions. The relative numerical errors were below 10^{-14}.

For the final calculation of the new asset we need to solve system (5.1). Solving this system is essentially like computing an equilibrium, except that the price and the portfolio positions of the new asset are held constant, and that instead two of the asset's payoffs are variables. Therefore, we can use a slight variation of the homotopy algorithm that we use for computing equilibria to solve this system. Maximal running times were less than ten seconds, maximal relative numerical errors were below 10^{-12}.

(given by the first two elements of r) by examining various step sizes and computing the resulting payoffs. If we make the first two elements too large, then we will not be able to solve the resulting system of equations.

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