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SYMMETRY EXTENSIONS OF "NEUTRALITY" I: ADVANTAGE TO THE CONDORCET LOSER

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ABSTRACT. This is the first of three papers introducing a theory for positional voting methods that determines all possible election rankings and relationships that ever could occur with a profile over all possible subsets of candidates for any specified choices of positional voting methods. As such, these results extend to all positional voting systems what was previously possible only for the Borda Count and the plurality voting systems. In this first part certain mathematical symmetries based on neutrality are used 1) to generalize the basic properties that cause the desired features of the Borda Count and 2) to describe classes of positional voting methods with new types of election relationships among the election outcomes. Some of these relationships generalize the well-known results about the positioning of a Condorcet winner/loser within a Borda ranking, but now it is possible for the Condorcet loser, rather than the winner, to have the advantage to win certain positional elections. Included among the results are axiomatic characterizations of many positional voting methods.

1. OVERVIEW AND SYMMETRY

How should the outcome of a positional voting election be interpreted? For instance, suppose four candidates are ranked by the positional voting procedure where three points is assigned to a voter's top-ranked candidate, one to a second-ranked candidate, and zero to all others. For a given profile, are there restrictions among the admissible rankings for this positional method and the plurality rankings of the four subsets of three candidates? (Yes.) Suppose c_1 wins in a four candidate election where six, three, one, and zero points are given, respectively, to a top, second, third and bottom ranked candidate. Does this fact impose any restrictions on how she would fare in two and three candidate plurality elections? (It does.) Are there any restrictions on the admissible choices of the plurality ranking of the four candidates and the anti-plurality (or the plurality) rankings of the four sets of three candidates of a profile? (No.) These questions form a small portion of many natural issues that can be raised about the interpretation of single-profile positional voting elections. The purpose of this study is to describe a theory to answer these kinds of questions for all possible positional voting procedures.

More precisely, this is the first of three articles to extract and use new implications about "neutrality" to completely characterize all possible election outcomes of positional voting methods.[1] In this first part I introduce the positional voting processes and some of their election relationships. As it will be shown, several new kinds of restrictions on election rankings emerge.

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[1]Consequently this study completes the goal of characterizing the set of all possible kinds of single-profile election rankings and relationships that ever could emerge over all subsets of candidates for any specified choices of positional voting methods.

For instance, while some of the election relationships involve extensions of the concepts of a "Condorcet winner" and a "Condorcet loser," we encounter the surprising conclusion that all positional voting methods except the Borda Count can place a Condorcet winner at a distinct disadvantage!

In the second paper (Saari [10]), I introduce other relationships and I describe the geometric structure of this set of voting methods. From this geometry, a partial ordering is defined over all positional voting methods; an ordering that characterizes which methods impose more restrictions on the election rankings than others. The third paper (Saari [11]) is more technical; here I describe the actual election rankings (normalized tallies) of the different subsets of candidates for a given profile. In this way, all of the election rankings associated with given sets of positional voting procedures are characterized. The proofs for the major conclusions of this three part study rely upon the technical details derived in this third part.

To start this first part, recall that a positional voting method for n candidates is defined by a *voting vector* of weights

$$\vec{w}^n = (w_1, w_2, \ldots, w_n), \quad w_i \geq w_{i+1}, \ i = 1, \ldots, n-1, \ w_1 > w_n = 0.$$

In the tally of a ballot, w_i points are assigned to the ith ranked candidate; each candidate's election ranking is determined by the sum of points assigned to her where "more is better." In this manner the plurality system is defined by the voting vector $(1, 0, \ldots, 0)$ while the Borda Count (BC) is defined by $\vec{B}^n = (n-1, n-2, \ldots, 0)$.

The goal is to determine what kind of relationships emerge among the election rankings when different positional voting methods are used. One issue is to see what happens with changes in the procedure used to tally the same n-candidate election; this is analyzed in Saari [12]. In this current series of papers I consider election relationships among the subsets of candidates.

When considering election relationships over different subsets of candidates, the Borda Count (BC) plays a critical role because, as we now know, the BC is the unique positional voting system to minimize the numbers and kinds of single profile "paradoxes" that ever could occur. (See (Saari [6, 7, 8]); to be denoted as [S] in what follows.) Restated, this assertion means that the BC is the unique positional voting system to maximize the number and kinds of relationships that occur among the election rankings of the different subsets of candidates. In particular, these relationships ([7]) prove that the BC is the unique system to maximize the consistency of election rankings over all possible subsets of candidates.

While the results of [S] resolve several theoretical issues from choice theory, they also introduce the new questions answered here. For instance, why does the BC have its desirable properties? If, in the above sense, the BC is the "best" choice of a positional voting method, are there "second-best," "third-best," methods? In fact, does any other positional voting method admit relationships among the election rankings of the subsets of candidates and, if so, why? As shown here, there are an infinite number of such methods.

These new classes of positional voting methods are derived by exploiting the standard symmetry assumption of *neutrality* (and, implicitly, *anonymity*). More specifically, the principal idea of this study is to show that when neutrality is combined with certain voting vectors we discover new kinds of symmetry that become apparent only when the election outcomes for different subsets of candidates are analyzed simultaneously. It is this "super-symmetry" that creates the election relationships.

To develop intuition about these "super neutrality" symmetry properties and the relationships they determine, I review the BC symmetry to explain why it causes the desirable BC properties; a short listing of these properties is given. Then, the symmetry ideas are modified to create the new classes of positional voting vectors. In Section 2, terms are formally defined and several new results about election rankings are given. In Section 3 applications and axiomatic representations are considered. In both Sections 2 and 3, examples are given to illustrate the basic results and to indicate what other kinds of conclusions can be obtained.

1.2 Neutrality and the Borda symmetry.

For $n = 3$ candidates, the BC is defined by the voting vector $\overrightarrow{B}^3 = (2,1,0)$. It turns out that by using $\overrightarrow{B}^3$, the weight assigned by a voter to each candidate equals the sum of votes he would assign to her over the three majority vote (voting vector $(1,0)$) elections. To see this, compare the following assignments of points for the preferences $\mathcal{A}_3 = c_1 \succ c_2 \succ c_3$.

Method	Set of candidates	$\{c_1\}$	$\{c_2\}$	$\{c_3\}$
Majority	$\{c_1, c_2\}$	1	0	
Majority	$\{c_1, c_3\}$	1		0
Majority	$\{c_2, c_3\}$		1	0
BC	$\{c_1, c_2, c_3\}$	2	1	0

Thus, the BC voting vector is the aggregated version of the majority vote vector $(1,0)$. With this "aggregated majority vote" interpretation, we see why the BC (not the plurality vote) is the "natural" extension of the majority vote. I underscore this dependency on $(1,0)$ (which may go back to Borda) with the notation $\overrightarrow{B}^3 = \overrightarrow{w}^3((1,0))$. This assignment phenomena extends to all values of $n \geq 3$ candidates; namely, with $\overrightarrow{B}^n = (n-1, n-2, \dots, 1, 0)$, a voter gives each candidate the same number of points as he would over the $\binom{n}{2}$ majority vote elections. Again, to emphasize the BC dependency on the majority vote voting vector $(1,0)$ I use the notation $\overrightarrow{B}^n = \overrightarrow{w}^n((1,0))$.

To express this relationship more formally, let S be a subset of candidates, let $\overrightarrow{w}^{|S|}$ be the positional voting vector assigned to S, and let $C_{j,S}(\mathbf{p}, \overrightarrow{w}^{|S|})$ be the number of points assigned to candidate $c_j \in S$ with profile $\mathbf{p}$. As an illustration, let $\mathbf{p}_{\mathcal{A}_n}$ denote the single voter profile where this voter has the ranking $\mathcal{A}_n = c_1 \succ c_2 \succ \cdots \succ c_n$. It follows that

$$C_{j,S^n}(\mathbf{p}_{\mathcal{A}_n}, \overrightarrow{B}^n) = \sum_{i \neq j} a_{i,j} C_{j,\{c_i,c_j\}}(\mathbf{p}_{\mathcal{A}_n}, (1,0)) \tag{1.1}$$

where the $a_{i,j}$ terms are scalars.

By itself, Eq. 1.1 is of minimal interest because, with appropriate choices of scalars $a_{i,j}$, *any voting vector admits such a representation for profile* $\mathbf{p}_{\mathcal{A}_n}$. For instance, the plurality voting vector is given by $a_{1,2} = 1$, and all other $a_{i,j} = 0$. At the other extreme, the anti-plurality vector $(1, \dots, 1, 0)$ is captured by $a_{1,2} = a_{2,3} = \cdots = a_{n-1,n} = 1$ and all remaining $a_{i,j} = 0$. What distinguishes $\overrightarrow{B}^n$ from other voting vectors $\overrightarrow{w}^n$ is that *the BC is the unique voting vector where all $a_{i,j}$ terms equal the same positive scalar.*

The assertion that all $a_{i,j} = a$ for some $a \neq 0$ is a symmetry property. To see this, recall that the traditional symmetry assumption of *neutrality* assures us that an election outcome is based only on the voters' profiles; the ranking is independent of the names of the candidates. Namely, if all voters interchange their rankings of Anneli and Katri, then the resulting election ranking manifests this permutation. More formally, for n candidates, if σ is a permutation of the names of the candidates, if $\mathbf{p}$ is a profile, and if $f_{S^n}(\mathbf{p}, \overrightarrow{w}^n)$ is the election ranking for the subset of n candidates S^n, then

$$f_{S^n}(\sigma(\mathbf{p}), \overrightarrow{w}^n) = \sigma(f_{S^n}(\mathbf{p}, \overrightarrow{w}^n)). \tag{1.2}$$

The close relationship between the requirement that all $a_{i,j} = a$ in Eq. 1.1 and neutrality can be seen by applying a permutation σ to both sides of Eq. 1.1. The conclusion is that nothing changes; the same equation holds because $a_{i,j} = a \forall i, j$. In other words, *for all permutations σ, the BC is the unique method to satisfy the conditions*

$$\sigma(C_{j,S^n}(\mathbf{p}_{\mathcal{A}}, \overrightarrow{B}^n)) = C_{j,S^n}(\sigma(\mathbf{p}_{\mathcal{A}}), \overrightarrow{B}^n) = \sum_{i<j} a_{i,j} C_{j,\{c_i,c_j\}}(\sigma(\mathbf{p}_{\mathcal{A}}), (1,0)) \tag{1.3}$$

or

$$\sigma(C_{j,S^n}(\mathbf{p}_{\mathcal{A}}, \overrightarrow{B}^n)) = \sum_{i \neq j} a_{\sigma(i,j)} C_{j,\sigma(\{c_i,c_j\})}(\mathbf{p}_{\mathcal{A}}, (1,0)). \tag{1.4}$$

Conversely, if these expressions hold, then all of the $a_{i,j}$ terms must equal a fixed constant; the voting vector must be the BC.

The thrust of Equations 1.3 - 4 is that the BC satisfies a more exacting form of neutrality where this stronger neutrality condition becomes apparent only if the outcomes of different subsets of candidates are compared simultaneously. In this sense, the BC is the unique method to enjoy this stronger version of neutrality over all subsets of candidates. As an immediate consequence of this Borda symmetry, it follows that for any profile $\mathbf{p}$

$$C_{j,S^n}(\mathbf{p}, \overrightarrow{B}^n) = a \sum_{i \neq j} C_{j,\{c_i,c_j\}}(\mathbf{p}, (1,0)) \tag{1.5}$$

where $a = a_{i,j} = 1$ is the common multiple. Thus, the election tally for a BC election is uniquely determined by summing the tallies for the different majority vote elections, and the BC is the unique method for which this is true. Consequently the BC is the unique voting system where the number of points assigned by a voter to each candidate is identical to what she would have received in the majority votes over all $\binom{n}{2}$ pairs of candidates.

To describe this "super-neutrality theme" in the more traditional terms of changing the names of the candidates, notice that this symmetry condition requires the assignment of points in Eq. 1.5 to be independent of

1. the names of the candidates *and*
2. the names of the opponents in each pairwise election.

Thus, only the BC respects this super-neutrality.

As another way to emphasize the BC symmetry, observe that when neutrality is applied to a a family (a collection) of subsets of candidates $\mathcal{F} = \{S_j\}$ where $\overrightarrow{w}^{|S_j|}$ is the voting vector assigned to a S_j election, where σ is a permutation of the candidates' names, and where $\mathbf{p}$ is the profile, we have that

$$\{f_{S_j}(\sigma(\mathbf{p}), \overrightarrow{w}^{|S_j|})\}_{S_j \in \mathcal{F}} = \{\sigma(f_{S_j}(\mathbf{p}, \overrightarrow{w}^{|S_j|}))\}_{S_j \in \mathcal{F}}. \tag{1.6}$$

Should this family include all two-candidate subsets and should some of the voting vectors $\overrightarrow{w}^{|S_j|}$ be BC vectors, then we must suspect (from Eq. 1.5, 1.6) that the Borda symmetry ensures the existence of election relationships over the different subsets of candidates. This happens; indeed, this is the source of the desirable BC properties.

1.3 Borda Symmetry and Condorcet Winners.

There are several ways to support my assertion that the desirable BC properties follow from the Borda symmetry (Eqs. 1.3-5). To start, recall the following standard definition.

Definition 1.1. Candidate c_j is a *Condorcet winner* if she wins all majority vote elections when compared with each of the other candidates. Candidate c_k is a *Condorcet loser* if she loses all majority vote elections when compared with each of the other candidates.

From Eq. 1.5, it can be shown, for instance, that a Condorcet winner never is BC bottom ranked, a Condorcet loser never is BC top-ranked, and a Condorcet winner always is BC ranked above a Condorcet loser. (See Smith [14], [S], etc.) With reflection (using the above discussion and Eq. 1.5), these conclusions must be expected, and, perhaps, they were known by Borda. After all, in order for c_j to be a Condorcet winner, she must win all of her $(n-1)$ pairwise elections, so her point total on the right hand side of Eq. 1.5 must exceed that of at least one other candidate. This happens, and an elementary proof can be fashioned from this description. To see this in the special case where there are $n = 3$ candidates, k voters, and c_1 is the Condorcet winner, we have that

$$C_{1,\{c_1,c_j\}}(\mathbf{p},(1,0)) = [\frac{1}{2} + \epsilon_{1,j}]k,$$
$$C_{j,\{c_1,c_j\}}(\mathbf{p},(1,0)) = [\frac{1}{2} - \epsilon_{1,j}]k,\ j = 2,3,$$
$$C_{2,\{c_2,c_3\}}(\mathbf{p},(1,0)) = [\frac{1}{2} + \alpha_{2,3}]k,$$
$$C_{3,\{c_2,c_3\}}(\mathbf{p},(1,0)) = [\frac{1}{2} - \alpha_{2,3}]k$$

where $\epsilon_{1,j} > 0$ is the (fractional) amount over $\frac{1}{2}$ of c_1's victory over c_j, and the sign of fractional difference (from $\frac{1}{2}$) $\alpha_{2,3}$ determines whether c_2 or c_3 wins their pairwise contest.

Substituting these values into Eq. 1.5 leads to

$$C_{1,\{c_1,c_2,c_3\}}(\mathbf{p},\overrightarrow{B}^3) = [1 + \epsilon_{1,2} + \epsilon_{1,3}]k > k;$$
$$C_{2,\{c_1,c_2,c_3\}}(\mathbf{p},\overrightarrow{B}^3) = [1 - \epsilon_{1,2} + \alpha_{2,3}]k;$$
$$C_{3,\{c_1,c_2,c_3\}}(\mathbf{p},\overrightarrow{B}^3) = [1 - \epsilon_{1,3} - \alpha_{2,3}]k.$$

Now, if c_1 is not BC top ranked, then another candidate, say c_2, is. This requires $\alpha_{2,3} > 2\epsilon_{1,2} + \epsilon_{1,3} > 0$. But the fact that $\alpha_{2,3} > 0$ ensures that c_3 is BC bottom ranked. A similar argument holds for all $n \geq 3$. Closely related arguments prove other assertions such as the Condorcet winner must be BC ranked strictly above a Condorcet loser, etc.

Although with Eq. 1.5 it is almost trivial to show that relationships among the Condorcet winners (losers) and the BC election rankings must hold, what is not obvious is the much deeper conclusion that *the BC is the only positional voting method where its rankings must reflect, in any manner whatsoever, the majority vote rankings of the pairs of candidates.* This assertion is one of several consequences of the Borda symmetry properties derived in [S]; many of these statements are generalized here.

In subtle ways the Borda symmetry admits other kinds of consequences. To illustrate, this symmetry condition plays a critical role in defining axioms to characterize the BC. For instance, the principal axiom in Young's important paper [17] characterizing the BC essentially asserts that if all pairwise elections end in tie votes, then the outcome of the full election must be a complete tie. By use of Eq. 1.5, this condition holds trivially for the BC, so Young's condition is a special case of the binary symmetry.[2] What is not obvious is that this condition does not hold for other voting systems. Young showed that this condition is unique to the BC; in [S] it is shown that *any* such condition is unique to the BC. Extension are in Section 3.

Alternatively, in (Saari [7,8]) I showed how to construct many other axiomatic characterizations for the BC. In each case the central axiom reflects either a "pairwise symmetry condition," or, more often, a property derived from the Borda symmetry conditions. As the BC is the only method manifesting the binary symmetry, any such axiom or condition immediately isolates the BC from all other positional voting methods. Using this idea, it can be shown (Saari [7]) that *almost any condition relating an election outcome (or outcome of a choice function) to what happens with the pairwise elections either plays a principal role in characterizing the BC, or it leads to an impossibility theorem.* For instance, Young's condition can be replaced with the much weaker axiom that if the pairwise elections all are tie votes, then a single candidate cannot be top-ranked, or with the significantly different requirement that a Condorcet loser can never be top-ranked. These axiomatic representations are generalized in Section 3.

In a different but related direction (Saari [8]), this Borda symmetry condition is used to characterize the families of subsets of candidates where election relationships can emerge. The idea is to use the Borda symmetry to define a "binary independence property" and the "cyclic dimension" of a family of subsets of candidates. These concepts are then used to characterize those families of subsets of candidates that admit election relationships. (Also see J. Kelly's column [4].) Generalization are in Section 3 and [10].

As a partial sample of still other consequences of this binary symmetry, consider the set of profiles defining a specified election outcome. One might correctly suspect that if the BC is used in an election, then the BC symmetry imposes a symmetry on these sets of profiles. To anticipate what consequences result from this fact, recall from geometry that the regions with a maximum volume or a minimum surface area are the "symmetry regions." This suggests that the BC symmetry imposes similar properties on the sets of profiles leading to certain election outcomes. This is the case.

An application of the "maximum volume" implication related to the BC is in J. Van Newenhizen's paper [16]. She significantly extends the nice Fishburn and Gehrlein result [3] asserting that the BC maximizes the probability that certain desired election rankings will occur. The Fishburn- Gehrlein proof is for $n = 3$ candidates; Van Newenhizen's result holds for all $n \geq 3$ candidates and for much larger classes of probability distributions.

As an example of the implications of the "minimal surface area" related to the BC, it is shown in (Saari [9]) that the BC either minimizes, or comes close to minimizing (depending on the value of n) the likelihood that a small group of voters can successfully manipulate an election outcome. A related result is that the BC minimizes or comes close to minimizing the likelihood that by changing the way they vote, a small group can change the election outcome.[3] These assertions involve properties of the boundaries of sets of profiles because if a small group of voters can alter an election outcome then the manipulated profile must be near

[2] More technically, because the pairwise elections end in ties, these outcomes are invariant with respect to any permutation of the names of the candidates. If the election outcome for n candidates is to preserve the same kind of neutrality, then it must be a complete tie.

[3] The difference between the two conclusions is that, for the first, the change must benefit the manipulating voters.

the boundary of those profiles defining the original outcome.

On the other hand, with a large, well coordinated group effort to manipulate an election outcome, the BC tends to be one of the more manipulable systems [9]. This is because with large, carefully coordinated group manipulations, the manipulated profile can be anywhere in the set of profiles leading to the sincere outcome. Consequently, this analysis involves the volume of the profile set associated with each outcome; a volume that tends to be maximized by the BC symmetry.

All of these results are direct consequences of the *binary symmetry* exhibited by the BC. What is important for the current discussion is that *all of these results extend to other classes of positional voting methods when the binary symmetry is replaced by the generalized symmetry considerations introduced below*. As such, this review of the BC conclusions serves as an outline of what to expect from the new classes of voting vectors; these extensions are given below and in [10, 11].

1.4 Extensions to four and more candidates.

For $n = 3$, it is known [8] that the BC is the only method to admit any kind of relationship among the election rankings. So, to extend the BC symmetry properties to other methods we need $n \geq 4$ candidates. The idea is to mimic the above construction with a k-fold symmetry - a symmetry that captures distinctions among the elections of k-candidate subsets. Start with $k = 3$ and a voter with the ranking $\mathcal{A}_4 = c_1 \succ c_2 \succ c_3 \succ c_4$ and find the number of points he gives to each candidate when $\overrightarrow{w}^3 = (w_1, w_2, 0)$ is assigned to each triplet of candidates. The following occurs where I use w_3 instead of zero to display the w_1, w_3 symmetry.

(1.7)

Subset of candidates	$\{c_1\}$	$\{c_2\}$	$\{c_3\}$	$\{c_4\}$
$\{c_1, c_2, c_3\}$	w_1	w_2	w_3	
$\{c_1, c_2, c_4\}$	w_1	w_2		w_3
$\{c_1, c_3, c_4\}$	w_1		w_2	w_3
$\{c_2, c_3, c_4\}$		w_1	w_2	w_3
Totals	$3w_1$	$w_1 + 2w_2$	$2w_2 + w_3$	$3w_3$

From 1.7, it is reasonable to conjecture that *there is a relationship among the $(w_1, w_2, 0)$ positional election rankings of the four sets of triplets and the election ranking of the four candidates with the voting vector*

$$\overrightarrow{w}^4 = \overrightarrow{w}^4(\overrightarrow{w}^3) = (3w_1, w_1 + 2w_2, 2w_2, 0).$$

This conjecture is true. Indeed, from the derivation of $\overrightarrow{w}^4(\overrightarrow{w}^3)$, we see that $\overrightarrow{w}^4(\overrightarrow{w}^3)$ is the aggregated outcome of the four $\overrightarrow{w}^3$ elections - thus $\overrightarrow{w}^4(\overrightarrow{w}^3)$ is the natural extension of $\overrightarrow{w}^3$ from three-candidate subsets to the set of four candidates. The mathematical support for the existence of relationships is that Eqs 1.3 - 5 extend immediately (Eq. 1.8) to this three-fold symmetry situation *if $\overrightarrow{w}^3$ is used with each three-candidate election*. Consequently the $\overrightarrow{w}^4(\overrightarrow{w}^3)$ - $\overrightarrow{w}^3$ pair generalizes the "BC-majority vote" connection.

Proposition 1.1. *For the candidates $S^4 = \{c_1, c_2, c_3, c_4\}$, let $\mathcal{F}_3$ denote the four subsets of three candidates where $\overrightarrow{w}^3 = (w_1, w_2, 0)$ is assigned to each of them. For any profile of voters,*

p, *the following holds.*

$$C_{j,S^4}(\mathbf{p}, \overrightarrow{w}^4(\overrightarrow{w}^3)) = \sum_{S \in \mathcal{F}_3} C_{j,S}(\mathbf{p}, \overrightarrow{w}^3). \tag{1.8}$$

An immediate consequence of this proposition is that there must exist relationships — restrictions on the admissible election rankings over the subsets of candidates — among, say, the plurality $(1,0,0)$ rankings of the four sets of three candidates and the $\overrightarrow{w}^4((1,0,0)) = (3,1,0,0)$ ranking of the four candidates. This particular "triplet-symmetry" generalization of the binary "Borda symmetry" justifies the first assertion in the introductory paragraph of this paper.

Using Proposition 1 and examining the kinds of election relationships that occur for the "BC–majority vote" pair, it is easy to conjecture what types of relationships are admitted by "$\overrightarrow{w}^4(\overrightarrow{w}^3)$–$\overrightarrow{w}^3$" pairs. For instance, suppose c_j is $\overrightarrow{w}^3$ top-ranked whenever she is involved in an election of a subset of three candidates. (In Section 2, this is called a "$\overrightarrow{w}^3$–Condorcet winner"). Mimicking what occurs with the BC when Eq. 1.5 is used, it is reasonable to conjecture that she cannot be $\overrightarrow{w}^4(\overrightarrow{w}^3)$ bottom-ranked in the set of four candidates. The reasoning is that a $\overrightarrow{w}^3$-Condorcet winner must win all three of the $\overrightarrow{w}^3$ elections in which she is involved, so she must garner more points in the aggregated $\overrightarrow{w}^4(\overrightarrow{w}^3)$ election than some other candidate. In other word, the tallies from her $\overrightarrow{w}^3$ victories add substantially to the terms in the summation of the right hand side of Eq. 1.8. The elementary proof showing that this conjecture is correct closely follows my earlier argument showing why a Condorcet winner cannot be BC bottom ranked, but now Eq. 1.8 replaces Eq. 1.5 to reflect the triplet-symmetry. What is surprising is that *this assertion holds only if all four sets of three candidates are tallied with $\overrightarrow{w}^3$ while the set of four candidates is tallied with $\overrightarrow{w}^4(\overrightarrow{w}^3)$. (Saari [11].) For any other choice of voting vectors for three candidate subsets, no relationships whatsoever exist!*[4] This means that rankings can be arbitrarily assigned to the four sets of three candidates and the set of four candidates, and a profile can be found where the assigned outcomes are the actual election rankings! This assertion is illustrated in the next example.

Example. We have sufficient information to design several new, intricate comparisons of election rankings. For instance, let $n = 4$ and let $\overrightarrow{w}^4 = (3,1,0,0)$ be used to tally the election for the set of candidates $\{c_1, c_2, c_3, c_4\}$. Let the BC ($\overrightarrow{B}^3 = \overrightarrow{w}^3((1,0)) = (2,1,0)$) be used to tally the elections for the four sets of three candidates. As $(3,1,0,0) \neq \overrightarrow{w}^4((2,1,0))$, *there need not be any relationship whatsoever among the election ranking of all four candidates with the BC election rankings for the four sets of three candidates.* Thus, for example, there exists a profile $\mathbf{p}$ so that the $(3,1,0,0)$ election ranking is $c_1 \succ c_2 \succ c_3 \succ c_4$, but each of the three-candidate BC election rankings is the natural restriction of the reversed ranking $c_4 \succ c_3 \succ c_2 \succ c_1$. In other words, even though c_1 is the BC-Condorcet loser, she wins the $(3,1,0,0)$ election. This must be viewed as a counter-intuitive conclusion.

Now consider what happens if instead of the BC, the plurality vote is used with the four sets of three candidates. In this situation, no profile exists that defines the above set of five rankings

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[4] This statement can be modified with certain restrictions. For instance, if $\overrightarrow{w}^3$ is used only with the three subsets $\{c_1, c_2, c_3\}, \{c_1, c_2, c_4\}, \{c_1, c_3, c_4\}$ that include c_1 and if c_1 is a $\overrightarrow{w}^3$-Condorcet winner, then she cannot be $\overrightarrow{w}^4(\overrightarrow{w}^3)$-bottom ranked. However, in this setting this assertion holds only for c_1; e.g., if a different voting vector is used with $\{c_2, c_3, c_4\}$, then, for any of these three candidates, there are profiles where the designated candidate wins all elections with three candidates, but she is $\overrightarrow{w}^4(\overrightarrow{w}^3)$-bottom ranked.

because $(3,1,0,0) = \overrightarrow{w}^4((1,0,0))$. Thus, for instance, with the above profile $\mathbf{p}$ ensuring c_1 as the $(3,1,0,0)$ top-ranked candidate for the set of all four candidates, it is impossible for her to be the plurality- Condorcet loser over the four sets of three candidates. Similar assertions apply to any $\overrightarrow{w}^4(\overrightarrow{w}^3) - \overrightarrow{w}^3$ pair.

Nevertheless, if the BC is used with the three-candidate elections, we ensure the existence of other kinds of regularity among the election outcomes; i.e., relationships that cannot occur with $(1,0,0)$ and $(3,1,0,0)$ election outcomes. This is because the $(1,0,0)$ and $(3,1,0,0)$ election outcomes are not related to the outcomes of the pairwise elections; only the BC outcomes are. So, with the same profile $\mathbf{p}$ and the above BC ($\overrightarrow{B}^3 = \overrightarrow{w}^3((1,0))$) election outcomes for the four sets of three candidates, we know that it is impossible for c_4 to lose a majority vote election to any two other candidates; it is impossible for c_3 to lose to both c_1 and c_2. This is because if c_3 lost to both candidates, she would be a Condorcet loser in the subset $\{c_1, c_2, c_3\}$, so she could not be top BC ranked in this subset. Likewise, if c_4 lost two of the pairwise elections, she would be the Condorcet loser for some subset of three candidates, and this would preclude her being BC top-ranked in this subset. □

To extend this approach of creating pairs of voting vectors to all integer values of s, n for $2 \leq s < n$, start with a voting vector $\overrightarrow{w}^s$. With the one voter profile $\mathbf{p}_{\mathcal{A}_n}$ (i.e., this voter has the ranking $\mathcal{A}_n = c_1 \succ \cdots \succ c_n$), determine the number of points he assigns to each candidate over the $\binom{n}{s}$ sets of s candidates where $\overrightarrow{w}^s$ is used. The sum of points assigned to candidate c_j is the jth weight (or component) for voting vector $\overrightarrow{w}^n(\overrightarrow{w}^s)$, $j = 1, \dots, n$. The resulting voting vector $\overrightarrow{w}^n(\overrightarrow{w}^s)$ is the aggregated version of the voting vector $\overrightarrow{w}^s$, so the $\overrightarrow{w}^n(\overrightarrow{w}^s)$ $-\overrightarrow{w}^s$ pair must reflect an s-fold symmetry among the election rankings; a symmetry ensured by using $\overrightarrow{w}^s$ to tally each of the $\binom{n}{s}$ sets of s candidates. These relationships are of type where a $\overrightarrow{w}^s$-Condorcet winner cannot be $\overrightarrow{w}^n(\overrightarrow{w}^s)$-bottom ranked, or where a $\overrightarrow{w}^s$-Condorcet winner is $\overrightarrow{w}^n(\overrightarrow{w}^s)$ ranked above a $\overrightarrow{w}^s$ Condorcet loser. A formal study of this is started in Section 2.

The $\overrightarrow{w}^n(\overrightarrow{w}^s)$-$\overrightarrow{w}^s$ pairs do not exhaust all ways there are to find positional voting methods with relationships among election rankings. To illustrate another possibility, let $n = 4$ and use the single voter profile $\mathbf{p}_{\mathcal{A}_4}$ to compute the number of points assigned to each candidate in the four $\overrightarrow{w}^3$ elections of three candidates *and* in the six binary elections. The sum of points assigned in this way to c_j defines the w_j values, $j = 1, \dots, 4$, for the voting vector for four candidates; it is

$$\overrightarrow{w}^4(\overrightarrow{w}^3, (1,0), \{\frac{1}{2}, \frac{1}{2}\}) = \frac{1}{2}(3w_1 + 3, 2w_2 + w_1 + 2, 2w_2 + 1, 0). \tag{1.9}$$

In defining this voting vector, equal emphasis is placed on the points assigned to the candidates from the binary elections as to those assigned in the elections of three candidates. (This is the meaning of the term $\{\frac{1}{2}, \frac{1}{2}\}$.) With another ratio where, say, twice as much weight is placed on the triplets as on the binaries, we obtain the different voting vector

$$\overrightarrow{w}^4(\overrightarrow{w}^3, (1,0), \{\frac{2}{3}, \frac{1}{3}\}) = \frac{1}{3}(6w_1 + 3, 4w_2 + 2w_1 + 2, 4w_2 + 1, 0).$$

The voting vector $\overrightarrow{w}^4(\overrightarrow{w}^3, (1,0), \{\frac{1}{2}, \frac{1}{2}\})$[5] is an aggregated form of the $\overrightarrow{w}^3$ and $(1,0)$ outcomes, so we must expect these voting vectors to exhibit still a different kind of neutrality -

[5] This notation is temporary; in Section 2 it will be slightly changed to reflect properties of the scalar weights.

again, this symmetry becomes apparent only by simultaneously examining the three different types of subsets of candidates. This symmetry can be expressed in a form similar to Eqs. 1.3-4, or its extension Eq. 1.8, which is

$$C_{j,S^4}(\mathbf{p}, \overrightarrow{w}^4(\overrightarrow{w}^3, (1,0), \{\frac{1}{2}, \frac{1}{2}\})) = \frac{1}{2}\sum_{S\in\mathcal{F}_3} C_{j,S}(\mathbf{p}, \overrightarrow{w}^3) + \frac{1}{2}\sum_{i<j} C_{j,\{c_i,c_j\}}(\mathbf{p}, (1,0)).$$

We know from this symmetry relationship that voting vectors of the type given by Eq. 1.9 must admit relationships among election rankings. To develop intuition about what kinds of election relationships to expect, consider the special case where the three candidate subsets are plurality ranked ($\overrightarrow{w}^3 = (1,0,0)$), and the set of four candidates is assigned the voting vector

$$\overrightarrow{w}^4 = \overrightarrow{w}^4((1,0,0), (1,0), \{\frac{1}{2}, \frac{1}{2}\}) = \frac{1}{2}(6,3,1,0).$$

Now suppose profile $\mathbf{p}$ is such that c_j is the Condorcet winner and c_k is the plurality-Condorcet winner. (Namely, c_k wins all plurality elections of three candidates in which she is involved.) As it turns out [11], c_j and c_k need not be the same candidate: there are profiles where c_j is the plurality-Condorcet loser and c_k is the Condorcet loser. In fact, there even exist profiles where the above occurs and these two candidates are tied as being bottom- ranked in the $\frac{1}{2}(6,3,1,0)$ election ranking! Thus this triplet ($\frac{1}{2}(6,3,1,0)$–$(1,0,0)$–$(1,0)$) of voting vectors does not admit election relationships of the type guaranteed by $\overrightarrow{B}^4 = (3,2,1,0) = \overrightarrow{w}^4((1,0))$ (where c_j, the top candidate from the binary elections, is assured of not being bottom ranked in the set of all four candidates), nor of the type guaranteed by $(3,1,0,0) = \overrightarrow{w}^4((1,0,0))$ (where c_k, the top candidate from the triplets, is not bottom ranked).

To see what kind of relationships $\overrightarrow{w}^4((1,0,0), (1,0), \{\frac{1}{2}, \frac{1}{2}\})$ does impose among the election rankings, we need the extra condition that $c_j = c_k$, (i.e., the same candidate is the Condorcet winner and the plurality-Condorcet winner). In this restricted situation, she cannot be bottom-ranked in the set of all four candidates when $\frac{1}{2}(6,3,1,0)$ is used. Moreover, unless a BC or a $\overrightarrow{w}^4(\overrightarrow{w}^3)$-$\overrightarrow{w}^3$ pair is used, this is the only way to ensure that such a relationship exists. Incidentally, this explains the second assertion of the introductory paragraph of this paper.

To give a hint about some of the surprises that can arise with the extension of this construction, let me note that the same voting vector $(6,3,1,0)$ can be derived by using point totals assigned to candidates over the three-candidate subsets using $(3,1,0)$ and the pairwise elections. With this information, it is reasonable to expect the above kind of assertion to hold where if c_j is both a $(3,1,0)$-Condorcet winner and a majority Condorcet winner, then she cannot be $(6,3,1,0)$ bottom ranked. However, *this assertion is false:* for many profiles ([11]) she is $(6,3,1,0)$ bottom ranked. Instead, the operative relationship is

> "if c_j is both a $(3,1,0)$-Condorcet winner and a majority Condorcet *loser*, then she cannot be $(6,3,1,0)$ bottom ranked."

A related relationship emphasizing the lack of respect shown to the Condorcet winner is:

> "if c_j is a majority Condorcet winner and the $(3,1,0)$ Condorcet loser, then she cannot be top- ranked in a $(6,3,1,0)$ election. However, it is possible for c_j to be both the $(3,1,0)$-Condorcet winner and the majority Condorcet loser, and $(6,3,1,0)$ top-ranked!"

These assertions are highly counter-intuitive. The reason the advantage in a $(6,3,1,0)$ election now goes to a Condorcet loser rather than a Condorcet winner is that the voting vector $(6,3,1,0)$ arises by finding how many points voter $\mathbf{p}_{\mathcal{A}_4}$ assigns to each candidate over the three-candidate sets with (3, 1, 0) (which gives the voting vector (9, 5, 2, 0)), and then *subtracting* the number of points assigned to the candidate over the majority election of pairs of candidates. It is this subtraction effect that reverses the role of the Condorcet winner and loser in analyzing (6, 3, 1, 0) elections. Consequently the aggregation effect of using (6, 3, 1, 0) forces the Condorcet winner to stack up huge point totals in the (3, 1, 0) elections in order for her to counter the majority vote totals that are being subtracted.

It is easy to use this construction to design many other unexpected conclusions. For instance, the voting vector (3, 3, 2, 0) is obtained by giving double weight to the $\mathbf{p}_{\mathcal{A}_4}$ majority vote outcomes, and then subtracting the $\mathbf{p}_{\mathcal{A}_4}$ three-candidate plurality vote totals. This creates a situation where the disadvantage in a (3, 3, 2, 0) election is shown to a plurality-Condorcet winner of the three candidate elections. Thus, for example, we obtain relationships such as

> "a candidate who is both a Condorcet winner and a (1, 0, 0) – Condorcet loser cannot be bottom ranked in a (3, 3, 2, 0) election; she is (3, 3, 2, 0) ranked above a candidate who is both a Condorcet loser and a (1, 0, 0)-Condorcet winner."

Returning to the basic theme of these three papers which is to characterize election relationships, it is instructive to consider these conclusions in terms of Arrow's Theorem. It is trivial to ensure complete agreement in the election rankings over the different subsets of candidates: first find a ranking of the n candidates, and then use the natural restriction of this ranking for each of the subsets of candidates. However, the IIA axiom in Arrow's theorem outlaws this approach. So, an alternative construction is to go in the opposite direction; i.e., maybe we should use the rankings of the subsets of candidates to determine the rankings for the set of all candidates. This is the approach developed here in the context of positional voting. (These results extend to more abstract settings.) To ensure that the aggregated procedure satisfies neutrality, I extend the symmetry considerations of neutrality over families of subsets of candidates. By exploiting the different versions of "super- neutrality," all positional voting voting methods where any sort of relationships, positive or negative, exist among the election rankings can be determined. Thus, at least for positional voting methods, this approach of aggregating the election outcomes of the smaller subsets of candidates is the only viable approach that exists to overcome the challenge of Arrow's Theorem. As a corollary, all classes of voting vectors that admit election relationships must involve $\{\overrightarrow{w}^k(\overrightarrow{w}^s)\}$ voting vectors. Thus these voting vectors can be thought of as constituting a "basis" for the space of all such classes of voting vectors. This is discussed in Section 2 as well as in the other two parts of this study.

2. Composite voting vectors

In this section, I describe the voting vectors that admit relationships among the election outcomes of subsets of candidates. Applications are in Section 3.

The discussion of Section 1 motivates the formal definition of the voting vector $\overrightarrow{w}^n(\overrightarrow{w}^s)$, $2 \le s < n$ for a given voting vector $\overrightarrow{w}^s = (w_1, w_2, \dots, w_{s-1}, 0)$. The jth component of $\overrightarrow{w}^n(\overrightarrow{w}^s)$ is the number of points assigned by a voter with preferences $\mathcal{A}_n$ to c_j over the $\binom{n}{s}$ elections of s candidates with the voting vector $\overrightarrow{w}^s$. This process leads to the following definition.

Definition 2.1. Let $2 \le s < n$ and voting vector $\overrightarrow{w}^s = (w_1, w_2, \dots, w_{s-1}, 0)$ be given. The *composite voting vector* is

$$(2.1) \qquad \overrightarrow{w}^n(\overrightarrow{w}^s) = (\binom{n-1}{s-1} w_1, \dots, \sum_{k=1}^{j} \binom{j-1}{k-1} \binom{n-j}{s-k} w_k, \dots, 0). \square$$

One might correctly expect $\overrightarrow{w}^n(\overrightarrow{w}^k, \overrightarrow{w}^s, \Lambda)$ to be defined as a weighted sum of the points cast by this voter with the voting vectors $\overrightarrow{w}^k$ and $\overrightarrow{w}^s$ over the appropriate k and s candidate subsets. This is the case, but we need additional results to describe the weights $\Lambda = \{\lambda_1, \lambda_2\}$.

Proposition 2.1. *The vector $\overrightarrow{w}^n(\overrightarrow{w}^s)$ is a voting vector with at least $n - s + 2 \geq 3$ distinct values for the defining weights. If the weights in $\overrightarrow{w}^s$ satisfy $w_k > w_{k+1}$, then the jth component of $\overrightarrow{w}^n(\overrightarrow{w}^s)$ is larger than the $(j+1)$th for*

$$k \leq j \leq n - s + k.$$

An immediate consequence is that the plurality and anti-plurality vectors, which have only two values for the weights, are *not* composite voting vectors. The significance of this comment becomes clear in the third article of this series [11] where it is proved that only a composite voting vector can admit relationships among its election rankings and those of the subsets of candidates. Thus such relationships are denied to the plurality and anti-plurality systems. This explains the last assertion of the introductory paragraph of this paper.[6]

In the following, Proposition 2.1 is verified and Eq. 2.1 is derived.

Proof. To derive Eq. 2.1, start with a one voter profile $\mathbf{p}_{\mathcal{A}_n}$ and determine how many points c_j receives over all s-candidate subsets where each subset is assigned the voting vector $\overrightarrow{w}^s$. In various subsets of s candidates, c_j may be top-ranked, second-ranked, ..., or jth-ranked; which situation prevails depends upon who else is in the subset. For instance, for c_j to be top-ranked, none of the $\mathcal{A}_n$ higher ranked candidates $A_j = \{c_1, c_2, \ldots, c_{j-1}\}$ can be included, so the remaining $s - 1$ candidates must come from $B_j = \{c_{j+1}, \ldots, c_n\}$. As there are $\binom{n-j}{s-1}$ such subsets and as $\binom{j}{0} = 1$, c_j receives $\binom{j}{0}\binom{n-j}{s-1}w_1$ first place points. To find the number of kth place points, observe that c_j is in kth place iff she is joined by $k-1$ candidates from A_j and $s-k$ candidates from B_j. As there are $\binom{j-1}{k-1}\binom{n-j}{s-k}$ subsets of this type, c_j receives $\binom{j-1}{k-1}\binom{n-j}{s-k}w_k$ points from kth place finishes. In total c_j receives

$$\sum_{k=1}^{j} \binom{j-1}{k-1}\binom{n-j}{s-k} w_k$$

points. This is Eq. 2.1.

To prove Proposition 2.1, observe that the weight assigned by $\overrightarrow{w}^n(\overrightarrow{w}^s)$ to a top-ranked candidate $(j = 1)$ is $\binom{0}{0}\binom{n-1}{s-1}w_1 = \binom{n-1}{s-1}w_1 > 0$ because $w_1 > 0$. The weight assigned to a bottom ranked candidate $(j = n)$ is $\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{0}{s-k}w_k = \binom{n-1}{s-1}w_s = 0$ because $w_s = 0$.

It remains to show that the weight assigned to the jth ranked candidate bounds the weight assigned to the $(j+1)$th ranked candidate; namely,

$$\sum_{k=1}^{j} \binom{j-1}{k-1}\binom{n-j}{s-k} w_k \geq \sum_{k=1}^{j+1} \binom{j}{k-1}\binom{n-(j+1)}{s-k} w_k.$$

[6] It is worth noting that this assertion holds for all of the voting vectors where a voter votes for his k top-ranked candidates. These are the vectors that define an approach known as "approval voting." (See, for instance, Brams and Fishburn [1].) Consequently, we must expect serious voting inconsistencies, with assertions that "anything can happen" to be associated with this procedure. This is correct; see, for example, Saari and Van Newenhizen [13, 14], Brams, Fishburn, and Merril [2], and Niemi [5].

There are two situations to consider with the $\mathbf{p}_{A_n}$ assignment process: the first is where c_j and c_{j+1} both are in the same s-candidate subset, and the second is where only one of them is in the subset. When both are in the same subset, then, because c_j is ranked above c_{j+1}, c_j receives at least as many points as c_{j+1} because $w_k \geq w_{k+1}$. As $\overrightarrow{w}^s$ is a voting vector, there is a value of k where $w_k > w_{k+1}$: here a kth ranked c_j receives more points than c_{j+1}. This positive difference ensures that the jth component of $\overrightarrow{w}^n(\overrightarrow{w}^s)$ is strictly larger than the $(j+1)$th whenever c_j is kth ranked in at least one s-candidate subset. To construct such a subset, we need $k-1$ higher ranked candidates in A_j, so $j-1 \geq k-1$. Also we need $s-(k+1)$ candidates ranked lower than c_{j+1}, so $n-(j+1) \geq s-(k+1)$. Thus the difference between w_k and w_{k+1} is manifested in the weights of $\overrightarrow{w}^n(\overrightarrow{w}^s)$ for values of j satisfying $k \leq j \leq n-s+k$.

The remaining case is where either c_j or c_{j+1} is in a s-candidate subset, but not both. By interchanging c_j and c_{j+1}, there is a one-to-one identification between those s-candidate subsets that include c_j but not c_{j+1} and those that include c_{j+1} but not c_j. Therefore, over these subsets the number of kth place rankings for each candidate is the same, so both candidates receive the same point total. This completes the proof. □

Example. The BC is a special case of this definition where $\overrightarrow{w}^2 = (1,0)$. For instance, $(2,1,0) = \overrightarrow{w}^3((1,0))$ and $\overrightarrow{w}^n((1,0)) = (n-1, n-2, \dots, 1, 0) = \overrightarrow{B}^n$.

For $n = 4$ candidates, $(3,1,0,0) = \overrightarrow{w}^4((1,0,0))$ and $(3,3,2,0) = \overrightarrow{w}^4((1,1,0))$. For $(1,0,0)$ we have that $w_1 > w_2 = w_3 = 0$, so $k = 1$ in Proposition 2.1. According to the proposition, there must be three distinct values for the weights of $\overrightarrow{w}^4((1,0,0))$ where the jth component is larger than the $(j+1)$th for $1 \leq j \leq 2$; this happens. For the anti-plurality vector, the first differences in weights occurs between w_2 and w_3, so the differences in weights in $\overrightarrow{w}^4((1,1,0))$ is where $2 \leq j \leq 3$. All of the weights in $(3,1,0)$ are distinct, so according to the proposition all of the weights in $\overrightarrow{w}^4((3,1,0)) = (9,5,2,0)$ must assume distinct values.

Another voting vector with distinct weights is $\overrightarrow{w}^3 = (2,1,0)$, the BC. Here we have that

$$\overrightarrow{w}^4((2,1,0)) = (6,4,2,0) = 2(3,2,1,0) = 2\overrightarrow{w}^3((1,0)). \square \tag{2.2}$$

Equation 2.2 can be generalized and simplified with a standard equivalence relationship for voting vectors. Namely, for a profile $\mathbf{p}$ the election rankings determined by voting vectors $\overrightarrow{w}^n$ and $\overrightarrow{w}^{n*}$ always are the same iff

$$\overrightarrow{w}^{n*} = a\overrightarrow{w}^n + b(1, \cdots, 1) \text{ for scalars } a > 0,\ b. \tag{2.3}$$

This is because the effect of the $b(1, \cdots, 1)$ vector is to add the same constant (b times the number of voters) to each candidate's tally while the a term creates a multiple of the final tallies. In this paper, two voting vectors are equivalent, $\overrightarrow{w}^{n*} \approx \overrightarrow{w}^n$, if they satisfy Eq. 2.3. From this relationship and Eqs. 2.2-3 we have that

$$\overrightarrow{w}^4(\overrightarrow{w}^3((1,0))) \approx \overrightarrow{w}^4((1,0)).$$

More generally, a voting vector $\overrightarrow{w}^n = (w_1, w_2, \dots, w_n)$ is (equivalent to) a BC vector iff

$$w_i - w_{i+1} \text{ is a fixed constant for } i = 1, \dots, n-1. \tag{2.4}$$

Equation 2.2 is generalized by the next assertion.

Proposition 2.2. *For $2 \le s < k < n$, and for any voting vector $\vec{w}^s$*

$$\vec{w}^n(\vec{w}^k(\vec{w}^s)) \approx \vec{w}^n(\vec{w}^s). \tag{2.5}$$

Conversely, if $\vec{w}^n(\vec{w}^s) \approx \vec{w}^n(\vec{w}^t)$ where $s > t \ge 2$, then $\vec{w}^s \approx \vec{w}^s(\vec{w}^t)$.

Thus if $\vec{w}^k$ has a s-fold symmetry because $\vec{w}^k \approx \vec{w}^k(\vec{w}^s)$, then $\vec{w}^n(\vec{w}^k)$ inherits the stronger s-fold symmetry because $\vec{w}^n(\vec{w}^k) \approx \vec{w}^n(\vec{w}^k(\vec{w}^s)) \approx \vec{w}^n(\vec{w}^s)$. To illustrate with the binary symmetry we have that $\vec{w}^n((4,3,2,1,0)) \approx \vec{w}^n((1,0))$ because $(4,3,2,1,0) \approx \vec{w}^5((1,0))$. Likewise, $\vec{w}^n((3,1,0,0)) \approx \vec{w}^n((1,0,0))$ reflects the three-fold symmetry guaranteed by $(3,1,0,0) \approx \vec{w}^4((1,0,0))$.

This proposition can be viewed as showing how a voting vector can be reduced to its basic symmetry; this is similar to reducing a fraction to lowest terms. For instance as $\vec{w}^4((2,1,0)) \approx \vec{w}^4((1,0))$, the form $\vec{w}^4((1,0))$ better advertises its stronger binary symmetry property.

Proof. Each subset of s candidates appears in $\binom{n-s}{k-s}$ of the k-candidate subsets. From the counting argument, we have that $\binom{n-s}{k-s}\vec{w}^n(\vec{w}^s) = \vec{w}^n(\vec{w}^k(\vec{w}^s))$. The conclusion follows immediately.

The proof in the other direction is similar, so it is left for the reader. □

Definition 2.2. A voting vector $\vec{w}^n(\vec{w}^s)$ is in its *reduced composite form* if there does not exist a voting vector $\vec{w}^t$, $t < s$, so that $\vec{w}^n(\vec{w}^s) \approx \vec{w}^n(\vec{w}^t)$.

A *solitary voting vector* $\vec{w}^n$ is one that cannot be expressed as a composite voting vector $\vec{w}^n(\vec{w}^s)$ for $s < n$. □

The connection between the reduced composite vectors and the solitary vectors is that if $\vec{w}^n(\vec{w}^t)$ is a reduced composite voting vector, then $\vec{w}^t$ must be a solitary voting vector.

2.2 Space of Borda Normalized Voting Vectors.

Using different values of a and b with Eq. 2.3, we can define different normalizations for the voting vectors to obtain a geometric representation of the vectors. The one used here is what I call the *Borda normalization* because it requires the number of points given to a top-ranked candidate to be the same for all voting vectors where the common value is determined by $\vec{B}^n$.

Definition 2.3. The space of *Borda normalized voting vectors* is

$$BVV^n = \{\vec{w}^n \mid w_1 = n-1 \text{ and } w_n = 0.\}$$

Let $\mathcal{W}(n,s) \subset BVV^n$ be the subset of Borda normalized voting vectors that are composite voting vectors of the form $\vec{w}^n(\vec{w}^s)$.□

Example. $BVV^3 = \{(2,s,0) \mid s \in [0,2]\}$. For instance, the values $s = 0,1,2$ correspond, respectively, to the plurality vote, the BC, and the anti-plurality vote. The set $\mathcal{W}(3,2) = \{\vec{B}^3\}$ is a point (the BC vector).

For $n = 4$, $BVV^4 = \{(3,s,t,0) \mid 0 \le t \le s \le 3\}$ while $\mathcal{W}(4,2) = \{\vec{B}^4\}$ is a point, and $\mathcal{W}(4,3) = \{(3,1+s,s,0) \mid s \in [0,2]\}$ is the line segment of normalized $\vec{w}^4(\vec{w}^3)$ vectors.

Observe that $\overrightarrow{B}^4$ separates $\mathcal{W}(4,3)$ into two disjoint line segments. These two components constitute the set of reduced composite voting vectors $\overrightarrow{w}^4(\overrightarrow{w}^3)$. □

Those voting vectors representing the instructions to the voters "to vote for s of the candidates" play a critical role in our analysis. The Borda normalized form is

$$\overrightarrow{E}^n_s = (\overbrace{n-1,\dots,n-1}^{s\text{ terms}},0,\dots,0), \quad s = 1,\dots,n-1.$$

The geometric importance of these vectors is that they are the vertices of BVV^n. Namely, $\{\overrightarrow{E}^n_s\}_{s=1}^{n-1}$ form a convex basis for BVV^n, so each voting vector $\overrightarrow{w}^n \in BVV^n$ has a unique convex representation

$$\overrightarrow{w}^n = \sum_{s=1}^{n-1} \gamma_s \overrightarrow{E}^n_s, \quad \gamma_s \geq 0, \quad \sum_{s=1}^{n-1} \gamma_s = 1. \tag{2.6}$$

For instance, the BC voting vector is the barycentric point of BVV^n because

$$\overrightarrow{B}^n = \frac{1}{n-1}\sum_{s=1}^{n-1} \overrightarrow{E}^n_s.$$

The composite voting vectors admit election relationships, so it is important to understand their relationship with the other voting vectors. This analysis is started with the following fundamental theorem that describes their geometric structure.

Theorem 2.1. *a. For $n > s \geq 2$, BVV^n is a compact, convex space with dimension $n-2$. A convex basis for BVV^n is $\{\overrightarrow{E}^n_s\}_{s=1}^{n-1}$.*

b. The set $\mathcal{W}(n,s)$ is a compact, convex subset of BVV^n with dimension $s-2$. The convex basis for $\mathcal{W}(n,s)$ is $\{\overrightarrow{w}^n(\overrightarrow{E}^s_j)\}_{j=1}^{s-1}$. Thus if $\overrightarrow{w}^s = \sum_{j=1}^{s-1}\gamma_j \overrightarrow{E}^s_j$, then

$$\overrightarrow{w}^n(\overrightarrow{w}^s) = \sum_{j=1}^{s-1} \gamma_j \overrightarrow{w}^n(\overrightarrow{E}^s_j). \tag{2.7}$$

Also

$$BVV^n \supset \mathcal{W}(n,n-1) \supset \mathcal{W}(n,n-2) \supset \cdots \supset \mathcal{W}(n,2) = \{\mathbf{B}^n\}. \tag{2.8}$$

c. The voting vector $\overrightarrow{w}^n(\overrightarrow{E}^s_k)$ is an interior point of the line segment connecting $\overrightarrow{w}^n(\overrightarrow{E}^{s+1}_k)$ and $\overrightarrow{w}^n(\overrightarrow{E}^{s+1}_{k+1})$ for $k = 1,\dots,s-1$. This point is $\frac{k}{s}$ of the way from $\overrightarrow{w}^n(\overrightarrow{E}^{s+1}_k)$ to $\overrightarrow{w}^n(\overrightarrow{E}^{s+1}_{k+1})$. Consequently,

$$\overrightarrow{w}^n(\overrightarrow{E}^{n-1}_k) = (\overbrace{n-1,\dots,n-1}^{k\text{ times}},k,0,\dots,0).$$

d. The set of reduced composite voting vectors $\vec{w}^n(\vec{w}^s)$ is $\mathcal{W}(n,s)\backslash\mathcal{W}(n,s-1)$. The set of solitary voting vectors is $BVV^n\backslash\mathcal{W}(n,n-1)$.

Proof. The definition of a $\vec{w}^n(\vec{w}^s)$ composite voting vector defines an one-to-one linear mapping $G^n_s : BVV^s \to BVV^n$ given by $G^n_s(\vec{w}^s) = \vec{w}^n(\vec{w}^s)$. As BVV^s is a compact, convex set, so is $\mathcal{W}(n,s) = G^n_s(BVV^s)$; both the choice of the convex basis and the dimension statement are immediate. Eq. 2.7 follows from the linearity of G.

It follows from Proposition 2.1 that

$$G^n_s = G^n_{n-1} \circ G^{n-1}_{n-2} \circ \cdots \circ G^{s+1}_s. \tag{2.9}$$

This relationship simplifies the proofs because, by using it, we only need to prove the assertions for BVV^{s+1} and $\mathcal{W}(s+1,s) = G^{s+1}_s(BVV^s)$; the relationship in Eq. 2.9 guarantees that the derived properties are transported to BVV^n.

First I show that $\vec{w}^j(\vec{E}^{j-1}_k)$ is on the line segment joining $\vec{E}^j_k$ and $\vec{E}^j_{k+1}$. Observe from Proposition 2.1 that the first k weights in $\vec{w}^j(\vec{E}^{j-1}_k)$ have the same value α, the $(k+1)$th has a different (smaller) value denoted by β, and the remaining components have the third value which must be zero. Therefore,

$$\vec{w}^j(\vec{E}^{j-1}_k) = \frac{1}{j-1}[(\alpha-\beta)\vec{E}^j_k + \beta\vec{E}^j_{k+1}]. \tag{2.10}$$

A straightforward computation using the Borda normalization and Eq. 2.1 shows that $\alpha = j-1$, $\beta = k$, so this point is $\frac{k}{j-1}$ of the way from $\vec{E}^j_k$ to $\vec{E}^j_{k+1}$. As G^n_{s+1} is an one-to-one linear mapping, it follows immediately that $\vec{w}^n(\vec{E}^s_k)$ is an interior point at the specified location on the line segment joining $\vec{w}^n(\vec{E}^{s+1}_k)$ and $\vec{w}^n(\vec{E}^{s+1}_{k+1})$. By use of Eq. 2.9 this relationship is preserved in all of the subspaces $\mathcal{W}(n,j)$.

To prove part d, note that a vector in $BVV^s\backslash\mathcal{W}(s,s-1)$ must be a solitary voting vector. The conclusion follows because $\mathcal{W}(n,s)\backslash\mathcal{W}(n,s-1) = G^n_s(BVV^s\backslash\mathcal{W}(s,s-1))$.

The remaining assertions follow from Eq. 2.9 and the above constructions. □

Example. I find it easier to use Theorem 2.1-c and Eq. 2.7 to compute $\vec{w}^n(\vec{w}^s)$ rather than the definition. For instance, $(3,1,1,0) = \frac{2}{3}\vec{E}^4_1 + \frac{1}{3}\vec{E}^4_3$. Therefore $\vec{w}^5((3,1,1,0)) = \frac{2}{3}\vec{w}^5(\vec{E}^4_1) + \frac{1}{3}\vec{w}^5(\vec{E}^4_3)$. By use of Eq. 2.7, this is $\frac{2}{3}(4,1,0,0,0) + \frac{1}{3}(4,4,4,3,0)$. □

2.3 Λ- Composite Voting Vectors.

With Theorem 2.1, we can define those voting vectors defined first by counting the points cast by $\vec{w}^k$ over the $\binom{n}{k}$ subsets for different choices of k, and then assigning different weights for the point totals. According to Proposition 2.1, given $z \geq 1$ voting vectors $\vec{w}^{s_1}, \ldots, \vec{w}^{s_z}$, the point totals obtained from $\mathbf{p}_{\mathcal{A}_n}$ with $\vec{w}^{s_j}$ is equivalent to $\vec{w}^n(\vec{w}^{s_j})$. This suggests that the weighting process can be defined with the vectors $\{\vec{w}^n(\vec{w}^{s_j})\}^z_{j=1}$ instead of using the direct count.

There is a further reduction; the goal of the definition is to extract the maximum symmetries enjoyed by the voting vector. This is best done by using the vectors in their reduced form.

For instance, if $\overrightarrow{w}^{s_j} = \overrightarrow{w}^{s_j}(\overrightarrow{w}^{t_j})$ is a reduced composite vector, then the reduced vector notation $\overrightarrow{w}^n(\overrightarrow{w}^{t_j})$ should be used rather than $\overrightarrow{w}^n(\overrightarrow{w}^{s_j})$ to reflect the stronger symmetry. Notice that $s_j \geq t_j$ and that even if the s_j's assume distinct values, the t_j's need not. For instance, $\overrightarrow{w}^4((1,0,0))$ and $\overrightarrow{w}^5((1,1,0))$ could be used to define a Λ form for $\overrightarrow{w}^6$.

Definition 2.4. Suppose given $z \geq 1$ solitary voting vectors $\overrightarrow{w}^{t_1}, \ldots, \overrightarrow{w}^{t_z}$ where $n > t_1 \geq \cdots \geq t_z \geq 2$. For a given set of nonzero scalars $\Lambda = \{\lambda_1, \ldots, \lambda_z\}$, the Λ-composite voting vector is

$$\overrightarrow{w}^n(\overrightarrow{w}^{t_1}, \ldots, \overrightarrow{w}^{t_z}; \Lambda) = \sum_{j=1}^{z} \lambda_j \overrightarrow{w}^n(\overrightarrow{w}^{t_j}). \tag{2.11}$$

The only restriction on the weights in Λ is that the summation on the right hand side of Eq. 2.11 remains in BVV^n. □

Because of the minimal restrictions on the λ_j's, a Λ composite vector need not be in $\mathcal{W}(n, n-1)$. To see that $\mathcal{W}(n, n-1)$ does not contain all linear combinations of composite voting vectors, observe that the relationship between BVV^n and $\mathcal{W}(n, n-1)$ changes with the value of n. For example, with $n = 3$ the point $\mathcal{W}(3,2) = \{\overrightarrow{B}^3\}$ divides BVV^3 into two components. This division property also holds for $n = 4$ because the line segment $\mathcal{W}(4,3) = \{(3, 1+s, s, 0) \mid s \in [0,2]\}$ divides the two dimensional triangle BVV^4 into two components. However, as depicted in Figure 1, this property fails to extend to $n \geq 5$ because $\mathcal{W}(n, n-1)$ fails to divide BVV^n into two components.

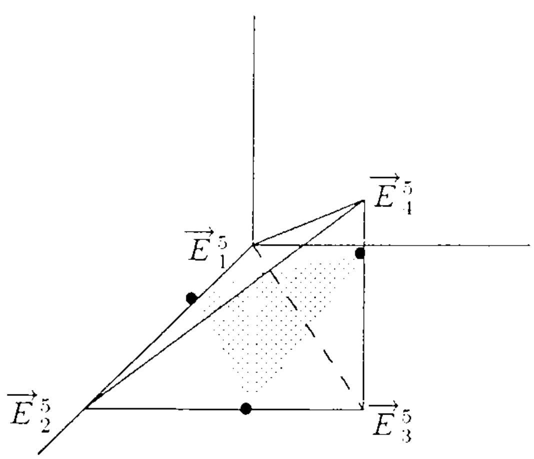

Figure 1

To interpret Figure 1, the four vertices[7] of the three- dimensional simplex are $\{\overrightarrow{E}^5_j\}_{j=1}^4$ and $\mathcal{W}(5,4)$ is the shaded triangle defined by the three dots on the edges. According to Theorem

[7] With the Borda normalization, the values of the first and the fifth weights are common to all $\overrightarrow{w}^5$ voting vectors, so a geometric representation in R^3 can be given by plotting only the middle three coordinates.

2.1, these vertices of $\mathcal{W}(5,4)$ are $\{\overrightarrow{w}^n(\overrightarrow{E}^4_j)\}^3_{j=1}$. The assertion now is obvious because $\mathcal{W}(5,4)$ does not intersect the BVV^5 edge defined by $\overrightarrow{E}^5_1$ and $\overrightarrow{E}^5_4$; $\mathcal{W}(5,4)$ fails to divide BVV^5 into two components.[8] This leads to the following definition.

Definition 2.4. The set of *extended composite voting vectors*, $\mathcal{L}W(n,s)$, is the intersection of the $s-2$ dimensional affine plane containing $\mathcal{W}(n,s)$ with BVV^n. □

$\mathcal{L}W(n,s)$ is one of the spaces of Λ composite voting vectors. Because $\mathcal{L}W(n,s)$ is defined in terms of intersections, this property is not transferred by G^n_j from one BVV^j set to another. Instead, we have that

$$G^n_j(\mathcal{L}W(j,s)) \subset \mathcal{L}W(n,s) \subset \mathcal{L}W(n,s+1).$$

Definition 2.5. A *scoring vector* $\overrightarrow{s} = (s_1, s_2, \ldots, s_n = 0)$ is a non-zero vector. When $\overrightarrow{s}^n$ is used to tally a ballot, s_j points are assigned to the jth ranked candidate. □

Example. The scoring vector $(4, 6, -2, 0)$ assigns four points to a top-ranked candidate, six to a second-ranked, a negative two points for a third-ranked candidate, and zero to a bottom-ranked candidate. Thus, a scoring vector need not be a positional voting vector, but a positional voting vector is a scoring vector that satisfies the "inequality" constraints on the magnitudes of the weights. □

As indicated at the end of the introductory section, if $\lambda_j \in \Lambda$ is negative, then the composite voting vector $\overrightarrow{w}^n(\overrightarrow{w}^{t_1}, \ldots, \overrightarrow{w}^{t_z}; \Lambda)$ introduces a negative prejudice against a candidate who did well in all $\overrightarrow{w}^{t_j}$ elections. Therefore it is important to understand which voting vectors admit such a representation. This is done in the next statement where $A\backslash B$ is the set of points in A that are not in B and where ∂A is the set of boundary points of A.

Theorem 2.2. *a. For solitary vectors* $\overrightarrow{w}^{t_1}, \ldots, \overrightarrow{w}^{t_z}$, $t_1 \geq t_2 \geq \cdots \geq t_z \geq 2$, *and* Λ,

$$\overrightarrow{w}^n(\overrightarrow{w}^{t_1}, \ldots, \overrightarrow{w}^{t_z}; \Lambda) \in \mathcal{L}W(n, t_1). \tag{2.12}$$

b. Conversely, if $\overrightarrow{w} \in \mathcal{L}W(n,t_1)\backslash\mathcal{L}W(n,t_1-1)$, *then* $\overrightarrow{w} = \overrightarrow{w}^n(\overrightarrow{w}^{t_1}, \ldots, \overrightarrow{w}^{t_z}; \Lambda)$ *for some choice of* $t_1 \geq t_2 \geq \cdots \geq t_z \geq 2$, *solitary voting vectors* $\overrightarrow{w}^{t_1}, \ldots, \overrightarrow{w}^{t_z}$ *and* Λ. *The vector* $\overrightarrow{w}^{t_1}$ *must be in* $BVV^{t_1}\backslash\mathcal{L}W(t_1, t_1-1)$.

c. If $\overrightarrow{w} \in \partial\mathcal{W}(n,t_1)\backslash\partial\mathcal{W}(n,t_1-1)$, *then* $\overrightarrow{w}$ *can be expressed as a reduced composite voting vector* $\overrightarrow{w}(\overrightarrow{w}^{t_1})$ *(where* $\overrightarrow{w}^{t_1} \in BVV^{t_1}\backslash\mathcal{L}W(t_1,t_1-1)$*) and as a* Λ *composite voting vector where some scalar in* Λ *must be negative.*

d. If $\overrightarrow{w}$ *is in the interior of* $\mathcal{W}(n,t_1)\backslash\mathcal{W}(n,t_1-1)$ *where* $t_1 > 2$, *then* $\overrightarrow{w}$ *can be expressed as a composite voting vector and as a* Λ *composite vector. There are choices of* $\overrightarrow{w}^{t_1}, \ldots, \overrightarrow{w}^{t_z}$

[8] According to Theorem 2.1, the first dot defining $\mathcal{W}(5,4)$ is $\frac{1}{4}$ the distance between $\overrightarrow{E}^5_1$ and $\overrightarrow{E}^5_2$, the second dot is $\frac{2}{4}$ the distance between $\overrightarrow{E}^5_2$ and $\overrightarrow{E}^5_3$, etc. Moreover, $\overrightarrow{w}^5(\overrightarrow{E}^3_1)$ must be $\frac{1}{3}$ of the way along the line connecting the first two dots, while $\overrightarrow{w}^5(\overrightarrow{E}^3_2)$ is $\frac{2}{3}$ along the line from the second dot to the third. These two points define $\mathcal{W}(5,3)$, and the BC vector is at the midpoint of this line. Therefore Theorem 2.1 permits an accurate representation of the $\mathcal{W}(n,s)$ sets and how they are nested. See Figure 2.

and Λ where all scalars in Λ are positive, and other choices where at least one scalar in Λ must be negative.

e. If a voting vector $\overrightarrow{w}$ is in $\mathcal{LW}(n,t_1)\backslash\mathcal{W}(n,t_1)$ but not in $\mathcal{LW}(n,t_1-1)$, then $\overrightarrow{w}$ can be expressed as a composite vector $\overrightarrow{w}^n(\overrightarrow{s}^{t_1})$ where $\overrightarrow{s}^{t_1}$ is a scoring vector, but not a positional voting vector. Alternatively, $\overrightarrow{w}$ can be expressed as a Λ composite vector, but at least one scalar in Λ must be negative.

f. The vector $\overrightarrow{B}^n$ is the only voting vector that cannot be expressed as a Λ-composite voting vector with a negative coefficient for the binary elections.

What does all of this mean? As shown in the third article of this series, if $\overrightarrow{w}^n \notin \mathcal{LW}(n,t_1)$ for all t_1, $2 \leq t_1 < n$, then $\overrightarrow{w}^n$ can never admit election relationships with subsets of candidates. On the other hand, the above theorem asserts that should $\overrightarrow{w}^n \in \mathcal{LW}(n,n-1)$, then there can be election relationships. However, if $\overrightarrow{w}^n \in \mathcal{LW}(n,n-1)$ is a non-BC vector, then the voting vectors assigned to the subsets of candidates can require a *negative* correlation among the election outcomes. (This occurs when some $\lambda \in \Lambda$ has a negative value.) Indeed, if $\overrightarrow{w} \in \mathcal{LW}(n,t_1)\backslash\mathcal{W}(n,t_1)$[9] and if voting vectors for the subsets of candidates are chosen so that $\overrightarrow{w}$ is Λ composite (so election relationships must occur), then some scalar in Λ must be negative so these negative correlations must arise! Moreover, the voting vectors in $\mathcal{LW}(n,t_1)\backslash\mathcal{W}(n,t_1)$ have the dubious distinction of being an aggregated version of the outcomes of scoring vectors that are not positional voting vectors!

We see that $\partial\mathcal{W}(n,s)$ serves as a bifurcation set: on one side of the boundary the voting vectors can admit both reduced composite and Λ representations with positive scalars. On the other side, there is no reduced composite representation (at least with voting vectors) and all Λ representations involve at least one negative scalar. The continuity is manifested by the vectors on the boundary still admitting a reduced composite representation, but where the Λ form must have a negative scalar.

How bad are these restrictions on election rankings? Is it possible to choose a subset of voting vectors so that, while they admit negative relationships, the admissible relationships are positive at least with respect to the majority vote outcomes? As asserted in part f of the theorem, only the BC has this property. In fact, any non-BC voting vector in $\mathcal{LW}(n,t_1)$ admits a negative correlation with any choice of a voting vector used for a particular subset of candidates. More precisely, choose a non-BC voting vector $\overrightarrow{w} \in \mathcal{LW}(n,s_1)$ and a voting vector $\overrightarrow{w}^{s_2}$, $s_2 \leq s_1$. There exist voting vectors $\{\overrightarrow{w}^{s_j}\}_{j=1}^{z}$ so that the voting vector is Λ composite and λ_{s_2} is negative. This is just one of the many different possible constructions admitted by the geometry defining the Λ-composite voting vectors. This geometry, which can be used to derive other relationships, is described in the proof of parts b and c.

Proof. Part a. This is an immediate consequence of the definition of a Λ composite voting vector which requires the composite voting vector to be in the linear span of the vectors $\{\overrightarrow{w}^n(\overrightarrow{w}^{t_i})\}_{i=1}^{z}$.

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[9] For instance, in Figure 1 $\mathcal{LW}(5,4)\backslash\mathcal{W}(5,4)$ is the part of the plane passing through the three dots that does not include the shaded triangle. This is the triangular region defined by the vertices $\overrightarrow{w}^5(\overrightarrow{E}_1^4)$, $\overrightarrow{w}^5(\overrightarrow{E}_3^4)$, and $(4,2,2,2,0)$. (Theorem 2.3)

Part b. The $\mathcal{LW}(n,s)$ sets define a coordinate system. The "origin" is the point $\mathcal{W}(n,2) = \{\vec{B}^n\}$. This point divides the straight line segment $\mathcal{W}(n,3)$ into two parts; one segment can be designated as the "positive axis," and the other as the "negative direction" of this first coordinate axis. Continuing, there is a unique line in the $s-1$ dimensional space $\mathcal{LW}(n,s+1)$ that passes through $\vec{B}^n$ and is orthogonal to the $s-2$ dimensional plane $\mathcal{LW}(n,s)$; this is the $(s-1)$th coordinate axis. As $\mathcal{LW}(n,s)$ divides $\mathcal{LW}(n,s+1)$ into two components, the coordinate axis can be assigned a positive and a negative direction. This coordinate system for $n=5$ is illustrated in Figure 2; the triangle from Figure 1 is replaced with the extended shaded plane $\mathcal{LW}(5,4)$, the line connecting edges of the shaded region is $\mathcal{W}(5,3)$ and the dot in the middle of the line is the "origin" $\vec{B}^5$. The second axis would be in the shaded area orthogonal to the $\mathcal{W}(5,3)$ line and passing through $\vec{B}^5$.

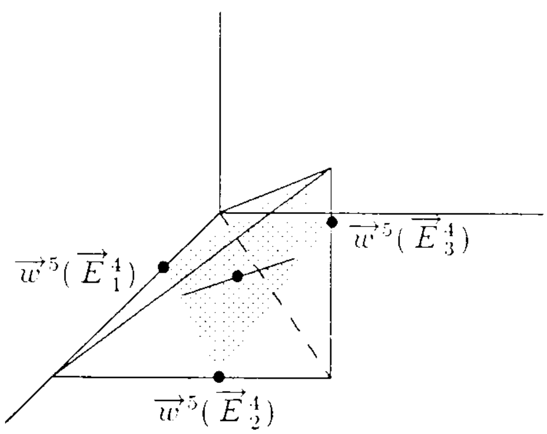

Figure 2

Using the coordinate system, if $\vec{w} \in \mathcal{LW}(n,s+1)\backslash\mathcal{LW}(n,s)$, then $\vec{w}$ must have a nonzero component along the $(s-1)$th coordinate axis. Consequently, it is impossible for $\vec{w}$ to be expressed as a linear combination of vectors from $\mathcal{LW}(n,s)$. So, with the assumptions in part b of the theorem, if $\vec{w} \in \mathcal{LW}(n,t_1)\backslash\mathcal{LW}(n,t_1-1)$ and if $\vec{w}$ has a Λ composite representation, then one of the defining vectors must be in $\mathcal{LW}(n,t_1)\backslash\mathcal{LW}(n,t_1-1)$. As this vector is in a reduced composite form, it must be in $\mathcal{W}(n,t_1)\backslash\mathcal{LW}(n,t_1-1)$.[10] Thus this vector must have the representation $\vec{w}^n(\vec{w}^{t_1})$ where $\vec{w}^{t_1} \in BVV^{t_1}\backslash\mathcal{LW}(t_1,t_1-1)$.

Now choose voting vectors from $\mathcal{W}(n,t_1)$ so that, along with $\vec{w}$ and $\vec{w}^n(\vec{w}^{t_1})$, they form a dependent set where $\vec{w}$ can be expressed as a linear combination of the other vectors. This is one choice of a Λ-composite representation for $\vec{w}$. Observe that if $\vec{w}$ has one Λ decomposition, it has an infinite number of different Λ decompositions.

Part c. In the construction of part b, there are two geometric possibilities. The first is that

[10] Observe that in a Λ representation, the sign of the scalar for this vector depends on whether this vector and $\vec{w}$ are both on the same component side of the $(s-1)$th axis.

the vectors define a convex hull where none of them are in the interior of the hull. The second is that some of the chosen voting vectors are in the interior. In either case, if $\vec{w} \in \partial\mathcal{W}(n, t_1)$ then, by construction, $\vec{w}$ is on the boundary of the convex hull. Thus when $\vec{w}$ is expressed as a linear combination of the other chosen voting vectors, one vector must have a negative coefficient. The only other possibility is that the convex set is degenerate – it is a point. In this admissible case, $\vec{w}$ can be expressed as the (reduced) composite vector $\vec{w}^n(\vec{w}^{t_1})$.

Part d. As $\vec{w}$ is in the interior of $\mathcal{W}(n, t_1)\backslash\mathcal{W}(n, t_1 - 1)$, the voting vectors can be chosen so that $\vec{w}$ is the sole interior point of the convex hull. This means that in the Λ composite form representation for $\vec{w}$, all of the scalars are positive. Similarly, it is possible to choose the other voting vectors so that $\vec{w}$ is one of the vertices of the hull. This means that when solving for $\vec{w}$, some of the scalars must be negative.

Part e. If $\vec{w} \in \mathcal{CW}(n, t_1)\backslash\mathcal{W}(n, t_1)$, then when the vectors are selected for the convex hull, $\vec{w}$ always must be a vertex. The conclusion follows.

Part f. This is an immediate consequence of the above construction and the fact that $\vec{B}^n$ is in the one point component $\mathcal{W}(n, 2)$. □

Example. First I'll show for $(4, 2, 2, 2, 0) \in \mathcal{CW}(5, 4)$ that its reduced composite representation is with the scoring vector $2\vec{E}^4_1 - \vec{E}^4_2 + 2\vec{E}^4_3 = (3, 1, 2, 0)$, rather than with a voting vector. It follows from this linear expression that $G^5_4((3, 1, 2, 0)) = 2\vec{w}^5(\vec{E}^4_1) - \vec{w}^5(\vec{E}^4_2) + 2\vec{w}^5(\vec{E}^4_3)$. Using Theorem 2.1-c to compute $\vec{w}^5(\vec{E}^4_k)$, we have that $G^5_4((3, 1, 2, 0)) = (4, 2, 2, 2, 0)$, so the assertion is verified. Incidentally, because $4, 2, 2, 2, 0)$ is on the line connecting $\vec{E}^5_1$ and $\vec{E}^5_4$, it is the missing vertex of $\mathcal{CW}(5, 4)$.

To find a class of Λ-composite representation for $(4, 2, 2, 2, 0) \approx (2, 1, 1, 1, 0)$, observe that any point on the line $t(3, 1, 2, 0) + (1 - t)(3, 2, 1, 0) = (3, 2 - t, 1 + t, 0)$ where $\frac{1}{2} \geq t \geq -1$ defines a voting vector $\vec{w}^4_t$. For instance, $t = \frac{1}{2}$ defines $(3, \frac{3}{2}, \frac{3}{2}, 0) \approx (2, 1, 1, 0)$; $t = 0$ defines $\vec{B}^4$; and $t = -1$ defines $\vec{E}^4_2$. Using the linearity of G^5_4 with this line, we have that

$$(4, 2, 2, 2, 0) = -\frac{1-t}{t}\vec{B}^5 + \frac{1}{t}\vec{w}^5(\vec{w}^4_t).$$

In this continuum of Λ composite forms for $(4, 2, 2, 2, 0)$, if $0 < t \leq \frac{1}{2}$, then the coefficient for the majority vote outcomes is negative and the $\vec{w}^5(\vec{w}^4)$ coefficient is positive. These roles are reversed for $-1 \leq t < 0$. The composite form is not defined at $t = 0$; the reason is explained in the proof of part b.

A geometric description of the design of Λ composite representations can be given with Figure 2. First, choose a voting vector $\vec{w}$ from the shaded region and then draw a line passing through $\vec{w}$ and $\vec{B}^5$. Next, choose a vector $\vec{w}_1 \neq \vec{B}^5$ that is on the line and in the shaded triangle region. Because $\vec{w}_1$ is in the triangle region, it is a composite voting vector. Now, if $\vec{w}$ is between $\vec{w}_1$ and $\vec{B}^5$, then both scalars must be positive in a Λ representation. If $\vec{w}_1$ is between the two vectors, then the scalar for the BC is negative, and that for $\vec{w}_1$ is positive. If $\vec{B}^5$ in the middle, then its scalar multiple is positive while that of $\vec{w}_1$ is negative.

To derive more complicated Λ-composite forms for the same vector $\overrightarrow{w}$, draw the line passing through $\overrightarrow{w}$ and the line $\mathcal{W}(5,3)$. Now choose a vector $\overrightarrow{w}_1$ on this line, another vector off this line but in $\mathcal{W}(5,3)$ and the BC. The signs of the scalars in Λ are based on which vector is in the interior (if any) of the convex hull defined by these points. □

To conclude this section, I describe what happens with $n = 4$.

Theorem 2.3. *a. $\mathcal{CW}(5,4)$ is defined by the vertices* $(4,1,0,0,0)$, $(4,4,2,0,0)$, $(4,4,4,3,0)$, *and* $(4,2,2,2,0)$.

The set

$$\mathcal{CW}(5,4)\backslash\mathcal{W}(5,4) =$$

(2.13)

$$\{\gamma_1(4,1,0,0,0) + \gamma_3(4,4,4,3,0) + \gamma(4,2,2,2,0) \mid \gamma_1 + \gamma_3 + \gamma = 1, \gamma > 0, \gamma_j \geq 0\}.$$

b. A necessary and sufficient condition for $\overrightarrow{w}^4 = (w_1, w_2, w_3, w_4)$ to admit choices of $\overrightarrow{w}^3$ that allow election relationships among the set of four candidates , the three candidate subsets and/or the binary majority vote elections is that

$$w_1 - 3w_2 + 3w_3 - w_4 = 0. \tag{2.14}$$

Proof. The proof of part a follows immediately from the computations in the example and from the properties of G^5_4. For part b, it is shown in [S] that Eq. 2.14 is a necessary condition for $\overrightarrow{w}^4$ to admit election relations. When $\overrightarrow{w}^4$ is written in a Borda normalized form, Eq. 2.14 becomes $(3, 1+s, s, 0)$, $s \in [0,2]$. As shown, this requires the vector to be in $\mathcal{W}(4,3)$; this establishes the sufficiency assertion. □

3. From $\overrightarrow{w}$ Condorcet Winners and Losers to Axioms

Now that we have the composite voting vectors, some of the election relationships they admit are described next.

Definition 3.1. Let $n \geq 3$, c_j, and s, $2 \leq s < n$, be given. Let $\overrightarrow{w}^s$ be assigned at least to all s-candidate subsets that include c_j. Candidate c_j is the $\overrightarrow{w}^s$ Condorcet winner if c_j is top-ranked in all of these elections. Candidate c_k is the $\overrightarrow{w}^s$ Condorcet loser if $\overrightarrow{w}^s$ is assigned at least to the s-candidate subsets that include c_k and in all of these elections c_k is bottom-ranked.

Suppose given the set of voting vectors $\{\overrightarrow{w}^{s_j}\}_{j=1}^{z}$ and the set of scalars $\Lambda = \{\lambda_1, \ldots, \lambda_z\}$. A candidate c_k is a Λ-Condorcet winner if for each $\lambda_j \in \Lambda$ where $\lambda_j > 0$, c_k is the $\overrightarrow{w}^{s_j}$ Condorcet winner, and for each $\lambda_j \in \Lambda$ where $\lambda_j < 0$, c_k is the $\overrightarrow{w}^{s_j}$ Condorcet loser; $j = 1, \ldots, z$.

A candidate c_k is a Λ-Condorcet loser if for $\lambda_j > 0$, c_k is the $\overrightarrow{w}^{s_j}$ Condorcet loser, and for $\lambda_j < 0$, c_k is the $\overrightarrow{w}^{s_j}$ Condorcet winner. □

The next theorem shows that the relationships between the $\overrightarrow{w}^n(\overrightarrow{w}^s)$ positional election rankings and the $\binom{n}{s}$ $\overrightarrow{w}^s$ positional election rankings mimic those obtained for the BC and majority vote elections. Indeed, if $\overrightarrow{w}^s = \overrightarrow{w}^2 = (1,0)$, we recover the earlier assertion relating the BC outcomes with the majority vote rankings.

Theorem 3.1. *a. Let $2 \leq s < n$ be given. Let $\vec{w}^s$ be a voting vector. The $\vec{w}^s$ - Condorcet winner cannot be $\vec{w}^n(\vec{w}^s)$ bottom-ranked; a $\vec{w}^s$ - Condorcet loser cannot be $\vec{w}^n(\vec{w}^s)$ top-ranked; and a $\vec{w}^s$-Condorcet winner is always $\vec{w}^n(\vec{w}^s)$ ranked above a $\vec{w}^s$-Condorcet loser.*

b. Let $\vec{w}^n = \vec{w}^n(\vec{w}^{t_1}, \dots, \vec{w}^{t_z}, \Lambda)$ be a Λ composite voting vector. Suppose $\vec{w}^{s_j}(\vec{w}^{t_j})$ is assigned to all s_j-candidate subsets, $j = 1, \dots, z$. Therefore the vectors $\{\vec{w}^{s_j}(\vec{w}^{t_j})\}_{j=1}^{z}$ is used along with the set Λ to define the Λ-Condorcet winners and losers. A Λ - Condorcet winner cannot be $\vec{w}^n(\vec{w}^{t_1}, \dots, \vec{w}^{t_z}, \Lambda)$ bottom-ranked; a Λ - Condorcet loser cannot be $\vec{w}^n(\vec{w}^{t_1}, \dots, \vec{w}^{t_z}, \Lambda)$ top-ranked; and a Λ-Condorcet winner is always $\vec{w}^n(\vec{w}^{t_1}, \dots, \vec{w}^{t_z}, \Lambda)$ ranked above a Λ-Condorcet loser.

If any other choices of (non-equivalent) voting vectors are used, then the conclusion does not hold.

Example. Suppose for $n = 5$ that $(6, 6, 5, 3, 0)$ is assigned to the five-candidate subset, $(3, 3, 2, 0)$ to the four-candidate subsets, and $(4, 3, 0)$ to the three-candidate subsets. The types of election relationships that can emerge is based on the fact that $(6, 6, 5, 3, 0) = \vec{w}^5((3, 3, 2, 0))$. Thus, a $(3, 3, 2, 0)$ Condorcet winner (loser) cannot be $(6, 6, 5, 3, 0)$ bottom-ranked (top-ranked).

The election relationships among the pairs and triplets with the other candidates is governed by $(6, 6, 5, 3, 0) = \vec{w}^5((4, 3, 0), (1, 0), \{\frac{1}{2}, -\frac{1}{2}\})$ and $(3, 3, 2, 0) = \vec{w}^4((4, 3, 0), (1, 0), \{\frac{1}{2}, -\frac{1}{2}\})$. (Because $(6, 6, 5, 3, 0)$ and $(3, 3, 2, 0)$ are, respectively, on the boundaries of BVV^5 and BVV^4, it follows from Theorem 2.2 that one of the scalars in the Λ representation must be negative.) Therefore if a candidate is both the Condorcet winner and the $(4, 3, 0)$-Condorcet loser, she cannot be top-ranked in any of the four candidate subsets nor in the five candidate subset; indeed, she would be ranked below any candidate who happened to be the Condorcet loser but the $(4, 3, 0)$-Condorcet winner. There are more refined relationships; for instance, c_1 may be the winner in all $(4, 3, 0)$-three-candidate elections involving c_2, c_3, c_4 and lose to each of these candidates in a pairwise elections. In this case, c_1 cannot be bottom-ranked in the $\{c_1, c_2, c_3, c_4\}$ election.

As $(4, 3, 0) \neq \vec{B}^3$, there need not be any relationships among the three-candidate and the two-candidate elections. □

For other examples, see the comments in the introductory section. Indeed, using the approach developed in Section 2 to create Λ-composite vectors, it is clear that an infinite number of different examples can be created.[11]

This theorem generalizes the Borda relationship between the majority vote outcomes and the Borda outcomes to much larger classes of voting vectors. However, it is important to remember that a Λ-Condorcet winner can, in fact, be required to be the $\vec{w}^s$ -Condorcet *loser* for several voting vectors. In fact, any non-BC voting vector that admits election relationships can always have this property even with respect to the binary elections. Nevertheless, when this surprising conclusion is viewed in terms of the extra symmetry introduced by the composite voting vectors, it becomes a result that must be expected. In fact, the reason for the relationships is that the outcome of the n-candidate subset is uniquely determined by the election tabulations

[11] Part of the purpose of the second article [10] of this series is to develop a "bookkeeping" approach to handle all of the possibilities.

of the subsets with various numbers of candidates. The following theorem gives this exact relationship.

Theorem 3.2. *Suppose given a voting vector* $\overrightarrow{w}^n(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)$. *Suppose the reduced composite vector* $\overrightarrow{w}^{s_j}(\overrightarrow{w}^{t_j})$ *is assigned to all* s_j*-candidate subsets,* $j = 1, \dots, z$. *For some fixed scalar value* a, *the election tally for* c_j *is given by*

$$C_{j,C^n}(\mathbf{p}, \overrightarrow{w}^n(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)) = a \sum_{k=1}^{z} \lambda_k \sum_{S_i, |S_i| = t_k} C_{j,S_i}(\mathbf{p}, \overrightarrow{w}^{t_k})$$

$$= a \sum_{k=1}^{z} \frac{\lambda_k}{\binom{s_k}{t_k}} \sum_{S_i, |S_i| = s_k} C_{j,S_i}(\mathbf{p}, \overrightarrow{w}^{s_k}) \tag{3.1}$$

Proofs. The proof of Theorem 3.2 follows the ideas of Section 1 along with Proposition 2.1, 2.1. The binomial term in Eq. 3.1 is explained in the proof of Proposition 2.2. An elementary proof of Theorem 3.1 follows that used in the first section to prove the assertions about the BC rankings of the Condorcet winners and losers. (A different proof is given in [11].) To show the ideas, I outline the proof that a Λ-Condorcet winner, c_k, must be $\overrightarrow{w}^n(\overrightarrow{w}^{t_1}, \dots, \overrightarrow{w}^{t_z}, \Lambda)$ ranked above a Λ-Condorcet loser c_i.

Assume that $\lambda_j \in \Lambda$ is positive, so c_k wins all s_j-candidate elections in which she is involved. This means that in each such election, she must receive more than $\frac{1}{s_j}$ of all points cast. On the other hand, c_i, as the loser, must receive less than $\frac{1}{s_j}$ of all votes cast. Thus, over all s_j-candidate elections, c_k receives more points than c_i. When these points are included in the right hand side of Eq. 2.13, they are scaled by the positive multiple λ_j, so c_k has a point total advantage over c_i.

On the other hand, for those elections where λ_j is negative, c_k loses the s_j-candidate elections, so she receives less than $\frac{1}{s_j}$ of the total vote while c_i receives more than $\frac{1}{s_j}$ of the total vote. However, when these points are included in the appropriate position in the right hand side of Eq. 2.13, these point totals are multiplied by a negative scalar. Thus the point total applied to c_k's total is strictly larger than that applied to c_i's. This completes the proof. □

3.2 Applications.

As an interesting application of Theorem 3.1, suppose we wish to design a run-off election procedure that always selects a $(1,0,0)$-Condorcet winner (when one exists). According to the theorem, the way to do this is to assign $\overrightarrow{w}^n((1,0,0))$ as the voting vector to rank the n candidates. Then the bottom ranked candidate is dropped and the remaining $n-1$ candidates are advanced to the next stage. The induction step with s candidates, $s = n-1, \dots, 4$, is to use $\overrightarrow{w}^s((1,0,0))$ to rank the set of s candidates, drop the bottom-ranked candidate,[12] and then advance the remaining $s-1$ candidates to the next stage. When only three candidates remain, select the winner of the plurality election.

[12] As a slight modification, at each stage with s candidates, one could drop all candidates who receives less than $\frac{1}{s}$ of the total tally. This is because as shown in the proof of Theorem 3.2, the $\overrightarrow{w}^s(\overrightarrow{w}^3)$-Condorcet winner always receives at least $\frac{1}{s}$ of the total tally.

To see why this procedure always elects the plurality-Condorcet winner, notice from Theorem 3.1 that at each stage she cannot be bottom-ranked, so she cannot be eliminated. At the last stage, she must be elected. On the other hand, if voting vectors not equivalent to the above are used at any stage, or if a fixed number of more than one candidate is to be dropped at any stage, then examples can be constructed to prove that the plurality-Condorcet winner need not be selected! This is because the plurality-Condorcet winner can be bottom ranked at some stage if a voting vector not equivalent to $\overrightarrow{w}^k((1,0,0))$ is used, or next to bottom ranked even if $\overrightarrow{w}^k((1,0,0))$ is used (yet she receives over $1/s$ of the total tally) at a stage where more than one candidate is eliminated.

Of course, there is nothing particular about the plurality vote; if the requirement is to select a $\overrightarrow{w}^s$-Condorcet winner, then the elimination process uses $\overrightarrow{w}^k(\overrightarrow{w}^s)$ at each stage. A similar argument applies to other elimination procedures. For instance, to generalize the idea of an agenda, one might take the first four candidates from a listing of all n candidates. Based on an election outcome, a certain number of candidates are dropped from further consideration, and they are replaced by the next candidates on the given listing. The process continues until only three candidates remain, and then a $\overrightarrow{w}^3$ election is held to determine the "winner." If the goal is to end up with, say, the $(3,1,0)$-Condorcet winner, then the voting vector assigned to the groups of four must be $\overrightarrow{w}^4((3,1,0)) = (9,5,1,0)$, and either all but the bottom-ranked candidate, or all candidates receiving at least $\frac{1}{4}$ of the total tally must be advanced. Of course, for the final group of three, the $(3,1,0)$ winner is determined. For the same reasons as given above, if a $(3,1,0)$ Condorcet winner exists, she wins.

These examples illustrate an important class of application for Theorems 2.2, 3.1. The idea is that these conclusions, which are of independent interest, also form a powerful tool for the analysis of different types of choice procedures. In fact, most of the kinds of conclusions found in the literature relating, in some way, the BC and the majority vote outcomes now can be extended from the $\overrightarrow{w}^n((1,0))$ $(1,0)$ pair to any $\overrightarrow{w}^n(\overrightarrow{w}^s)$ $\overrightarrow{w}^s$ pair.

3.3 Axiomatic Representations.

To further underscore this assertion about the kinds of results one now can expect, recall that one application of the relationship between the BC and the majority vote elections is the axiomatic characterizations of the BC. This type of result extends to all choices of composite and A composite voting vectors. Suppose, for instance, that we want an axiomatic representation for $(9,5,1,0)(= \overrightarrow{w}^4((3,1,0)))$, or for $(3,3,2,0)(= \overrightarrow{w}^4((1,1,0)))$, or for $(6,3,1,0,0)(= \overrightarrow{w}^5((1,0,0)))$. As I now illustrate, for any composite voting vector, it is easy to find many sets of characterizing axioms! The following is one such choice that extends Young's [16] characterization of the Borda Count to all composite voting vectors.

Theorem 3.3. *Let $n > k \geq 2$ candidates be given. Suppose F is a choice procedure that is anonymous, neutral, consistent, and somewhat faithful.*[13] *Furthermore, suppose for a specified $\overrightarrow{w}^k$ the choice procedure satisfies the $\overrightarrow{w}^k$-cancellation property whereby whenever all $\binom{n}{k}$ of the $\overrightarrow{w}^k$ elections end in a complete tie, then all n candidates are selected. The procedure is*

[13] A choice procedure is consistent if $f(\mathbf{p}) \cap f(\mathbf{p}') \neq \emptyset$, then $f(\mathbf{p} + \mathbf{p}') = f(\mathbf{p}) \cap f(\mathbf{p}')$. A choice procedure is somewhat faithful if for the one voter profile $\mathbf{p}$, $f(\mathbf{p})$ does not include the voter's bottom ranked candidate. These terms are discussed in greater detail in [7, 8, 17].

equivalent to choosing the top- ranked candidate(s) from a $\vec{w}^n(\vec{w}^k)$ election. When $k = 2$, this is a characterization of the BC.

Thus, for example, suppose we want a procedure with the first set of properties that chooses all candidates in those situations where, whenever subsets of five candidates are considered, $\frac{1}{5}$ of the voters have each candidate top-ranked. This procedure must be equivalent to selecting the top- ranked candidates from a $\vec{w}^n((1,0,0,0,0))$ election. Theorem 2.4 follows by slightly modifying the argument in [7] and using the fact that the $\vec{w}^n(\vec{w}^k)$ is the only voting vector to satisfy the cancellation property.

For other axiomatic generalizations of $\vec{w}^n((1,0)) - (1,0)$ pair to more general classes, recall my assertion from Section 1 that "almost any condition relating an election outcome to what happens with the pairwise elections either plays a principal role in characterizing the BC, or it leads to an impossibility theorem." A similar assertion hold for other composite voting vectors. Namely, most axioms relating a choice procedure outcome with what happens with the elections associated with a particular $\vec{w}^3$ either leads to a characterization of $\vec{w}^n(\vec{w}^3)$, or an impossibility theorem. The following is one of many possible examples of an impossibility theorem and a characterization theorem.

Theorem 3.4. *Let $n > 3$ candidates be given. Suppose F is a choice procedure that is anonymous, neutral, consistent, and somewhat faithful.*

a. *Suppose the procedure is to select a $(1,0,0)$ – Condorcet winner whenever one exists. No such procedure exists.*

b. *Suppose there is a candidate c_j so that whenever she is in a subset of three candidates, more voters have her bottom ranked than any other candidate. Whenever such a candidate exists, suppose the procedure must not select her. This procedure is equivalent to selecting the top ranked candidates from a $\vec{w}^n((1,1,0))$ election. If more than one candidate is top-ranked, the procedure can require a series of Young run-off elections to make a refined selection among these candidates C. This is where a specified positional election is held for all candidates. The resulting election rankings provides a relative ranking of the candidates in C. From this ranking, the top-ranked candidates are selected.*

c. *Suppose the procedure never selects a candidate who is both a Condorcet loser and a $(1,0,0)$-Condorcet winner. The procedure is equivalent to choosing a $\vec{w}^n((1,0,0),(1,0),\Lambda)$ vector where the scalar for the plurality election is negative and that for the majority elections is positive. The top-ranked candidate is selected. A series of Young run-off elections can be applied to break ties.*

Outline of the proof. The basic set of axioms define a procedure equivalent to choosing the top-ranked candidate with a positional voting method. There is the option of breaking ties with an iterated sequence of the positional voting procedures. (This conclusion is an extension [7] of Young's nice work [17].) The goal is to identify the positional voting methods. For part 1, if such a procedure exists, it must be equivalent to first using the $\vec{w}^n((1,0,0))$. However, this election procedure cannot guarantee that the plurality-Condorcet winner will be top- ranked, only that she will not be bottom-ranked. Thus, an impossibility theorem follows.

For the second part, the requirement about c_j means she is a $(1,1,0)$ – Condorcet loser. Using a procedure equivalent to a $\vec{w}^n((1,1,0))$ election means she must be dropped at the first stage. The procedure could require an iterated Young run-off to refine who is selected. Thus, here a possibility theorem emerges.

For part c, the conditions identify a Λ composite voting vector. However, there is no particular choice; thus this is an axiomatic representation for a class of voting vectors. □

Another example is to replace condition 1 of Theorem 3.3 with "if all $\overrightarrow{w}^3 \neq \overrightarrow{B}^3$ elections end up in a tie vote, then more than one candidate is selected." This leads to a process where the top-ranked candidates in a $\overrightarrow{w}^n(\overrightarrow{w}^3)$ election are kept, but now a Young run-off is admitted.

3.4 Ternary inclusion property.

As a last illustration of these results, recall from (Saari [8]) that if a family $\mathcal{F}$ of subsets of candidates admits relationships among the BC election rankings, then $\mathcal{F}$ must have a subfamily $\mathcal{F}_1$ that satisfies the *binary inclusions property* (bip). This is where at least one subset of candidates in $\mathcal{F}_1$ has three or more candidates, and if $S_j \in \mathcal{F}_1$ is such that $|S_j| \geq 3$, then there is another subset of candidates $S_k \in \mathcal{F}_1$ so that two of the candidates from S_j are also in S_k. For instance, no subfamily of $\mathcal{F} = \{\{c_1, c_2, c_3\}, \{c_3, c_4, c_5\}, \{c_5, c_6, c_1\}\}$ satisfies bip, so there need not be any relationship whatsoever among the election rankings of these three subsets of candidates for any choice of positional voting methods. That is, arbitrarily choose a ranking for each subset. Then, for any choice of positional voting methods, there exists a profile so that the election rankings are the selected rankings.

What happens if the BC is not used for any subset of candidates? Will the bip condition suffice as an indicator of whether a family will admit election relationships? The answer is no: a more exacting requirement manifesting the more crude symmetry requirements is needed.

Definition 3.2. A family $\mathcal{F}$ satisfies the *ternary inclusion property*, tip, if

1. each subset in $\mathcal{F}$ has at least three candidates and one has at least four candidates, and
2. if $S_j \in \mathcal{F}$ is such that $|S_j| \geq 4$, then there is a different set $S_k \in \mathcal{F}$ and a triplet of candidates from S_j that are also in S_k. □

Theorem 3.5. *If no subfamily of $\mathcal{F}$ satisfies the tip and if the BC is not assigned to any subset in $\mathcal{F}$, then for any assignment of voting vectors, there are no relationships among the election rankings. That is, the election rankings for each subset of candidates can be selected in any desired manner. For any choice of non-BC voting vectors assigned to these subsets, there exists a profile so that the selected rankings are the sincere election rankings.*

Example. Let $n \geq 4$ and let $\mathcal{F}_3$ be the family of all $\binom{n}{3}$ sets of three candidates. No subfamily of $\mathcal{F}$ satisfies tip, so, if the BC is not assigned to any of these subsets, then there are no relations among the election rankings. On the other hand, election relationships are admitted if the BC is used. To see what they are, see (Saari [7, 8]).

Theorem 3.2 can be extended to reflect other symmetries. For instance, should the assigned voting vectors not reflect any binary (BC) or ternary ($\overrightarrow{w}^k(\overrightarrow{w}^3, (1,0), \Lambda)$) symmetry, then even more strict inclusion properties are required. In particular, an s-fold inclusion property (sip) forms the necessary conditions should none of the assigned voting vectors admit any form of a k-fold symmetry, $k < s$. This is a further illustration of the critical role symmetry plays in the analysis of voting processes. □

The proof of this theorem follows that in Saari [8]; a different proof is in [11].

Somewhat surprisingly, simple symmetry properties can be used to significantly extend the basic properties of the Borda Count to a large number of other classes of positional voting procedures. In this manner it is seen that the connection between the Condorcet winner and the Borda Count rankings are not unique: closely related connections arise with the election rankings of sets of s candidates and k candidates, $s < k$, should appropriate choices of positional voting vectors be used. In particular, the relationship between the choices of voting vectors that need to be used manifest the kind of relationship that exists between the Borda Voting Vector and the majority vote vectors. For the conclusions derived here, the positional voting vector for the larger sets of candidates must be an "aggregated" version of the voting vectors assigned to the other sets of candidates.

One of the basic messages of this article, then, is the critical role symmetry – extensions of neutrality – plays in understanding the kinds of election outcomes that occur with different choices of positional voting vectors.

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