Discussion Paper No. 25

A GENERAL FORMULATION OF THE MULTIATTRIBUTE DECISION PROBLEM: CONCEPTS AND SOLUTION ALGORITHMS

by

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November 28, 1972

Section 1: Statement of the Problem

1.1 A large number of real-world decision problems cannot be properly assessed from a single viewpoint: a firm attempting to compare a set of alternative investment projects might want to rate them on the basis of (1) the net discounted profit expected from each investment (2) the payoff period and (3) the market share. An economist trying to assign a precise quantitative content to such expressions as the "rate of growth of the general price level" would want to compare prices of a set of commodities over several periods of time; similarly, in system analysis, the question of how to take into account multiple criteria often arises; in the field of social choice theory the same problem is encountered and voting mechanisms are but one possible way of resolving it. To set the stage for our analysis it is convenient to adopt a few definitions to capture the essential similarity between the various problems we have just mentioned.

1.2 Basic Definitions The "alternative set" A is a finite set of well-specified objects (e.g. investment projects, candidates in an election, etc.).

(1) $A = \{a_1, a_2, \ldots, a_i, \ldots, a_m\}$

We are also given a finite class $\mathscr{A}$ of "criteria" (e.g. characteristics, features, voters)

(2) $\mathscr{A} = \{S_1, S_2, \ldots, S_h, \ldots S_\ell\}$

Now each individual criterion $S_h \in \mathscr{A}$ is itself a set endowed with a certain structure, algebraic, topological or both as the case may be.

For instance we could have

(3) $S_h = N$ (the set of natural numbers)

(4) $S_h = \{0,1,2,\ldots,n\}$, the finite set of the first n integers

(5) $S_h = \{0,1\}$ or $\{\text{yes, no}\}$

(6) $S_h = \mathbb{R}$ or $\mathbb{R}_+$

More generally S_h could be a metric space or a topological space.

These criteria now give us a basis for representing the m objects of A. This "representation" process can be viewed as a set of ℓ mappings φ_h:

(7) φ_h: $A \to S_h$ $(h=1,2,\ldots,\ell)$

In general we would not expect these mappings to be identical (if they were, we would be faced with a single criterion decision problem and no aggregation would be necessary). Each object a_i is thus described by an ℓ-dimensional image:

(8) $[\varphi_1(a_i); \varphi_2(a_i);,,,; \varphi_h(a_i);\ldots;\varphi_\ell(a_i)]$

1.3. <u>The Aggregation Problem</u>: Informally we would like to "combine" the ℓ-dimensional images of the m objects in a certain "best" way.

Letting the image set of A be denoted by $\Phi(A)$ (where $\Phi(A) \subset \prod_{h=1}^{\ell} S_h$) the aggregation problem consists in finding a mapping σ that maps $\Phi(A)$ into a one-dimensional "aggregate space" 0:

(9) σ: $\Phi(A) \to 0$

Now the nature of the aggregate space 0 will vary depending upon the problem at hand. For instance if all the criterion sets $S_h = \{1,2,\ldots,n\}$ and the representation mappings φ_h are permutations of S_h, we may want to require that

(10) $0 = \{1,2,\ldots,n\}$

and $\sigma \in \mathcal{S}$ (where $\mathcal{S}$ denotes the set of permutation operators, i.e. the group of permutations).

Clearly a very large number of mappings σ could be chosen. To discriminate among them some "goodness of fit" criterion is needed. Intuitively we would like the aggregate representation mapping σ to respect as much as possible the individual mappings φ_h. The question then revolves around the choice of an objective function that will evaluate the goodness of fit between the "extensive" image $\Phi(A)$ of the set A and its aggregate image $\sigma[\Phi(A)] \subset 0$. Once such an objective function has been chosen, the problem is then to search for a class of aggregation mappings σ that meet this optimality requirement. Clearly the answer to the first question, i.e. the choice of a goodness of fit index is partly dependent upon the choice of a structure for the criterion sets S_h. In the next sections we will illustrate this approach by using a simple linear form as our objective function to be maximized. The metric interpretation of this solution concept will also be discussed as well as a simple solution algorithm.

Section 2: Aggregating a Set of (ℓ) Complete Strict Orderings of m Objects

2.1 Introduction and Background. We shall now assume that the individual representation mappings φ_h of the m objects $a_i \in A$ are permutation operators, i.e.

(11) $S_h = \{1,2,\ldots,i,\ldots,m\}$

(12) $\forall_h = 1,2,\ldots,\ell \; : \varphi_h \in \mathcal{S}_m$ where $\mathcal{S}_m$ denotes the group of permutation of the first m integers

As we know each operator φ_h can be represented by a permutation matrix $\mathbb{P}$, i.e. an $m \times m$ nonnegative matrix each row and column of which has only one entry equal to 1 (and the others are 0). And finally we want to find an aggregate mapping $\sigma \in \mathcal{S}_m$ to represent the individuals mappings φ_h.

2.2 Maximizing Agreement Among the Various Ranking Criteria.

Definition 1: An agreement matrix Π is a square $(m \times m)$ nonnegative matrix whose entries π_{ij} represent the number of individual orderings where the ith alternative (of the reference order) is placed in the jth position:

(13) $\pi_{ij} = k \Leftrightarrow \exists\, K \subset \{1,2,\ldots,\ell\} \ni |K| = k$

and $\varphi_h(a_i) = a_j$ iff $h \in K \;\; \forall h \in \{1,2,\ldots,\ell\}$. It is clear that we may agree to assign unequal weights w_h to each individual criterion S_h; in this case this amounts to assuming that the hth individual ordering is replicated w_h times.

Definition 2: The agreement index I_A is a real-valued linear mapping such that:

(14) $I_A = \sum_{i,j} \pi_{ij}\, p_{ij}$

where the p_{ij}'s are the entries of an $(m \times m)$ permutation matrix P, representing the hth ordering.

The first formulation of the aggregation problem in this framework is then:

Find $P^* \in \mathscr{P}_m$ such that

(15) $\sum_{i,j} \pi_{ij} p_{ij} \leq \sum_{i,j} \pi_{ij} p^*_{ij}$

for all P matrices of the mth order. Of course, the first solution method we can think of is simply to enumerate the m! permutation matrices P and choose that matrix P* which maximizes I_A. Clearly, this is computationally inefficient and even infeasible as m becomes large. An alternative formulation of the problem is now proposed, which will greatly reduce this computational burden.

The second formulation of our problem consists in allowing fictitious stochastic orderings. More specifically we want to find a bistochastic solution matrix $[b_{ij}]$ (**) which

(16) $\text{Max} \sum_{i,j} \pi_{ij} b_{ij}$

subject to

$$\left.\begin{aligned} &(17) \quad \sum_{j=1}^{m} b_{ij} = 1 \\ &(18) \quad \sum_{i=1}^{m} b_{ij} = 1 \\ &(19) \quad b_{ij} \geqq 0 \end{aligned}\right\} \qquad i, j=1,2,\ldots,m$$

We can readily recognize the b_{ij}'s of this formulation as the entries of an $(m \times m)$ bistochastic matrix B(*). This new problem is, of course, a simple linear programming problem.

(**) A bistochastic matrix B is a nonnegative $(m \times m)$ matrix whose coefficients satisfy the following properties:

(i) $\forall i, \forall j \quad b_{ij} \geq 0 \ (i,j=1,2,\ldots,m)$ (ii) $\forall i \ \sum_{j=1}^{m} b_{ij} = 1$ (iii) $\forall j \ \sum_{i=1}^{m} b_{ij} = 1$

(*) The idea for this formulation was first proposed by T. C. Koopmans and M. Beckmann in the context of a location problem [11].

The crucial point of this formulation, however, is the fact that any solution to this second problem will necessarily be a solution to the first one. The proof of this result is obvious: (i) it is a well known fact of linear programming theory that if there exists an optimal solution, there will always be at least one solution at a vertex of the polyhedral feasible region; and (ii) this vertex is nothing else but a permutation matrix P, according to the Birkhoff-von Neumann theorem.(*) Hence, solving problems (16-19) will give us all the solution(s) to problem (15) as we had claimed. This second approach, however, eliminates the computational limitation described before.

2.3 A Minimal Distance Algorithm. Another approach to the aggregation problem in the context of ℓ individual complete strict orderings on A, will now be presented.

The basic idea now is to exploit the geometrical properties of the set $\mathcal{P}_m$ and $\mathcal{B}$ as described in Theorem(*) below. In order to do that we must first prove a simple result on agreement matrices Π as defined previously.

Lemma 1. Let Π be an $(m \times m)$ agreement matrix. Then the following relations always hold:

$$\begin{cases} \sum_{j=1}^{m} \pi_{ij} = \ell & \forall\ i=1,2,\ldots,m \quad (20) \\ \sum_{i=1}^{m} \pi_{ij} = \ell & \forall\ j=1,2,\ldots,m \quad (21) \end{cases}$$

Intuitively, this result is obvious if we realize that each row of Π defines a (different) partition of the set of criteria $\{1,2,\ldots,\ell\}$.

Proof: It suffices to prove (20) for any row i since the labeling of the alternatives (the reference order in the set A) is entirely arbitrary.

(i) By contradiction, let us suppose just that $\sum_{j=1}^{m} \pi_{ij} > \ell$.

(*) Theorem (Birkhoff-von Neumann): The set $\mathcal{B}$ of bistochastic matrices of order m forms a convex polyhedron in $\mathbb{R}^{m^2}$, whose vertex set is identical with the set $\mathcal{P}_m$ of permutation matrices.

Then $\exists$ j and k $\in \{1,2,\ldots,m\}$ and h $\in \{1,2,\ldots,\ell\}$ such that $\varphi_h(a_i) = \{a_j a_k\}$ with $j \neq k$ contrary to our assumption that the φ_h are strict orderings.

(ii) Now suppose $\sum_{j=1}^{m} \pi_{ij} < \ell$.

Then $\exists$ i $\in \{1,2,\ldots,m\}$ and h $\in \{1,2,\ldots,\ell\}$ such that $\varphi_h(a_i) = \emptyset$ contrary to our completeness assumption for the φ_h mappings. Hence we must have $\sum_{j=1}^{m} \pi_{ij} = \ell$. The proof of (21) for any column j is exactly parallel to that of (20).

Q.E.D.

We now define a normalized agreement matrix Π^{norm}.

Definition 3: A normalized agreement matrix Π^{norm} is an agreement matrix Π the rows and columns of which have all been divided by ℓ: $\Pi^{norm} = \frac{1}{\ell}\Pi$.

An immediate corollary to Lemma 1 can now be stated.

Corollary 1: Any normalized $(m \times m)$ agreement matrix Π^{norm} is a bistochastic matrix $\mathcal{B}$ of the same order.

Proof: This follows directly from Lemma 1.

We can now make use of the geometrical characterization of the sets $\mathcal{S}_m$ and $\mathcal{B}$ afforded by the Birkoff-von Neumann theorem. We know from Corollary 1 that any agreement matrix Π-- obtained from the individual strict orderings $\varphi_h \in \mathcal{S}_m$ as explained above-- can be transformed through a simple normalization operation into an element $B \in \mathcal{B}$, the convex polyhedron of all bistochastic matrices of the mth order. In a sense one can view this normalized agreement matrix as defining a complete stochastic (aggregate) ordering on the set of alternatives A.

In this context an aggregation process could thus be considered as a mapping σ from the interior of $\mathcal{B}$ onto the set $\mathcal{S}_m$, i.e. the set of vertices of this convex polyhedron $\mathcal{B}$ by theorem (*)(p. 5). Such a σ mapping would clearly not be bijective since $\mathcal{S}_m$, the set of vertices of $\mathcal{B}$ (the permutation operators φ) is finite whereas it is very easy to show that the set $\mathcal{B}$ has the power of the continuum.

The solution concept we shall now propose could be thought of as a vertex projection method: given some normalized agreement matrix B, we can search for the vertex $\hat{P}$ which is "closest" to B under some metric in $\mathbb{R}^{m^2} \supset \mathcal{B}$.(*) Here the goodness of fit criterion for determining the "best" aggregate ordering in the set $\mathcal{S}_m$ can be of a metric nature because of the properties of $\mathcal{B}$ and $\mathcal{S}_m$.

Under this solution concept, the aggregation problem can thus be stated:

(22) Find $\hat{P} \in \mathcal{S}_m$

such that $d(\hat{P},B)$ is minimum $\hat{P}, B \in \mathbb{R}^{m^2}$; $B \in \mathcal{B}$; $\mathcal{S}_m \subset \mathcal{B} \subset \mathbb{R}^{m^2}$. Let us now choose as our metric d

(23) $d(\hat{P},B) = \sum_{i,j} | p_{ij}-b_{ij} |$.

In this case the <u>decision rule</u> to use in order to minimize d could be stated simply as follows:

<u>Notational Conventions</u>:

K_o refers to the set of "active" columns, i.e. columns which have already been used in a previous step and, thus, are no longer available.

$b_{i\cdot}$ refers to the ith row vector of B.

$b_{\cdot j}$ refers to the jth column vector of B.

The ith step of the solution algorithm for (23) is now stated in full.

0. Initialize by setting $p_{ij} = 0 \ \forall \ i, j=1,\dots,m \ \ K_o = \emptyset$; $i = 1$
1. If $i = m+1$, stop
2. Put $\ell_o \in K_o$
3. Let $\bar{b}_{i\cdot} = b_{K\cdot}$ where $K = \{1,2,\dots,m\} - K_o$
4. Find $b_{i\ell_o} = \max_{j \in K} \bar{b}_{i\cdot}$
5. Find $b_{i_o\ell_o} = \max_{i \in K} b_{\cdot\ell_o}$
6. If $i_o = i$ go to 10
7. Find $b_{i_o\ell^*} = \max_{j \in K} \bar{b}_{i_o\cdot}$

(*) The rationale for such a "minimal distance" algorithm could also be given in decision-theoretic terms.

8. If $b_{i_o \ell_o} < b_{i_o \ell *}$ go to 10

9. Go to 2

10. Set $p_{i\ell_o} = 1$

11. Set $i = i+1$

12. Go to 1

The above sketch of a solution algorithm for finding a minimal distance solution to our problem will now be briefly justified.

Take the first row $b_{1}.$ of the normalized agreement matrix $[b_{ij}]$. We first find the maximal element in this row b_{1j_o}, say.

- If b_{1j_o} also happens to be the maximal element in the column j_o then we can go ahead and single out this entry $(1,j_o)$ by setting $p_{1j_o} = 1$ and $p_{1j} = 0$ ($\forall$ $j=1,2,\ldots,m$ and $j \neq j_o$).
- If b_{1j_o} is <u>not</u> the maximal element in the column j_o, we still may be able to use this entry $(1,j_o)$ and set $p_{1j_o} = 1$ as before. Everything depends upon the following check:
- Letting $b_{i'j_o}$ denote the maximal element in column j_o, and having assumed that $b_{1j_o} < b_{i'j_o}$, we must now check the maximal element in row i', which we denote $b_{i'j*}$. If $b_{i'j*} > b_{i'j_o}$ we can choose b_{1j_o} and set $p_{1j_o} = 1$ in the first row

without running the risk of suboptimizing since $b_{i'j_o}$ is not the maximal element in row i' --even though it is the maximal element in column j_o. Suboptimization would occur simply because of the fact that once a non-zero (one) entry is entered in row i and column j, these rows and columns are to be deleted and are no longer available to be assigned a non-zero entry. Clearly if the above inequalities become equalities, i.e. if there are several maximal elements, there will be multiple optima to problem (23).

A simple illustration will now be outlined:

<u>Example</u>

Let $m = 4$, $\ell = 7$.

Let the individual orderings of the alternatives be as follows: $\varphi_1 = (a\ b\ c\ d)$, $\varphi_2 = (a\ b\ d\ c)$, $\varphi_3 = (a\ d\ b\ c)$, $\varphi_4 = (a\ d\ c\ b)$, $\varphi_5 = (d\ a\ c\ b)$, $\varphi_6 = (d\ c\ a\ b)$, $\varphi_7 = (d\ c\ b\ a)$.

This lead to the following normalized agreement matrix (assuming, for computational convenience that all criteria S_h are weighted equally: $w_h = 1\ \forall h = 1,2,\ldots,\ell$).

$$B = \begin{bmatrix} 4/7 & 1/7 & 1/7 & 1/7 \\ 0 & 2/7 & 2/7 & 3/7 \\ 0 & 2/7 & 3/7 & 2/7 \\ 3/7 & 2/7 & 1/7 & 1/7 \end{bmatrix} \tag{24}$$

- In row 1, the maximal element is the first entry = 4/7. So we set $p_{11} = 1$ and $p_{12} = p_{13} = p_{14} = 0$. (4/7 was also the maximal element in the first column so there was no need to consider another choice.)
- Now B becomes

$$B^1 = \begin{bmatrix} 2/7 & 2/7 & 3/7 \\ 2/7 & 3/7 & 2/7 \\ 2/7 & 1/7 & 1/7 \end{bmatrix} \tag{25}$$

(after deleting row 1 and column 1 as a result of the first step)

We choose 3/7 as the maximal element in the first row of B^1, again it is also maximal for its column (#3 in B^1). We enter a 1 as the (2,4) entry of the optimal permutation matrix $\hat{P}$ and 0's everywhere else in the second row.

- Proceeding in the same fashion we enter a one for both the (3.3) and the (4.2) entries in $\hat{P}$.

And we obtain

$$\hat{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{26}$$

If the reference order was (a,b,c,d) then under this rule, the optimal aggregate ordering would be

(27) $(a,b,c,d) \cdot \hat{P} = (a\ d\ c\ b)$.

As we can see, this decision rule will lead to (at least) one optimal vertex $\hat{P} \in \mathcal{S}_m \subset \mathcal{B}$ in a finite number of steps. The following lemma will show that this rule does yield a minimum for the objective function $d(\hat{P},B)$ as defined in equation (23) above.

Lemma 2: Under the above decision rule the objective function $d(\hat{P},B) = \sum_{i,j=1}^{m} |p_{ij}-b_{ij}|$ is minimized.

Proof: For a given row i, there are three mutually exclusive cases to be considered.

Case 1. $b_{ij_o} = \text{Max}_j\ b_{i.}$ (where $b_{i.}$ denotes the ith row vector of B). And $b_{ij_o} = \text{Max}_i\ b_{.j_o}$ (where $b_{.j_o}$ denotes the j_oth column vector of B).

Case 2: $b_{ij_o} = \text{Max}_j\ b_{i.}$

$b_{ij_o} < \text{Max}_i\ b_{.j_o} = b_{i'j_o}$

but $b_{i'j_o} < \text{Max}_j\ b_{i'.} = b_{i'j*}$

hence $b_{ij_o} < b_{i'j_o} < b_{i'j*}$

Case 3: $b_{ij_o} = \text{Max}_j\ b_{i.}$

$b_{ij_o} < \text{Max}_i\ b_{.j_o} = b_{i'j_o}$

but $b_{i'j_o} = \text{Max}_j\ b_{i'.} = b_{i'j*}$

hence $b_{ij_o} < b_{i'j_o} = b_{i'j*}$.

In case #1, b_{ij_o} is maximal for both row i and column j. In case #2 b_{ij_o} is maximal for i but not for j_o; the column maximum for j_o (denoted $b_{i'j_o}$) is not maximal for row i' however--which it is in case #3.

Now 0 is a lower bound for d and it is attained if and only if $\hat{P} = B$ by the metric properties of d. If $B \neq \hat{P}$ then we are to choose exactly m non-zero elements for $\hat{P}$--one for each row and each column of $\hat{P}$. Thus (m^2-m) elements in the sum

(28) $\sum_{i,j=1}^{m} |\ p_{ij}-b_{ij}\ |$

will be equal to $b_{ij} \in [0,1]$ and m elements will be equal to $|1-b_{ij}| \in [0,1]$.

Take any row sum, e.g.

(29) $\Sigma_1 = |p_{11}-b_{11}| + |p_{12}-b_{12}| + \ldots + |p_{1m}-b_{1m}|$

(i) Consider Case #1 and #2.

By contradiction, suppose we take

(30) $\begin{cases} P_{1k} = 1 & \text{for } k \neq j_o \\ P_{1t} = 0 & \text{for } t \neq k;\ t=1,2,\ldots,m \end{cases}$.

Then Σ_1 becomes

(31) $\Sigma_{1k} = b_{11} + b_{12} + \ldots + (1-b_{1k}) + \ldots + b_{1j_o} + \ldots + b_{1m}$.

But, since $(1-b_{1k}) > (1-b_{1j_o})$:

(32) $\Sigma_{1k} > \Sigma_{1j_o} = b_{11} + b_{12} + \ldots + b_{1k} + (1-b_{1j_o}) + \ldots + b_{1m}$

and (k) is not optimal.

(ii) Consider now Case #3. Again, by contradiction, suppose we take:

(33) $\begin{cases} P_{1j_o} = 1 & \text{for some } j_o \\ P_{1t} = 0 & t \neq j_o\ ;\quad t = 1,2,\ldots,m \end{cases}$

Now at stage (i'), the row sum for (i') would be

(34) $\Sigma_{i'} = |p_{i'1} - b_{i'1}| + \ldots + |p_{i'j_o-1} - b_{i'j_o-1}| +$
$|p_{i'j_o+1} - b_{i'j_o+1}| + \ldots + |p_{i's} - b_{i's}| + \ldots +$
$|p_{i'm} - b_{i'm}|$.

Let

(35) $b_{i's} = \underset{i}{\text{Max}}\ (b_{i'1},\ldots,b_{i'j_o-1}, b_{i'j_o+1},\ldots,b_{i'm})$

Then under the above choice (33) we would have

(36) $\Sigma_{i's} = b_{i'1} + \ldots + b_{i'j_o-1} + b_{i'j_o+1} + \ldots + (1-b_{i's}) + \ldots + b_{i'm}$

And since $b_{i's} < b_{i'j_o}$ in this case:

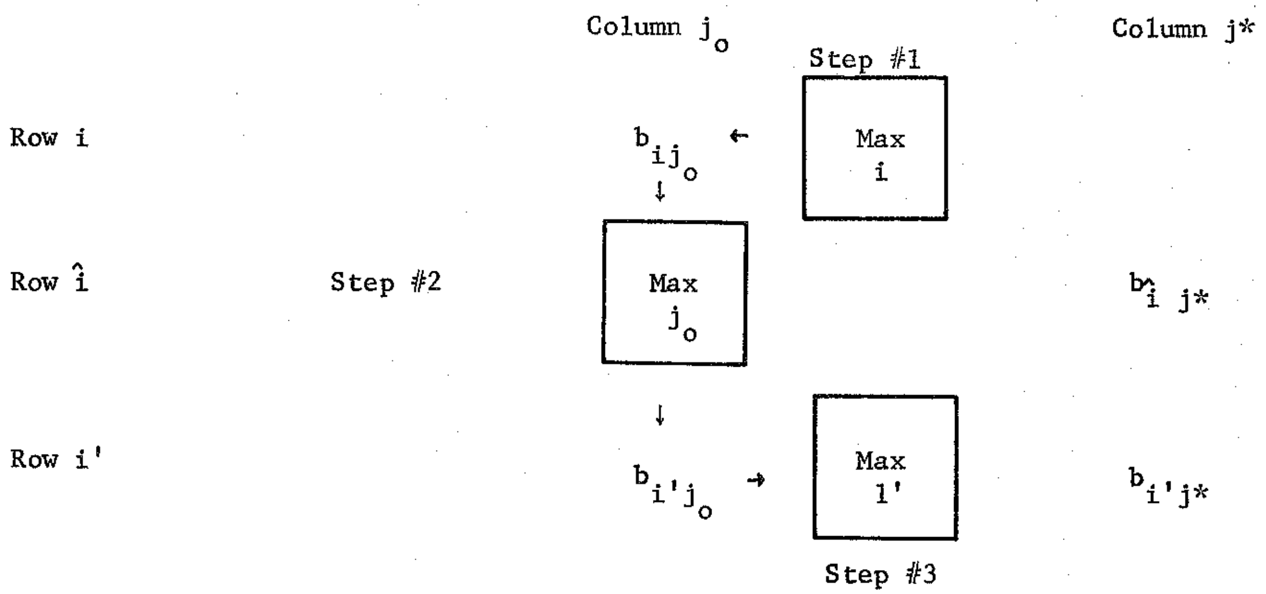
(37) $\Sigma_{i'j_o} = b_{i'1} + \ldots + (1-b_{i'j_o}) + \ldots + b_{i's} + \ldots + b_{i'm}$

(38) $\Sigma_{i's} > \Sigma_{i'j_o}$

which shows that in Case #3, choice (33) would have been suboptimal. Apart from the heavy notational burden involved it is easily shown that the argument can be extended to all row sums of $|P-B|$.

Q.E.D.

Regarding the proposed algorithm and Lemma 2 above, the following important point should be noted: it may seem that the suboptimization problem does not end after the three sequential checks devised in our method. The following diagram will illustrate our discussion.



In Case #2 of our proof we have

$$b_{ij_o} < b_{i'j_o} < b_{i'j*}$$

and we set $P_{ij_o} = 1$ under the claim that this choice will not involve suboptimization at step i' since the row maximum in i' is $b_{i'j*} \neq b_{i'j_o}$ and thus column j* will still be available at step i'. One might object, however, that the upper portion of column j* (i.e. all elements between i and i') must also be checked. If at step $\hat{i}$, for instance, it turns out that the $\max_j b_{\hat{i}.} = b_{\hat{i}j*}$ we might, in fact, want to set $p_{\hat{i}j*} = 1$, but the argument of the algorithm also applies there: either $b_{\hat{i}j*} = \max_i b_{.j*} > b_{i'j*}$ and there is no problem; or $b_{\hat{i}j*} < b_{i'j*} = \max_{\hat{i}} b_{.j*}$ and we would not want to set $p_{\hat{i}j*} = 1$ anyway as this would ential suboptimization when we reach step i'.

Another obvious point deserves a few comments, viz. the fact that both the maximal agreement approach and the minimal distance approach lead to the same solution(s). This can be stated as a simple corollary to Lemma 2.

Corollary: Under the assumptions of this aggregation model the maximal agreement problem (equation 15) and the minimal distance problem (equation 23) always lead to the same solution.

Proof: As we recall the maximal agreement problem consisted in maximizing the agreement index I_A over the set $\mathcal{P}_m$ of all permutation matrices:

$$(39) \quad \underset{[p_{ij}]\in\mathcal{P}_m}{\mathrm{Max}} \quad I_A = \sum_{i,j} \pi_{ij} p_{ij}$$

where $[\pi_{ij}]$ is the agreement matrix defined in 2.2 (Def. 1). Clearly, any optimal solution $[p^*_{ij}]$ to (39) is also an optimal solution to:

$$(40) \quad \underset{[p_{ij}]\in\mathcal{P}_m}{\mathrm{Max}} \quad I'_A = \sum_{i,j} \pi^{norm}_{ij} \quad p_{ij} \quad .$$

Let $[p^*_{ij}]$ be a maximal solution to (40). Then $[p^*_{ij}]$ is also a minimal solution to:

$$(41) \quad \underset{p_{ij}\in\mathcal{P}_m}{\mathrm{Min}} \sum_{i,j} \left| 1 - \pi^{norm}_{ij} \right| \cdot p_{ij}$$

which is an alternative way of writing the minimal distance problem.

Q.E.D.

Clearly, the maximal agreement and the minimal distance problems are dual problems. And, this constitutes the basic difference with respect to the linear assignment problem as posed by Koopmans and Beckman.

In the linear version of the assignment problem the profitability matrix, which indicates the expected profit from assigning resource i to location j, has no useful properties in itself; whereas in this case the agreement matrices do possess a basic property which allows us to simultaneously normalize their rows and columns, and thus obtain a bistochastic matrix.

Section 3: Concluding Comments

The above approach can be extended in a number of directions, some of which will now be briefly discussed.

3.1 From a probabilistic standpoint, first, one could study the implications of various types of probability measures on the set $\mathcal{S}_m$ of vertices of the convex polyhedrom $\mathcal{B}$, in regard to the minimal distance (maximal agreement) concept we have discussed. In particular the following point is worth noting: if we assume a uniform distribution of the orderings in $\mathcal{S}_m$, it is easy to show that the $(m \times m)$ normalized agreement matrix will have all entries $\pi_{ij}^{norm} = \frac{1}{m}$.

For example for m = 3 we have

$$\text{(42)} \qquad \pi_{3,3}^{norm} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

In this case the minimal distance (or the maximal agreement) method are of little help inasmuch as they point out that all elements of $\mathcal{S}_m$ are equally "good" to represent $\pi_{ij}^{norm} \in \mathcal{B}$. This occurs as π_{ij}^{norm} constitutes a center of gravity for the $\mathcal{B}$ polyhedron.

It is easily shown that the so-called "paradox of voting," i.e. the fact that an intransitive group ordering may arise if individual transitive orderings are aggregated by majority voting, occurs precisely if and only if the individual orderings lead to such a point π^{norm} at equal distance from all the vertices of $\mathcal{S}_m$!

For instance, if we had only three orderings

$$\varphi_1(A) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}; \qquad \varphi_2(A) = \begin{bmatrix} b \\ c \\ a \end{bmatrix}; \qquad \varphi_3(A) = \begin{bmatrix} c \\ a \\ b \end{bmatrix}$$

and we were to decide on an aggregate ordering by majority voting over each pair of alternatives we would obtain the following intransitive order: (a b c a). In such a case, however, the normalized agreement matrix π^{norm} is given by equation (42) above; and our approach clearly indicates that any one of the 3 individual orderings is "optimal." The occurrence of any intransitivity was only a poor indicator of such an indeterminacy![*] A possible unique solution could then be reached through a completely randomized choice.

<u>3.2</u> On the other hand, it is also clear that our approach can be viewed in a much more general framework - the problem of finding the appropriate permutation matrix $\hat{P}$ to aggregate ℓ individual orderings is intimately related to the problem of matrix approximation. This relationship which brings out many interesting aspects, will be presented elsewhere.

<u>3.3</u> Along similar lines the minimal distance approach can also be formulated in terms of many other metrics besides the sum of absolute values metric we used in this paper. Whatever our choice of a metric happens to be in a given problem, it might be interesting to investigate an efficient stochastic algorithm for its solution. In this problem such an algorithm has, in fact, been devised and will be discussed in a forthcoming paper.

Finally, we should note that the very notion of an agreement index and an agreement matrix can also be generalized through the use of other clustering techniques.

[*] For a discussion of this problem as well as other related problems, see [6],[7].

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