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IGNORING IGNORANCE AND AGREEING TO DISAGREE

by

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[3-3]

Abstract

The prevailing information structure in the literature of common knowledge is the partition of the set of states. This structure is based on the implicit assumption that agents are aware of what they do not know. We study more general information structure in which agents are allowed to ignore their own ignorance. The main result is that Aumann's [1976] famous result on the impossibility of agreeing to disagree, which was proved for partitions, still holds for the more general structure. Hence, people cannot agree to disagree even when they are ignorant of their ignorance.

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1. Introduction

In his seminal paper, "Agreeing to Disagree," Aumann [1976] has shown that agents who have the same prior distribution over the states of the world cannot agree to disagree. More precisely, if their posteriors for a certain event are common knowledge then these posteriors must coincide even though they are based on different information.

The way in which the posterior of a given event E is computed by agent i is as follows. With each state ω there is associated a set $P_i(\omega)$ of states that are indistinguishable from ω at ω. The posterior of E for i is the conditional probability of E given $P_i(\omega)$. A basic assumption in Aumann's paper and in all the literature that followed is that the sets $P_i(\omega)$ (when ω ranges over all states of the world) form a partition.

The main purpose of this paper is to show that Aumann's result can be extended to some information structures (given by the family of sets $P_i(\omega)$) which are more general than partitions. Further, we examine the underlying assumptions on knowledge which are required to guarantee Aumann's result for these general information structures. To do so we introduce the knowledge of the agents as formal components in the model. The objects that are known in our model are proposition, and for each proposition ϕ we assume the existence of propositions $K_i\phi$ which are interpreted as "agent i knows that ϕ." The use of a knowledge operator on proposition is standard in epistemic logic and was also used to study the interaction of several agents' knowledge (see, e.g., Halpern [1986] for a survey of such works and also some recent papers by Gilboa [1986] and Kaneko [1977]). Bacharach [1985] used a similar approach where he applied the knowledge operator to events. Milgrom [1981] characterized common knowledge by using a common knowledge operator on events.

There are three properties of knowledge in our model that together imply a partition of the states of the world into indistinguishability classes: (K1) when an agent knows a proposition he knows he does; (K2), any proposition known by an agent is true; (K3) when an agent does not know a proposition he knows he does not.

In our main theorem we show that the more general information structure which is implied by (K1) and (K2) is enough to guarantee Aumann's result. In other words, it is impossible to agree to disagree even when agents are allowed to ignore their own ignorance. Moreover, with some plausible assumption on the prior distribution (K1) alone is enough. We note they (K2) is what gives knowledge its good name. When a false proposition is known we would rather call it belief. So when we give up (K2) it is beliefs that we are talking about. "We" of course as external observers; the agent who believes consider his belief to be knowledge and thus his prior would assign probability zero to states in which his belief is not knowledge. If this is indeed the case then (K1) alone suffices to prevent agreeing to disagree.

The structure of this paper is as follows. We define in Section 2 states of the world as lists of true propositions. We then examine the implication of (K1)-(K3) on the information structure of the space of states. In Section 3 we define what is for a proposition to be common knowledge in a state. We show that if a proposition is common knowledge in a state it is automatically true and common knowledge in a whole group of states. This provides a link between the definitions of common knowledge in terms of propositions and in terms of events. In Section 4 we define a natural topology of events starting from the simplest events "proposition ϕ is true." The results of this section are used later in Section 6 to show that all the events required to the study of knowledge and common knowledge are measurable. In Section 5 the notion of finitely generated knowledge is introduced. Informally it reflects the assumption that our (possibly infinite) knowledge is derived from finitely many proposition (a posteriori knowledge) by (a priori) deduction rules. This property of knowledge implies some restrictions on the information structure which are essential to derive the results of Section 6. For the special case of partitions, the required restriction is the countability of the partition, which is assumed in Aumann [1976]. In Section 6 we prove that (K1) and (K2) are enough to guarantee the impossibility of agreeing to disagree, and show under what condition (K1) alone suffices. In Section 7 we discuss various aspects of the model used here.

2. Proposition and States

Let Φ and I be two countable sets. We interpret elements of Φ as propositions describing a certain environment of interest. Alternatively one may think of Φ as a set of well formed formulas in some language. But since the structure of such a language plays no role in our study we prefer the less technical notion of propositions to describe the primitives of the theory. Elements of I are interpreted as agents. For each agent $i \in I$ there exists a mapping $K_i: \Phi \to \Phi$, where for each $\phi \in \Phi$ the proposition $K_i\phi$

is interpreted as saying "i knows Φ." There exists also a mapping $\sim: \bar{\Phi} \to \bar{\Phi}$ such that for each Φ, $\sim\Phi \neq \Phi$ and, where $\sim\Phi$ is interpreted as "not Φ."

Consider the set $\Sigma = \{0,1\}^{\bar{\Phi}}$. Each element of Σ can be thought of as an assignment of the truth values to the propositions: 1 for true and 0 for false. An element ω of Σ is called a <u>state of the world</u> (or a <u>state</u>) if for each $\Phi \in \bar{\Phi}$, $\omega(\Phi) + \omega(\sim\Phi) = 1$. The set of states is denoted by Ω_0. We identify the state ω with the set of propositions $\{\Phi|\omega(\Phi) = 1\}$. Thus we write $\Phi \in \omega$ instead of $\omega(\Phi) = 1$ and $\Phi \notin \omega$ for $\omega(\Phi) = 0$. We write $\Psi \subset \omega$ for a set of proposition Ψ if for each $\Phi \in \Psi$, $\Phi \in \omega$. The phrase "Φ is true in ω" is also used to mean $\omega(\Phi) = 1$.

The <u>epistemic content of a state ω for agent i</u> is the set $K_i^-(\omega)$ of all propositions known by i in ω, i.e.,

$$K_i^-(\omega) = \{\Phi | K_i\Phi \in \omega\}.$$

We fix now a subset Ω of Ω_0. Define for each $i \in I$ a binary relation p_i on Ω by: $\omega' p_i \omega$ whenever $K_i^-(\omega) \subset \omega'$. We say in this case that <u>ω' is possible in ω for i</u>. This relation expresses the compatibility of the state ω' with the knowledge i has in ω; each proposition known by i in ω is true in ω'. For each i and $\omega \in \Omega$ let $P_i(\omega)$ be the set of all possible states in ω for i, i.e.,

$$P_i(\omega) = \{\omega' | \omega' p_i \omega\}.$$

For a set of propositions Ψ, $P_i(\Psi)$ is the set $\{P_i(\Phi) | \Phi \in \Psi\}$.

Consider now three properties of knowledge in state ω.

(K1) For each $\Phi \in \Phi$ and $i \in I$, if $K_i\Phi \in \omega$ then $K_iK_i\Phi \in \omega$.

(K2) For each $\Phi \in \Phi$ and $i \in I$, if $K_i\ \Phi \in \omega$ then $\Phi \in \omega$

(K3) For each $\Phi \in \Phi$ and $i \in I$, if $\sim K_i\Phi \in \omega$ then $K_i{\sim}K_i\Phi \in \omega$.

Condition (K1) says that in state ω, if i knows Φ he knows he does. (K2) says that every proposition known by i in ω is true in ω. (K3) says that if i does not know Φ in ω he knows he does not.

We denote by:

Ω_1 the set of all states which satisfy (K1);

Ω_2 the set of all states which satisfy (K1) and (K2);

Ω_3 the set of all states which satisfy (K1), (K2) and (K3).

Clearly, $\Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \Omega_0$.

The following theorem describes the relation p_i between states in terms of relation between epistemic content.

<u>Theorem 1</u>: For each $i \in I$:

(a) For $\Omega \subset \Omega_2$, $\omega' p_i \omega$ iff $K_i^-(\omega') \supset K_i^-(\omega)$

(b) For $\Omega \subset \Omega_3$, $\omega' p_i \omega$ iff $K_i^-(\omega') = K_i^-(\omega)$.

<u>Proof</u>: (a) Suppose $K_i^-(\omega') \supset K_i^-(\omega)$. By (K2), $\omega' \supset K_i^-(\omega')$ and therefore $\omega' p_i \omega$. Conversely, assume $\omega' p_i \omega$. If $\Phi \in K_i^-(\omega)$ then by (K1), $K_i\Phi \in K_i^-(\omega)$. Thus, $K_i\Phi \in \omega'$ and $\Phi \in K_i^-(\omega')$.

(b) If $K_i^-(\omega') = K_i^-(\omega)$ then $\omega' p_i \omega$ by (a). Suppose $\omega' p_i \omega$ then by (a) $K_i^-(\omega') \supset K_i^-(\omega)$. Let $\Phi \in K_i^-(\omega')$, i.e., $K_i\Phi \in \omega'$. Suppose $\Phi \notin K_i^-(\omega)$. Thus, $K_i\Phi \notin \omega$ and $\sim K_i\Phi \in \omega$ which by (K3) implies $K_i{\sim}K_i\Phi \in \omega$ and therefore $\sim K_i\Phi \in \omega'$, a contradiction. Q.E.D.

Theorem 2: For each $i \in I$,

(a) If $\Omega \subset \Omega_1$, p_i is transitive.

(b) If $\Omega \subset \Omega_2$, p_i is transitive and reflexive.

(c) If $\Omega \subset \Omega_3$, p_i is transitive, reflexive and symmetric.

Proof: (b) and (c) follow directly from parts (a) and (b) of Theorem 1, correspondingly. We omit the simple proof of (a). Q.E.D.

A proof of Theorem 2 in a slightly different setup is found in Hughes and Cresswell [1984].

Consider now the following three properties of Ω in terms of P_i.

(P1) For each $i \in I$ and $\omega \in \Omega$, if $\omega' \in P_i(\omega)$ then $P_i(\omega') \subset P_i(\omega)$.

(P2) For each $i \in I$ and $\omega \in \Omega$, $\omega \in P_i(\omega)$.

(P3) For each $i \in I$ and $\omega \in \Omega$, if $\omega' \in P_i(\omega)$ then $P_i(\omega) = P_i(\omega')$.

The following is an immediate corollary of Theorem 2.

Corollary 1:

(a) If $\Omega \subset \Omega_1$ then Ω satisfies (P1).

(b) If $\Omega \subset \Omega_2$ then Ω satisfies (P1) and (P2)

(c) If $\Omega \subset \Omega_3$ then Ω satisfies (P1), (P2) and (P3). In particular, $\{P_i(\omega) | \omega \in \Omega\}$ is a partition of Ω into equivalence classes with respect to equality of epistemic content.

3. Common Knowledge

A proposition *Φ is common knowledge in ω* if for each $n \geq 1$ and each sequence of agents, $i_1, \ldots, i_n$, $K_{i_1}, \ldots, K_{i_n} \Phi \in \omega$.

The state <u>ω' is commonly possible in ω</u> if there exists $n \geq 1$ and a sequence of agents $i_1,\ldots,i_n$ such that $\omega' \in P_{i_1}(P_{i_2}(\ldots,P_{i_n}(\omega),\ldots))$. The set of all states which are commonly possible in ω is denoted by $P(\omega)$, i.e., $P(\omega) = \cup P_{i_1}(P_{i_2}(\ldots,P_{i_n}(\omega),\ldots)$ where the union ranges over all finite sequences of agents.

Common knowledge and common possibility are related as follows.

<u>Theorem 3</u>: If Φ is common knowledge in ω_0 then Φ is true in every state ω' which is commonly possible in ω_0. Moreover, Φ is common knowledge in such ω'.

<u>Proof</u>: Let ω' be common probability in ω. Then there exists $n \geq 1$, a sequence $i_1,\ldots,i_n$ and states $\omega_1,\ldots.\omega_n$ such that $\omega_j P_{i_j} \omega_{j+1}$ for $j = 0,\ldots,n-1$ and $\omega_n = \omega'$. If Φ is common knowledge in ω_0 then $K_{i_1},\ldots,K_{i_n}\Phi \in \omega_0$. It follows immediately by induction on j that $K_{i_j},\ldots,K_{i_n}\Phi \in \omega_{j-1}$ and thus $K_{i_n}\Phi \in \omega_{n-1}$ which implies $\Phi \in \omega_n = \omega'$.

To show that Φ is common knowledge in ω', we observe that for each $n \geq 1$ and sequence $i_1,\ldots,i_n$, $K_{i_1},\ldots,K_{i_n}\Phi$ is also common knowledge in ω and therefore true in ω'. Q.E.D.

The relation between $P(\omega)$ and $P_i(\omega)$ is given in the next lemma. The simple proof is omitted.

<u>Lemma 1</u>: For each $i \in I$ and $\omega \in \Omega$, $P_i(P(\omega)) \subset P(\omega)$. Moreover, if $\Omega \subset \Omega_2$, then $P_i(P(\omega)) = P(\omega)$. If $\Omega \subset \Omega_3$ then $P(\omega)$ is the minimal element of the joint of the partitions $\{P_i\}_{i\in I}$ which contains ω.

We recall that the joint of the partitions $\{P_i\}_{i\in I}$ is the finest partition of Ω which is coarser than each P_i. In Aumann's model where common knowledge is an attribute of events, an event is common knowledge at ω if it contains this minimal element of the joint, $P(\omega)$.

4. Topology of Events

The product topology on $\Sigma = \{0,1\}^{\Phi}$ is generated by the family of sets $\{A_\phi\}_{\phi\in\Phi}$ where $A_\phi = \{\sigma \in \Sigma | \sigma(\phi) = 1\}$. A_ϕ can be interpreted as the event that ϕ is true. This topology induces topology on each of the spaces Ω_i $i = 0,\ldots,3$. Moreover, as subset of Σ:

Lemma 2: Each space Ω_i, $i = 0,\ldots,3$, is closed in Σ and therefore compact.

Proof: For a given ϕ the set $\{\omega | \omega(\phi) + \omega(\sim\phi) = 1\}$ is closed. $\Omega_0 = \cap_{\phi\in\Phi} \{\omega|\omega(\phi) + \omega(\sim\phi) = 1\}$ and thus Ω_0 is closed. $\Omega_1 = \Omega_0 \cap [\cap_{i\in I} \cap_{\phi\in\Phi} \{\omega|\omega(K_iK_i\phi) \geq \omega(K_i\phi)\}]$. Therefore, Ω_1 is closed. The proofs for Ω_2 and Ω_3 are similar. Q.E.D.

We assume from now on that the space $\Omega \subset \Omega_0$ is closed in Σ.

Lemma 3: If A is a closed subset of Ω then $P_i(A)$ is closed for each $i \in I$. In particular, for each $\omega \in \Omega$, $P_i(\omega)$ is closed.

Proof: Suppose $\{\omega_n\} \subset P_i(A)$ and $\omega_n \to \omega$. There exists a sequence $\{\omega'_n\} \subset A$ such that for each n, $\omega_n \in P_i(\omega'_n)$. Since Ω is compact we may assume without loss of generality that $\omega'_n \to \omega'$, and since A is closed, $\omega' \in A$.

It is enough to show that $\omega \in P_i(\omega')$, i.e., that $\bar{K}_i(\omega') \subset \omega$. Indeed, suppose $K_i\Phi \in \omega'$. Then for some N, $K_i\Phi \in \omega'_n$ for all $n > N$. Thus $\Phi \in \omega_n$ for $n > N$ and therefore $\Phi \in \omega$. Q.E.D.

Corollary 2: For each $\omega \in \Omega$, $P(\omega)$ is a countable union of closed sets.

Proof: The proof follows from the definition of $P(\omega)$, Lemma 3, and the countability of I. Q.E.D.

5. Finitely Generated Knowledge

The epistemic content of a state for an agent may be in general of infinite size. Indeed, if anything is known to agent i in state ω in Ω_1, then i knows all the propositions $k_i\Phi, k_ik_i\Phi, \ldots$, etc., which can be an infinite set. This kind of knowledge is acquired by i without any effort; it is derived from a single known proposition by a priori deduction rules which are independent of the state of the world. In other spaces, Ω, more deduction rules may be used and therefore more propositions can be derived. Thus in Ω_2 i derives from $k_i\Phi$ also propositions like $\sim K_i\sim\Phi$, $\sim K_iK_i\sim\Phi, \ldots$. In general, Ω may include, of course, consistency requirements and deduction rules beyond (K1)-(K3).

This leads us to the following definition. Let Ψ be a subset of $\bar{K}_i(\omega)$. We say that Ψ generates $\bar{K}_i(\omega)$ if for each $\omega' \in \Omega$, $\Psi \subset \bar{K}_i(\omega')$ implies $\bar{K}_i(\omega) \subset \bar{K}_i(\omega')$. That is, whenever i knows all the propositions in Ψ he also knows all the propositions in $\bar{K}_i(\omega)$.

There is, of course, knowledge that cannot be derived from other knowledge and may be considered therefore as a posteriori knowledge which depends on experience. We require that such knowledge is finite. Formally, we say that <u>knowledge is finitely generated in Ω</u> if for each $i \in I$ and $\omega \in \Omega$, $K_i^-(\omega)$ is generated by a finite set of propositions.

Consider now the equivalence relation $\sim_i$ defined on Ω by: $\omega \sim_i \omega'$ iff $K_i^-(\omega) = K_i^-(\omega')$. Let π_i be the partition of Ω to equivalence classes with respect to $\sim_i$. Let also $\Delta_i = \{P_i(\omega) | \omega \in \Omega\}$. By Corollary 1(c) when $\Omega \subset \Omega_3$, $\Delta_i = \pi_i$ but when $\Omega \subset \Omega_2$ these sets are not necessarily the same.

The following theorem relates Δ_i to π_i and to finitely generated knowledge for $\Omega \subset \Omega_2$.

<u>Theorem 4</u>: Suppose $\Omega \subset \Omega_2$ and knowledge is finitely generated in Ω. Then for each $i \in I$, the sets Δ_i and π_i are countable. Moreover, the σ-fields generated by these two sets coincide.

<u>Proof</u>: Let $A(\omega)$ be the element in π_i which contains ω. Let $\Psi(\omega)$ be a finite generator of $K_i^-(\omega)$. Since for each $\omega' \in A(\omega)$, $K_i^-(\omega') = K_i^-(\omega)$, $\Psi(\omega)$ generates also $K_i^-(\omega')$. Consider now a map $\Psi\colon \Omega \to 2^{\Phi}$ which assigns for each ω a finite generator of $K_i^-(\omega)$ such that Ψ is constant on elements of the partition π_i. Suppose now that $\Psi(\omega) = \Psi(\omega')$ then $\Psi(\omega) \subset K_i^-(\omega')$ and therefore $K_i^-(\omega') \supset K_i^-(\omega)$. By symmetry $K_i^-(\omega') = K_i^-(\omega)$, i.e., ω' and ω are in the same element of the partition. Thus there is a one-to-one correspondence between π_i and the values of Ψ. Since Φ is countable, π_i must also be countable.

Now observe that by Corollary 1(b), $P_i(\omega) = P_i(\omega')$ iff $\omega \in P_i(\omega')$ and $\omega' \in P_i(\omega)$ and this holds by Theorem 1(a) iff $K_i^-(\omega) = K_i^-(\omega')$.

Thus the map $P_i(\omega) \to \{\omega' | \omega' \sim_i \omega\}$ is a well defined one-to-one map from Δ_i onto π_i, which shows that Δ_i is countable. We show now that π_i is contained in the σ-field generates by Δ_i. Let $A(\omega)$ be the element of π_i which contains ω. Consider the set $B = P_i(\omega) \setminus \cup P_i(\omega')$ where the union ranges over all ω' such that $P_i(\omega') \subsetneqq P_i(\omega)$. Then $B = \{\omega'' | K_i^-(\omega'') \supset K_i^-(\omega)\} \setminus \{\omega'' | K_i^-(\omega'') \supsetneqq K_i^-(\omega)\} = \{\omega'' | K_i^-(\omega'') = K_i^-(\omega)\} = A(\omega)$. This, with the countability of Δ_i, shows that $A(\omega)$ is in the field. Now it is enough to show that no subset of $A(\omega)$ is in that field. For this it suffices to show that for each $P_i(\omega)$, $P_i(\omega') \cap A(\omega)$ is either $\emptyset$ or $A(\omega)$. Indeed, if $\omega'' \in P_i(\omega') \cap A(\omega)$ then $P_i(\omega'') \subset P_i(\omega')$ and $P_i(\omega'') \subset P_i(\omega)$. But since $\omega'' \in A(\omega) = B$ the last inclusion implies $P_i(\omega'') = P_i(\omega)$. Therefore, $P_i(\omega') \supset P_i(\omega)$ and $P_i(\omega') \cap A(\omega) = A(\omega)$. Q.E.D.

Another useful implication of finitely generated knowledge is the following:

<u>Theorem 5</u>: Suppose $\Omega \subset \Omega_2$ and knowledge is infinitely generated in Ω. Let $\{\omega_n\}$ be a sequence in Ω such that $P_i(\omega_{n+1}) \subset P_i(\omega_n)$ for $n \geq 1$. Then, for large enough n and m, $P_i(\omega_n) = P_i(\omega_m)$.

<u>Proof</u>: Since $P_i(\omega_n)$ is decreasing, $K_i^-(\omega_n) \subset K_i^-(\omega_{n+1})$ for all $n \geq 1$. Let $\Psi(\omega_n) \subset K_i^-(\omega_n)$ be a finite generator of $K_i^-(\omega_n)$ for each n. We may assume without loss of generality that $\omega_n \to \omega$. The closedness of each $P_i(\omega_n)$ implies $\omega \in P_i(\omega_n)$ for each n and thus $K_i^-(\omega) \supset K_i^-(\omega_n)$, i.e., $K_i^-(\omega) \supset \cup_n K_i^-(\omega_n)$. If $\Phi \notin \cup_n K_i^-(\omega_n)$ then for all n, $\sim K_i\Phi \in \omega_n$. Therefore, $\sim K_i\Phi \in \omega$ and $\Phi \notin K_i^-(\omega)$. Hence $K_i^-(\omega) = \cup_n K_i^-(\omega_n)$. Let $\Psi \subset K_i^-(\omega)$ be a finite

generator of $K_i^-(\omega)$. Clearly, for large enough n, $\Psi \subset K_i^-(\omega_n)$; but then $K_i^-(\omega_n) = K_i^-(\omega)$. Q.E.D.

Let us finally look at two simple conditions which imply finitely generated knowledge.

Theorem 6: Knowledge is finitely generated in each of the following cases:

(a) Ω in finite.

(b) For each $i \in I$, Δ_i is finite.

Proof: Clearly (a) implies (b). Assume now that Δ_i is finite. By Theorem 4, π_i is also finite. Let $\omega \in \Omega$. For each element $A(\omega')$ in π_i such that $K_i^-(\omega)\backslash K_i^-(\omega') \neq \emptyset$, choose a proposition in the latter set. The set Ψ of all such propositions is finite, $\Psi \subset K_i^-(\omega)$ and for any ω' such that $\Psi \subset \omega'$ it must be the case that $K_i^-(\omega) \subset K_i^-(\omega')$. Q.E.D.

6. Agreeing to Disagree

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space were $\mathcal{B}$ is the Borel σ-field on Ω on μ probability measure. The measure μ is interpreted as a prior distribution on Ω which is common to all agents.

Lemma 4: For each $i \in I$ and $\omega \in \Omega$, $P_i(\omega)$ and $P(\omega)$ are measurable.

Proof: Follows from Lemma 3 and Corollary 2.

Assume now that knowledge in Ω is finitely generated. Fix an element X in $\mathcal{B}$. For each $\omega \in \Omega$ and $i \in I$ such that $\mu(P_i(\omega)) > 0$ denote by $q_{i,\omega}$ the posterior probability of X given the knowledge of agent i, that is

$$q_{i,\omega} = \mu(X|P_i(\omega)) = \mu(X \cap P_i(\omega))/\mu(P_i(\omega)).$$

Let $Q_i = \{q_{i,\omega} | i \in I, \mu(P_i(\omega)) > 0\}$. By Theorem 4, Q_i is countable. We assume now that for each $q \in Q_i$ there is a proposition in Φ denoted by $x_i(q)$. This proposition is interpreted as saying that the posterior probability of X for i is q.

Consider now the countable set $\Psi_i = \{x_i(q) | q \in Q_i\}$. We assume further that for each ω and $i \in I$, if $\mu(P_i(\omega)) > 0$ then $\Psi_i \cap \omega = x_i(q_{i,\omega})$ and otherwise $\Psi_i \cap \omega = \emptyset$. Thus the only proposition from Ψ which is in ω is the one that properly describes the posterior of X in ω for i.

We say that in Ω it is <u>impossible to agree to disagree</u> if for each $\omega \in \Omega$ the following holds:

> If for each $i \in I$, $x_i(q_{i,\omega})$ is common knowledge in ω then for each $j,k \in I$, $q_{j,\omega} = q_{k,\omega}$.

<u>Theorem 7</u>: If $\Omega \subset \Omega_2$ then it is impossible to agree to disagree in Ω.

<u>Proof</u>: Suppose $x_i(q_{i,\omega})$ is common knowledge at ω. Then by Theorem 3 $x_i(q_{i,\omega}) \in \omega'$ for each $\omega' \in P(\omega)$. That means that $q_{i,\omega} = q_{i,\omega'}$ for each $\omega' \in P(\omega)$. Let us denote this common value by q_i. We will show that $\mu(x|P(\omega)) = q_i$. Therefore, if $x_i(q_{i,\omega})$ is common knowledge for each i, then for any $k,j \in I$. $q_j = \mu(X|p(\omega)) = q_k$.

Now by Lemma 1, $P(\omega) = \cup_{\omega' \in R(\omega)} P_i(\omega')$. By Theorem 4 it follows that there is a subset Γ_i of π_i such that $P(\omega) = \cup_{A \in \Gamma_i} A$. Since Γ_i is a countable partition of $P(\omega)$ it suffices to show that for any $A \in \Gamma_i$ either $\mu(A) = 0$ or $\mu(X|A) = q_i$. (Clearly there are A's with positive probability since $\mu(P(\omega)) > 0$.)

Consider the family of all subsets Γ of Γ_i which satisfy: for each $A \in \Gamma$ either $\mu(A) = 0$ or $\mu(X|A) = q_i$. This family is not empty ($\emptyset$ is such a subset) and is ordered by inclusion. The requirements of Zorn's lemma are trivially satisfied and thus there exists a maximal element Γ' in this family. We show that $\Gamma' = \Gamma_i$. Denote $G = \cup_{A \in \Gamma'} A$ and suppose $\Gamma \neq \Gamma_i$, there must exist $\omega_0 \in P(\omega)$ such that $P_i(\omega_0)\backslash G \neq \emptyset$. Also there should be a point $\bar{\omega}$ in that latter set such for each $\omega' \in P_i(\bar{\omega})\backslash G$, $P_i(\omega')\backslash G = P_i(\bar{\omega})\backslash G$, otherwise we can construct infinite sequence $\omega_0, \omega_1, \ldots,$ such that $P_i(\omega_n) \supsetneq P_i(\omega_{n+1})$ for all $n \geq 1$, contradicting Theorem 5.

Now for each $\omega' \in P_i(\bar{\omega})\backslash G$ we have $\bar{\omega} \in P_i(\omega')\backslash G$ and therefore $\omega' \sim_i \bar{\omega}$. Also, any ω' which satisfies this equivalence must be in $P_i(\bar{\omega})$ and cannot be in G. Thus $P_i(\bar{\omega})\backslash G = \{\omega' | \omega' \sim_i \bar{\omega}\}$ which is an element of Γ_i not in Γ'. Moreover, $P_i(\bar{\omega})\backslash G = P_i(\bar{\omega})\backslash \cup_{A \in \Gamma', A \subset P_i(\bar{\omega})} A$. Now $\mu(X|P_i(\bar{\omega})) = q_i$ and also either $\mu(X|A) = q_i$ or $\mu(A) = 0$ for each $A \in \Gamma'$. Therefore, either $\mu(P_i(\bar{\omega})\backslash G) = 0$ or $\mu(P_i(\bar{\omega})\backslash G) = q_i$. This contradicts the maximality of Γ' and completes the proof. Q.E.D.

It is possible to extend Theorem 6 to $\Omega \subset \Omega_1$ provided that we restrict Ω and the prior distribution μ as follows. We say that <u>μ is consistent with Ω</u> if $\mu(\Omega\backslash\Omega_2) = 0$. We note that for each $\omega \in \Omega\backslash\Omega_2$ there exists an agent i and a proposition Φ such that i knows Φ in ω but Φ is not true in ω.

Clearly this implies that ω is impossible for i in ω. The consistency on μ guarantees that the prior distribution reflects this (mistaken) impossibility.

A state ω is dead end in Ω if for all $i \in I$, $P_i(\omega) \neq \emptyset$.

Theorem 7: If $\Omega \subset \Omega_1$, μ is consistent with Ω and Ω does not have dead ends then it is impossible to agree to disagree in Ω.

Proof: Suppose $x_i(q_{i,\omega})$ is common knowledge in ω. Since Ω does not have dead ends $P(\omega) \neq \emptyset$ for each ω. The proof then continues as the proof of Theorem 6 by applying Theorem 3, concluding that $x_i(q_{i,\omega}) \in \omega'$ for each $\omega' \in P(\omega)$. We define now $P_i'(\omega) = P_i(\omega) \cap \Omega_2$ and $P'(\omega) = P(\omega) \cap \Omega_2$. $P_i'(\omega)$ is exactly the set of possible states in ω for i in the space $\Omega' = \Omega \cap \Omega_2$. Further it is easy to see that $P_i'(P'(\omega)) = P'(\omega)$. The proof now follows that of Theorem 6.

7. Discussion

Some Benefits of Epistemic Models. The main result of this paper can be stated in a model that, unlike our model, does not introduce knowledge explicitly. The primitive of such model is a measurable space Ω of states. Information structure is given by measurable sets $P_i(\omega)$ (for each agent i and state ω) which satisfy Corollary 1(b) (or 1(a)) and the property guaranteed by Theorem 5. Common knowledge is defined as the set $P(\omega)$ of all commonly possible states ω. Yet we preferred to introduce knowledge explicitly for obvious reasons; it enables us to reveal the underlying assumptions about knowledge that give rise to the specific properties of the sets $P_i(\omega)$. Specifically, we showed that (K1), (K2) and the finite generation of knowledge provide all the properties required for Theorem 7.

It is also possible in our model to give clear answers to questions concerning the scope of agents' knowledge required for the study of common knowledge. The answers to these questions are in accordance with the informal discussion in Aumann [1985]. First of all, what agent i does know in state ω is strictly defined. He knows every proposition in $\bar{K}_i(\omega)$. Does he know his information structure? This depends on the set of proposition Φ. If there are propositions in Φ which describe such structures (or interpreted as doing it) then agent i either knows his information structure or he does not depending on whether these propositions are in $\bar{K}_i(\omega)$. Moreover, if Φ does not contain propositions about information structure then knowing this structure in the model is simply meaningless. In fact, in general it is impossible to specify in Φ the possibility relation p_i since Φ is only countable while there is usually a continuum of states. In any case knowledge of one's own information structure (let alone other's information structure) is irrelevant to the theory of common knowledge as it is presented here. Indeed, for the definition of common knowledge and the results in Sections 2-5 there is no requirement of any specific content of knowledge.

For the "agreeing to disagree" result in Section 6 we require that some propositions of the form $x_i(q)$ are true in certain states. These propositions are interpreted as saying that "q is the posterior of X for agent i." If these propositions are common knowledge then the posteriors are the same, claims Theorem 6. Do the agents have to know probability theory or at least their prior? Must it be common knowledge that they have the same prior? The answers to all these questions is in the negative. True, the agents know the propositions $x_i(q)$. But these propositions are only interpreted in the model as saying that the posteriors are such and such (and rightly so because $x_i(q)$ is true in state ω iff q is the posterior of X given $P_i(\omega)$). For the agents, x_i is just a predicate of numbers that may be read as "q is an essential number of i." These propositions obey of course all the rules of probability but this is something else than to say that the agents know it. The source of their knowing these propositions, be it a probability book or a list of numbers in the newspaper, is irrelevant to the results of Section 6.

<u>Common Knowledge and Epistemic Logic</u>. The basic features of the model presented here are common in the literature of formal modal logic and epistemic logic but it is worth noting that for our purposes we do not need the full body of these theories. First of all unlike modal logic systems we do not start with a language but rather with a set of formulas or propositions. There is no requirement on the structure of the language that produces these propositions. Also the epistemic operators K_i which are not restricted in any way, may have properties that similar operators in epistemic logic never have. For example K_i is not necessarily one-to-one while if K_i is an operator within a formal language, $K_i\Phi \neq K_i\Psi$ for any $\Phi \neq \Psi$. Thus in our model the same proposition may express simultaneously that i knows two different propositions. Moreover we do not restrict the relation between K_i's for different agents. It is possible to have a proposition Φ such that $K_i\Phi = K_j\Phi$ for each two agents i and j. It may also be the case that for each i and Φ,

(*) $$K_i K_i \Phi = K_i \Phi.$$

This means that knowing Φ and knowing that Φ is known are the same. When this is the case, requirement (K1) is automatically satisfied in each state and $\Omega_1 = \Omega_0$. In such a model it is possible sometimes to verify that a certain proposition is common knowledge without resorting to infinite application of K_i's. This may happen for example if the only source of knowledge in our model is the newspaper and the propositions $K_i\Phi$ for all i are the same proposition: "Φ is in the newspaper." In such a case it is enough that all the agents know Φ in ω in order that Φ is common knowledge in ω. Such models formalize ideas of Lewis [1969] and Clark and Marshall [1981] which try to eliminate infinite processes of verifying common knowledge.

This liberty in shaping the proposition set and the operators K_i distinguishes this model from the models of Kaneko [1987] and Gilboa [1986]. In these two papers a high power language is developed that enables the agents to speak freely about states, events, common knowledge and more. With such a language the agents are, at least partially, omniscient. In this paper the whole theory of common knowledge is neutral to the language structure and to the content (or the possible content) of the agents' knowledge.

Another feature of the theory of common knowledge here is the lack of any "logical" restriction on knowledge. Beyond (K1)-(K3) there is no required relation between knowledge and propositional calculus or other logical structure. Unlike modal and epistemic logic we do not require that agents have any deductive tools. Our agents do not necessarily know all tautologies. In short the whole theory is indifferent to logic. This conclusion is a bit surprising when we consider Bacharach's [1985] model in which epistemic operators $\bar{K}_i$ are applied to events in order to justify partitions. In his model there are requirements $\bar{K}1$, $\bar{K}2$, $\bar{K}3$ which correspond to our (K1), (K2) and (K3). But he has also an additional requirement that for each agent i and events $E_1, E_2, \ldots$.

$$(\bar{K}4)\ \bar{K}_i(E_i \cap E_2 \cap \ldots) = \bar{K}_iE_1 \cap \bar{K}_iE_2 \cap \ldots .$$

This is interpreted as saying that knowing a conjunction is equivalent to knowing each conjungant. Where did this requirement disappear in our model? To answer it we define in our model event operators $\bar{K}_i$. Agent i knows that even E happened in state ω if any possible state in ω is in E. Thus the event that "i knows event E," denoted by $\bar{K}_iE$ is simply the event $\{\omega | \text{for each } \omega' \text{ with } \omega' p_i \omega,\ \omega' \in E\}$. It is easy to see that $\bar{K}1$, $\bar{K}2$ and $\bar{K}3$ of Bachrack are satisfied due to (K1), (K2) and (K3). But $\bar{K}4$ also follows immediately by set theoretic considerations which has nothing to do with the relation of knowledge to conjunctions. If like Bacharach we start with events and identify each event with a proposition then we have to introduce conjunctions to take care of the intersection of events and hence $\bar{K}4$ is also required. When we start with propositions then events are defined by sets of propositions, conjunction is simply replaced by intersection and $\bar{K}4$ is automatically gained.

Updating and Learning. Aumann's "Agreeing to Disagree" has also dynamic variations in which agents interchange information until their posteriors become common knowledge at which point they must coincide (see, e.g., Geanakoplos and Polemarachakis [1982] and Bacharach [1985]). In these dynamic models knowledge increases in each step and as a result the partitions are refined. When one tries to apply such a procedure in our model one faces a difficulty. Suppose $\Omega \subset \Omega_3$ and i does not know Φ in ω, i.e., $\sim K_i\Phi \in \omega$. If i gains some new information and he knows Φ then the state of the world is no longer ω. Change in i's knowledge results in a change in the state of the world. Moreover, the partition of Ω cannot change at all. It depends on the relation between the states and cannot change as a result of the moving from one state to another. And worse, information cannot increase. Suppose we are now in a new state ω' where $K_i\Phi \in \omega'$. Alas by moving to ω', i lost some knowledge; he knew $\sim K_i\Phi$ in ω while of course he cannot know it in ω' since it is false there.

To solve this apparent paradox one has to introduce time into the model. A state of the world should be a description of the whole history of the world. In particular knowledge is now time dependent. Formally we have for each t, t = 1,2,..., epistemic operators $K_{i,t}$ which are interpreted as "i knows at time t that. . . ." Correspondingly we have for each period t and state ω epistemic contents $\bar{K}_{i,t}(\omega)$ and sets of possible states for i, $P_{i,t}(\omega)$. Properties (K1)-(K3) should be applied now to each $K_{i,t}$ and Ω_1, Ω_2 and Ω_3 are defined mutatis mutandis. We add now a new requirement for each agent i proposition Φ, time t, and state ω.

(K0) If $K_{i,t}\Phi \in \omega$ then $K_{i,t+1}\Phi \in \omega$.

This simply says that agents do not forget what they know. K0 guarantees that for each t, i, and ω, $K^{-}_{i,t+1}(\omega) \supset K^{-}_{i,t}(\omega)$, i.e., knowledge does not decrease. This happens even for $\Omega \subset \Omega_3$. If in ω, i does not know Φ at time t but does know it at t + 1 then i does not lose any knowledge at time t+1. He still may know at t+1 that he did not know Φ at time t, i.e., $\sim K_{i,t}\Phi$, $K_{i,t+1}\Phi$, $K_{i,t+1}K_{i,t+1}\Phi$ and $K_{t+1}\sim K_t\Phi$ can all be true in ω. As a result of the growing knowledge, information structure is refined in time, that is for each i, ω and t, $P_{i,t}(\omega) \supset p_{i,t+1}(\omega)$. In particular for $\Omega \subset \Omega_3$ this means that the partitions of the agents are refined.

Using this model for dynamic processes of information exchange enables careful examination of the conditions under which the exchange leads to or ends in common knowledge. Results analogous to those of Bacharach [1985] can be obtained now for any information structure.

Why Ω_2? We end the discussion with an argument based on bounded ability of the agents to process information. This argument supports rejection of (K3) while it enables acceptance of (K1). Suppose we have a measure of complexity on $\bar{\Phi}$, comp: $\bar{\Phi} \to R^+$ such that $\{\text{comp}(\Phi) | \Phi \in \bar{\Phi}\}$ is unbounded. We assume that for each i and Φ

(C1) $\text{comp}(K_i\Phi) \geq \text{comp}(\Phi)$

(C2) $\text{comp}(\sim\Phi) \geq \text{comp}(\Phi)$.

Assume furthermore that

(C3) $\text{comp}(K_iK_i\Phi) = \text{comp}(K_i\Phi)$.

(An extreme case of this is when $K_i K_i \Phi = K_i \Phi$). If knowledge in our model involves the ability to produce the known proposition or to use it in a deductive process then it is natural to assume that knowledge of an agent in a given state is bounded by complexity. Formally this means that for each i and ω there exists a bound $M_{i,\omega}$ such that for each $\Phi \in K_i^-(\omega)$, $\text{comp}(\Phi) \le M_{i,\omega}$. Under this assumptions Ω cannot satisfy (K3). Indeed for Φ with $\text{comp}(\Phi) \ge M_{i,\omega}$, $\text{comp}(\sim K_i \Phi) \ge \text{comp}(\Phi) \ge M_{i,\omega}$ by (C1) and (C2) and therefore $\sim K_i \Phi \notin K_i^-(\omega)$. On the other hand, by (C2), (K1) can be satisfied notwithstanding the bounded knowledge. Note that it is not the resemblance of (C3) to (K1) that gives (K1) an advantage over (K3). Indeed the previous result still holds, even if we add the assumption

(C4) $\text{comp}(K_i \sim K_i \Phi) = \text{comp}(\Phi)$

which corresponds to (K3) in the same way (C3) corresponds to (K1).

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