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STRONG FORM INFORMATIONAL EFFICIENCY IN STOCK MARKETS WITH DISEQUILIBRIUM TRADING

by

Lawrence R. Glosten

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Northwestern University
Graduate School of Management
Evanston, Illinois 60201

1. INTRODUCTION

Among the virtues that have been attributed to an efficiently operating financial market is the ability of such a market to incorporate the information of insiders in the equilibrium price. For example, common stocks have random future prices and dividends which to some extent reflect uncertainty in the real operations of the firm. Insiders, officers in the firm for example, may have better information about the future production capabilities and hence superior information regarding the future price and dividends of the stock. The strong form of the efficient markets hypothesis maintains that this information will be reflected in the price of the security through the market operations of the insiders[1]. This paper formulates a model of trade on the stock exchange to examine the question of when such superior information will be revealed through the price sequence generated by out-of-equilibrium trades.

The model considered is one of a stock market in which a specialist maintains a record of the arrival of buy and sell orders and matches up traders who can agree to trade. Briefly, traders send in orders consisting of a price and quantity to buy or sell. These arrive randomly at the specialist's post. When a buy order arrives with a bid price higher than an ask price already registered with the specialist (or vice versa) then the specialist arranges a trade between the two investors. The price at which the trade takes place is then made public. For simplicity, I assume that traders are divided into two groups -- the insiders, or informed, and the uninformed[2]. Within each group, expectations are homogenous. The sequence of prices becomes part of the data set of the uninformed as trade continues. The question addressed is, then, will this sequence of prices reveal the information of the insiders.

The analysis shows that if the informed group has a "large enough impact" on the trading, their information will be revealed. Furthermore, if the uninformed are risk neutral, then "large enough impact" is a necessary and sufficient condition for the sequence of prices to reveal the information of the informed. This result is consistent with that of Feiger (1978) and Figlewski (1978) obtained within a temporary equilibrium context. In the case in which the uninformed are risk averse, the analysis shows that the impact of the informed, measured by the mean order size per unit time, necessary to imply that the information will be revealed is smaller than in the case of risk neutral uninformed. Furthermore, in this case, the condition on impact is sufficient but not necessary for this information to be revealed. This tends to add further support to the conjecture made by Figlewski that the more risk averse investors are, the better the chances of informational efficiency.

2. TRADE PROCESS

The mathematical model of the trading process proposed here is an abstraction of the specialist type market found on the floor of the New York Stock Exchange. The active participation of the specialist in the trading of stocks (buying and selling on his own account) is ruled out, and to this extent, the model is applicable to markets in which the role of the specialist is played by a computer. Before introducing the formal model used in the analysis, I will present a verbal description of the workings of a specialist type market, focussing attention on the fundamental characteristics of the trading process. For simplicity, the variety of types of orders possible on the exchange is not discussed.

The specialist is situated at the area on the floor of the exchange where the particular stock is being traded. The "tool of his trade" consists of a ring binder in which orders to sell or buy are entered (hereafter referred to as the specialist's book). An order consists of a specification of the price at which the agent wishes to buy or sell and the quantity to be transacted, along with the name of the agent. One half of the book is devoted to buy (or bid) orders and the other half to sell (or ask) orders.

For example, suppose that currently the lowest ask price on the book is 49, the highest bid price is 48, and that a broker has just received a sell order of one unit at a price of 47. The broker will approach the specialist and ask for the spread (highest bid, lowest ask). In this case, the specialist will respond "48 to 49". Typically, there is a group of traders around the specialist, and in this case the broker with the sell order of 47 will announce "ask 48.5" (or some amount between 48 and 49). If one of the traders wishes to buy at 48.5 the deal will be consummated without the specialist; otherwise, the broker will say "ask 48" and the specialist will arrange a trade between the broker and the agent with the bid price of 48. That agent's name and his listed price are then removed from the book, and the trade at the price of 48 is announced. The case in which the broker arrives with a buy order of 49 or more is analogous. If the broker arrives with a bid less than 49 or an ask greater than 48, then after checking with the traders around the specialist, he will inform the specialist of the order, and the specialist will enter it in his book.

The model of trade used here assumes that all trades are handled by the specialist. That is, all orders go directly to the specialist, and are either transacted or are entered in the book. Thus, if a sell order arrives lower than the highest buying price, then the order is transacted at the highest buying price (i.e. the price already entered in the book).

STATE SPACE

Intuitively, the state of the market will be a summary of the specialist's book. This consists of two collections of points and a number associated with each point. One collection of points represents buy prices, the other represents sell prices, and the number associated with a price is the quantity to be transacted at that price. Formally, the most convenient way to represent this is with a pair of purely atomic measures; one measure representing buy prices, the other representing sell prices registered with the specialist.

Let E be the set of purely atomic finite measures on R. If $m \in E$ then m might be represented as:

$m(.)=\sum a_k \delta_{x_k}(.)$, $a_k > o$, $x_k \in R$ for all k.
where δ_x is the Dirac measure putting unit mass on the point x.

If the measure m shown above represents the buy orders registered with the specialist, then there are buy orders at prices $x_1, x_2, \ldots, x_n$, and a buy order (or buy orders) at price x_k is for (totals) an amount a_k. At any particular time, all buy orders in the book are less than all sell orders; otherwise a transaction would be made and an order erased.

If $m \in E$ let S_m indicate the set of atoms of m. Thus, in the above example, $S_m=\{x_1,x_2,x_3,\ldots x_n\}$. Let $F=\{(m^b,m^a) \in E \times E \ : \ x \in S_{m^b},\ y \in S_{m^a} \Rightarrow x < y\}$
Thus, F is the state space of the market process, and if $\mathcal{F}$ is a σ-algebra on F, then $(F,\mathcal{F})$ is the measurable space corresponding to it. That is, if $e \in F$ is the state of the market, then $e=(M^b,M^a)$ where M^b and M^a can be represented as:
$M^b(.)= \sum B^k \delta_{X^k}(.)$, $M^a(.)= \sum A^k \delta_{Y^k}(.)$
Thus, $\{X^1,X^2,\ldots \}$ is the set of buy orders, and orders at a price of X^k total B^k shares, $\{Y^1,Y^2,\ldots \}$ is the set of sell orders and orders at a price of Y^k total A^k shares. Furthermore, $X^k < Y^l$ for k and l. It will be useful later to define

the functions a: $F \to R$ and b: $F \to R$ where a(e) is the lowest asking price corresponding to the state e and b(e) is the highest buying price corresponding to e. If $e=(m^b, m^a)$ and $m^b=o(m^a=o)$ so that no buying (selling) orders are registered then

$$b(e) = -\infty \quad (a(e) = +\infty)$$

The above description of the state of the market is a slight generalization of that given in Garman ( ). The set of allowable prices can be either unrestricted (i.e., R) or restricted to some discrete set by putting further conditions on the state space. Furthermore, trades can be restricted to unit amounts, integral amounts, amounts in some discrete set or can be left restricted only to finite amounts.

MOTION OF THE SYSTEM

In the models that follow, it will be assumed that there are an infinite number of participating agents. Agents submit buy orders and sell orders by calling their brokers, who in turn report to the specialist. A natural assumption in this case is that the arrival of buy orders forms a Poisson process and the arrival of sell orders forms another independent Poisson process. This is justified by the usual arguments for arrival processes (Prussian soldiers getting kicked in the head by horses, bombs falling on London or, more prosaically, calls arriving at a central telephone exchange). The arrival of bid and ask orders represents the decisions of many agents acting independently, none of whom individually have a large impact on the trading. Furthermore, it may be assumed that there are no "schedule" effects. That is, orders arrive more or less uniformly throughout the time the market is in session. This neglects the fact that there may be a surge of orders at the beginning or end of the day, These conditions correspond roughly to the sufficient conditions for convergence to a Poisson process as the number of traders becomes large (For a more technical treatment see Cinlar (1972)).

In order to fix the idea of the motion of the market and to introduce the methodology used in the remainder of this paper, suppose that there is no informational updating on the part of investors, and that all orders are submitted for unit lots. It will be assumed that an investor drawn at random from the infinite population is characterized by a (vector valued) random variable C. defined on a probability space $(\Omega, \mathcal{A}, P)$, taking values in some finite dimensional space. This (random) vector of characteristics might include parameters of the investor's (von Neumann-Morgenstern) utility function, endowment and initial information. Without specifying the derivation of buy and sell prices, the selling price and buying price of an investor with a vector of characteristics c are given

by $F_a(c)$ and $F_b(c)$ respectively. In the absence of informational updating, and assuming stationarity of preferences and beliefs about the future, F_a and F_b might reasonably be assumed to be independent of time. Furthermore, even though trades are in general occurring over time, any finite number of trades will not affect the distribution of characteristics in the economy[3]. Therefore, if orders are randomly drawn, one will see a random sample of buying and selling prices. The distributions of buying and selling prices, P_b and P_a respectively, are induced by the distribution of characteristics in the market, the probability measure P and the functions F_a and F_b; i.e., define
$P_b(t)=P\{F_b(C)\leq t\}, P_a(t)=P\{F_a(C)\leq t\}$.
Thus, if an investor is drawn at random, his buying and selling prices will follow the distributions P_b and P_a respectively.

As noted above, it is assumed that buy prices and sell prices arrive according to two independent Poisson processes. This randomness reflects the time it takes for an order to reach a broker and the time it takes the broker to get to the specialist. By a well known result, the superposition of these two processes is a Poisson process with arrival rate equal to the sum of the arrival rates of the two independent Poisson processes (Cinlar (1975)). The market process is thus a pure jump process, and let $X(t)$ be the state at time t. The state stays constant until the arrival of either a bid or ask price (an arrival in the superimposed process), at which point there is a change in the state. If a buy order comes in higher than the lowest ask price, or a sell order comes in lower than the highest bid price, then a trade takes place. Otherwise, no trade takes place but an order is entered on the specialist's book. If the functions F_b and F_a are independent of time (stationarity of preferences and beliefs), and independent of the past (no informational updating), then the memorylessness of the exponential distribution implies that the resulting market process, $X(t)$, is Markovian.

To formalize the above, let $(N^b(T))$ and $(N^a(t))$ be, respectively, the arrival processes of buy orders and sell orders with respective arrival rates r^b and r^a. Then, if $N(t)$ defined by $N(t)=N^b(t) + N^a(t)$ is the superposition of $(N^b(t))$ and $(N^a(t))$, $(N(t))$ is Poisson with arrival rate $r=r^b+r^a$. Given any past history, the probability that the next arrival is a buy order is given by $q=r^b/r$. Given any past history, the probability that the next arrival is a selling price less than or equal to s is given by $(1-q)P_a(s)$. Suppose that the current state is $e=(m^b, m^a)$ and $a(e)=a, b(e)=b$. The next state will be $(m^b-\delta_b, m^a)$ if an asking price arrives less than b. It will be $(m^b, m^a-\delta_a)$ if a buying price arrives greater than a. The new state will be $(m^b+\delta_x, m^a)$ if a buying price arrives equal to x for $x<a$ and $(m^b, m^a+\delta_y)$ if a selling price arrives equal to y for $y>b$. This describes the motion both of the imbedded Markov chain and the (exponential) distribution of the arrival times. The above discussion of the characterization of the Markov market process X(t) is summarized in Proposition 1.

Proposition 1

Suppose that buy orders arrive according to a Poisson process with rate r^b and sell orders arrive according to a Poisson process with rate r^a and that these two processes are independent. Furthermore, suppose the characteristics of the agents in the market follow a distribution independent of time and the past states of the market, and let P_b and P_a be, respectively, the induced distributions of buy and sell prices. Then, the infinitesimal generator of the market process (X(t)) is given by:

$A(e,de') = -rI(e,e') + rQ(e,de')$ where

$$I(e,e')=\begin{cases}1 & e=e'\\ 0 & e\neq e'\end{cases}$$

and Q is defined, given $e=(m^b,m^a)$, $a(e)=a$ and $b(e)=b$, by

$$Q(e,de')=\begin{cases}P_a(b)(1-q) & e'=(m^b-\delta_b,m^a)\\ q(1-P_b(a)) & e'=(m^b,m^a-\delta_a)\\ qP_b(dx) & e'=(m^b+\delta_x,m^a)\\ (1-q)P_a(dy) & e'=(m^b,m^a+\delta_y)\end{cases}$$

Remark:

In principal, the infinitesimal generator can be used to calculate the transition probability P_t through the Chapman-Kolmogorov backward and forward equations:

$$\int A(e,de')P_t(e',B)=\frac{d}{dt}P_t(e,B)\ ;\quad e\in F,\ B\in\mathcal{F}$$

$$\int P_t(e,de')A(e',B)=\frac{d}{dt}P_t(e,B);\quad e\in F, B\in\mathcal{F}$$

and the invariant measure m*, if it exists, by:

$$\int m^*(de)A(e,B)=0;\quad B\in\mathcal{F}\quad \text{(Cinlar (1975))}^4.$$

The fact that F is a rather complicated space implies that these calculations would be difficult in practice.

The above points out the importance of the imbedded Markov chain and indeed most of the following analysis will deal with this discrete time process. The analysis will rely upon more specific assumptions regarding the derivation of buy and sell prices and the simplification of the motion of the market that will result.

Except for the stationarity of preferences and beliefs, and times between order arrivals, the above discussion makes no assumptions about the behavior of the agents and the type of uncertainty they face. The period of trading activity to be described might best be thought of as the Hicksian "week". During this week, there is no new exogenously supplied information or other changes in the exogenous environment. At the beginning of the week, insiders (the informed) receive information concerning the random "value" of the stock being traded.

There are two possible ways to look at this situation. The first is to suppose that a share of stock represents a claim to the income from some short term operation. At the end of the week, the firm will be liquidated and some random amount, V, per share will be dispersed to all share owners. Ignoring discounting then, if an agent holds s shares and w dollars of money his random wealth is $w + sV$. Information in this case consists of some information about the random variable V, say the realization of a random variable correlated with V, or some bounds on the value that V will take on.

Another possible view is to suppose that agents know that at the end of the week, an announcement will be made concerning the firm; stock split, new technology or news of bankruptcy proceedings are examples. They do not know exactly what this news will be and they conclude that the effect will be to lead to some random evaluation V of a unit of the stock; i.e., V will be the equilibrium price after the announcement. By obtaining some information about what the news will be, they obtain some information about the random variable V. In this case, an agent with w dollars and s units of stock views his random wealth to be $w + sV$ as before.

In general it will be assumed that investors will submit their true marginal evaluations of the stock given their information, as their bid and ask prices. That is, if an investor has a (von Neumann-Morgenstern) utility function for wealth U(.), an endowment of w dollars and s units of stock and information represented by a σ-algebra, $\mathcal{F}$

then his (random) bid and ask prices for one unit of stock are respectively:

$$\text{1a)}\sup\left\{b\in R: E(U(w-b+(s+1)V)\mid \mathcal{J}) \geq E(U(w+sV)\mid \mathcal{J})\right\}$$

$$\text{1b)}\inf\left\{a\in R: E(U(w+a+(s-1)V)\mid \mathcal{J}) \geq E(U(w+sV)\mid \mathcal{J})\right\}$$

This assumption is in part justified by the results concerning Vickrey auctions (Vickrey (1961)). In a Vickrey auction, the object being auctioned is sold to the highest bidder at a price submitted by the second highest bidder. In the absence of collusion, then, there is a separation between the individual's bid and what he will have to pay. The following discussion shows that a similar, but not complete, separation exists in the model of stock market trading proposed here. As noted in the description of the trading process, when a trade takes place, it occurs at the price already on the book. In the absence of knowledge of the specialist's book, if an agent could agree to trade with someone on the book it would be in his best interest to submit his true evaluation. To see this, note that the payoff to submitting a bid b is

$$V(w-A,s+1)I_{\{b\geq A\}} + V(w,s)I_{\{b<A\}}$$

where $v(w,b) = E(u(w+sv)\mid \mathcal{J})$, A is the lowest ask price (in the absence of knowledge of the specialist's book it is a random variable) and I_B is the indicator function of the event B. Maximization of the expected payoff implies that the optimal bid is the solution to 1a). The same argument holds for the asking price. The problem arises, of course, out of the fact that no trade may occur immediately, in which case the agents bid or ask price will be entered in the book. At some later date, he may end up trading at this price, and it might have been to his advantage to submit a somewhat higher ask or lower bid than his true evaluation. However, if agents are myopic, the truthful strategy is optimal. A more compelling reason for the above assumption lies in its simplicity and the nature of the results which follow: the fact that efficiency is difficult to obtain even if agents do not try to obscure their information indicates that it may be even more difficult if agents follow some more complicated strategy.

3. RISK NEUTRAL AGENTS

The first model to be considered consists of two groups of risk neutral traders. The first group, the informed, or insiders, have some information about the random variable V. If V is defined on some probability space $(\Omega,\mathcal{A},P)$ then the information these insiders have is represented by $\mathcal{J}$, a sub-σ-algebra of $\mathcal{A}$. According to the above discussion of the bid and ask prices, the informed bid and ask prices are equal to Z, the conditional expected value of V given their information. Then z is defined by $z=E(V|\mathcal{I})$ (For the remainder of this section, consider the realization of the information to be fixed and put $z(w) = E(V|\mathcal{I})(w) = m_I$ for $w \in \Omega$.

Initially, the uninformed bid and ask prices are equal to $m_o=E(V)$. It will be assumed in this model that two informed will not trade together nor will two uninformed. Trade will occur if and only if one person is made strictly better off and the other is not hurt by the trade given their information.

Initially, trade will take place at either m_o or m_I. If $m_I > m_o$ ($m_I < m_o$) informed will buy from (sell to) the uninformed. Furthermore, if trade occurs at m_o then the informed can gain no information from the price. In the absence of quantity information, the uninformed cannot know whether trade is occurring or not since the price is unchanged. Hence, they cannot tell whether or not there is new information in the market.

As soon as a trade takes place at m_I, the uninformed simultaneously learn that new information is available, and, due to the assumption of risk neutrality, as much as they need to know about this information. The first statement is self-evident, the second is shown by the following:

$$\text{for } w \in \Omega, \; E(V \mid Z)(w)=E(E(V \mid \mathcal{I}) \mid Z)(w)=E(Z \mid Z)(w)=Z(w)=m_I.$$

Thus, after one trade takes place at m_I all agents will agree on the conditional expected value of V, namely m_I. The analysis now moves to the consideration of when this jump in the price will occur.

For concreteness (and without loss of generality) suppose that $m_I > m_o$ i.e., informed initially buy from uninformed. As long as there is a surplus of sell orders from uninformed on the specialist's book, trade will continue to occur at m_o, the uninformative price. The model of the motion of the market is somewhat expanded by assuming that uninformed buy and sell orders arrive according to independent Poisson processes with respective rates r_u^b and r_u^a, informed buy and sell orders arrive according to independent Poisson processes with rates r_I^b and r_I^a, respectively (these rates may depend upon the realization of information). The quantity specified with each uninformed buy and sell order and each informed buy and sell order are independent random variables. These are assumed to be independent of the arrival processes and to follow, respectively, the distributions Ψ_u^b, Ψ_u^a, Ψ_I^b, Ψ_I^a with respective means n_u^b, n_u^a, n_I^b, n_I^a . Define S(t) to be the amount of unfilled sell orders from the uninformed on the specialist's book at time t. If $S(t) > 0$ then at time t there are sell orders at a price of m_o amounting to S(t) shares. If the next arrival is an informed buy order for Y shares, then immediately after this arrival there will be S(t)-Y unfilled uninformed sell orders as long as $Y \leq S(t)$. If $Y > S(t)$ then some portion of the arriving buy order will be left unfilled namely $Y-S_t$ shares. Thus, $S(t) < o$ implies there are unfilled informed buy orders amounting to -S(t) shares. As soon as S(t) becomes negative, the next arrival of an uninformed sell price will lead to a trade at a price of m_I and the information will be revealed.

To describe this time probabilistically, define $T_1 = \inf\{t : S(t) < 0\}$ $T_2 = \inf_{t > T_1}\{t : S(t+) - S(t-) > 0\} - T_1$, and $T^* = T_1 + T_2$. Thus, T^* is the time at which the price m_1 is announced and information is revealed. Notice that T_1 and T_2 and hence T^* are arrival times of the superimposed Poisson arrival process and hence if the arrival times of this superimposed process are denoted (T^k), then $T_1 = T^N$ where N is the (random) time at which $(\hat{S}(n))$, the imbedded discrete time process, first goes negative. Furthermore, since T^k is an arrival time of a Poisson process, $T^k < \infty$ a.s. and $T^k \to \infty$ a.s. as $k \to \infty$ The events, $\{N < \infty\}$ and $\{T_1 < \infty\}$ are equivalent, and since T_2 is exponentially distributed, $P\{N < \infty\}$ and $P\{T^* < \infty\}$ are equal.

Now $\hat{S}(n)$ can be represented as the sum of S_o and n i.i.d. random variables, i.e., $S(n) = S_o + \sum Y_m$ where $Y_m(w) = 0$ if for the outcome $w \in \Omega$ an uninformed buy price or an informed sell price arrives at step m, and $Y_m(w) = y$ if the for the outcome $w \in \Omega$

1) an uninformed sell order arrives for y units $(y \geq o)$

2) an informed buy order arrives for y units $(y < o)$

Thus, $(\hat{S}(n))$ is a generalized random walk with

$$E(Y_m) = r_u^a/(r_u^a + r_u^b + r_I^a + r_I^b) \int y \Psi_u^a(dy) - r_I^b/(r_u^a + r_u^b + r_I^a + r_I^b) \int y \Psi_I^b(dy) = q_u^a n_u^a - q_I^b n_I^b = \theta .$$

Now, if $S_0 < 0$, then the first trade will reveal the information, and $T^* < \infty$ a.s. If $S_0 \geq 0$ and $\theta \leq 0$ then standard results in the theory of random walks show that $T^* < \infty$ a.s. (Feller (1966)) Finally, if $S_0 \geq 0$ and $\theta > 0$ the results of random walk theory show that there is a positive probability that $\hat{S}(n)$ and hence $S(t)$ will stay above zero indefinitely. In this case the price will stay at m_0 and the information will never be revealed. Note that $\theta > 0$ if and only if $r_u^a n_u^a > r_1^b n_1^b$, i.e., the mean sell order size of the uninformed times their arrival rate (the uninformed mean sell order size per unit of time) is greater than the informed mean buy order sizetimes the arrival rate of informed buy orders. Furthermore, the case in which $m_I < m_0$ is completely symmetric. The above arguments establish the following result.

<u>Theorem 1.</u>

Suppose $S_0 \geq 0$ then,

i) if $m_I > m_0$ and $r_u^a n_u^a > r_I^b n_I^b$ then there is a positive probability that the information will not be revealed.

ii) if $m_I > m_0$ and $r_u^a n_u^a \leq r_I^b n_I^b$ then the information will be revealed in an a.s. finite amount of time.

iii) if $m_I < m_0$ and $r_u^b n_u^b > r_I^a n_I^a$ then there is a positive probability that the information will not be revealed.

iv) if $m_I < m_0$ and $r_u^b n_u^b \leq r_I^a n_I^a$ then the information will be revealed a.s.

Intuitively,the above results show that the informed agents must have a large enough impact on the market in order that their information be revealed. This impact is measured both through their arrival rate and the mean size of orders. This suggests that if the group of insiders is small relative to the rest of the market, and credit limitations place a constraint on the size of each insider's trade, then these insiders may trade to their advantage indefinitely by not affecting the price at which trade occurs.

Clearly, the specifics of the above model derive from the assumption of risk neutrality. The next two models show that risk aversion on the informed side decreases the possibility of efficiency while risk aversion on the part of the uninformed weakens the conditions for informational efficiency. This suggests that the issue of informational efficiency revolves about domination of the market. It will be seen that risk aversion on the informed side decreases the aggregate impact of informed traders while risk aversion on the uninformed side weakens the aggregate uninformed impact and thus increases the relative impact of informed traders.

4. RISK AVERSE UNINFORMED TRADERS

In this model, it is assumed that while the informed investors are risk neutral, the uninformed are risk averse. This may be a reasonable scenario if the informed are more likely to be the larger institutional investors or the more wealthy. These groups are more likely to act as if they were risk neutral, while the uninformed, the rest of the market, might be thought to act as if they were risk averse. Alternatively, it could be assumed that $\mathcal{I}=\sigma(v)$, i.e., the informed know V exactly.

As in the specification of the motion of the market, the distributions of the uninformed buy and sell prices in the absence of market information are respectively P_b and P_a. If information is unanticipated then as soon as a trade takes place twice at the informed price, the conditional mean will be revealed. It cannot be assumed that the uninformed know which prices are due to which traders, but if characteristics are continuously distributed then the appearance of the same price twice indicates that this price is significant. It must be the conditional mean of the informed since the probability that two uninfo r med prices will be the same is zero. To a certain extent, this is taking the assumption of a continuous price set a little too seriously. The same intuition would hold if the price set were not continuous, however. With a discrete price set, one may see the same price several times, but as long as trades take place at the informed mean, in the long run, more trades will take place at that price than would be expected if there were no informationin the system.

Note that under the assumption that the informed are risk neutral, the uninformed can learn no more than the conditional mean; they cannot in general learn the entire conditional distribution. However, if the information consists of the observation of a random variable X then the monotone class theorem (Neveu(1965) implies that the conditional expectation, Z, is given by $Z=G(X)=E(V|X)$. If $x \rightarrow G(x)$ is a one to one function then knowledge of the conditional mean is equivalent to knowledge of the entire conditional distribution since for any function H, $E(H(V)|G(X))= E(H(V)|X)$. As in the risk neutral case, if information is unanticipated, then no information can be revealed until trades take place at the informed conditional mean. Thus, until the second trade at the informed price, the distributions of uninformed buying and selling prices remain respectively P_b and P_a. At that time the information is revealed and the distributions change to P_b^x and P_a^x if x is the realization of the information. Unlike the risk neutral case, trade will in general continue due to the random effects of endowments.

Suppose as above (and without loss of generality) that the realization of information is "good news" i.e. $E(V|\mathcal{J})(w) > m_0$. To simplify the analysis, it will be assumed in this case that the informed only submit buy orders which arrive with rate r_I^b. Furthermore, assume that all orders are for unit lots. As was exhibited in the first model, some further implications may be drawn from considering arbitrary order sizes, but the fundamental results do not depend upon this generality.

Analogously to the above model, define $\hat{S}(n)$ to be the number of uninformed sell orders less than the informed buy price, i.e., the conditional mean m_I. With $q_I^b=r_I^b/(r_u^a+r_u^b+r_I^b)$, $q_u^a=r_u^a/(r_u^a+r_u^b+r_I^b)$, $q_u^b=1-q_u^a-q_I^b$, $X(n)$ the state of the market after the nth jump, $(\mathcal{H}_n)$ the history generated by $(X(n))$, and $a(X(n))$ and $b(X(n))$ respectively the lowest selling price and highest buying price at the nth successive state,(by definition, $a(X(n))$ must be an informed selling price), the distribution of the change in $S(n)$ is given by:

$$P\left\{\hat{S}(n+1)-\hat{S}(n)=k \mid \mathcal{H}_n\right\} = \begin{cases} q_u^a(P_a(m_I)-P_a(b(X(n)))) & k=1 \\ q_I^b+q_u^b(1-P_b(a(X(n)))) & k=-1 \\ q_u^a(1-P_a(m_I))+q_u^aP_a(b(X(n)))+q_u^bP_b(a(X(n))) & k=0 \end{cases}$$

Define $Y(n+1)=\hat{S}(n+1)-\hat{S}(n)$, then $S(n)=S_0+\sum Y(m)$ and suppose that $S_0 > 0$. Define a sequence of i.i.d. random variables $(Y^*(m))$ on the same probability space with the following distribution:

$$P\left\{Y^*(n)=k\right\} = \begin{cases} q_u^aP_a(m_I) & k=1 \\ q_I^b & k=-1 \\ q_u^b+q_u^a(1-P_a(m_I)) & k=0 \end{cases} .$$

Comparison of the distribution of $Y^*(n)$ with the distribution of $Y(n)$ shows that $Y^*(n)$ behaves "as badly" as $Y(n)$ in the following sense. It is more likely that $Y^*(n)$ be +1 than $Y(n)$ and it is less likely that $Y^*(n)$ be -1 than $Y(n)$. Thus, the sequence $(Y^*(n))$ provides a standard of comparison since $(Y^*(n))$ is an i.i.d. family of random variables. Note that $P\{Y^*(n)=1\} \geq P\{Y(n)=1 \mid \mathcal{H}_n\}$ for all histories and hence $P\{Y^*(n)=1\} \geq P\{Y(n)=1\}$ Furthermore, $P\{Y^*(n)=-1\} \leq P\{Y(n)=-1 \mid \mathcal{H}_n\}$ for all histories and hence $P\{Y^*(n)=-1\} \leq P\{Y(n)=-1\}$. Define $S^*(n)=S_0+\sum Y^*(m)$ and note that as discussed above, if $q_I^b \geq q_u^a P_a(m_I)$, $S^*(n)$ will go negative in an a.s. finite amount of time. However, if $(S^*(n))$ goes negative in an a.s. finite amount of time, so must $(\hat{S}(n))$ since $P\{Y(n)=1\} \leq P\{Y^*(n)=1\} \leq P\{Y^*(n)=-1\} \leq P\{Y(n)=-1\}$. But, if $(\hat{S}(n))$ goes negative in an a.s. finite amount of time, then one trade will take place at the informed price, and hence two trades will take place at the informed price in an a.s. finite amount of time. Thus, the informed conditional mean will be revealed to the uninformed. As in the first model, these results are symmetric for the case in which the informed are selling. Hence, the above proves the following result.

Theorem 2.

If the informed are risk neutral and the uninformed are risk averse, then
i) if the informed are buying and $r_I^b \geq r_u^a P_a(m_I)$ then the information will be revealed in an a.s. finite amount of time.
ii) if the informed are selling and $r_I^a \geq r_u^b(1-P_b(m_I))$ then the information will be revealed in an a.s. finite amount of time.

There are two important characteristics of these results. First, the presence of risk aversion on the part of the uninformed leads to sufficient conditions for efficiency that are somewhat weaker than in the risk neutral case. The fact that uninformed will now in general trade with one another, diminishes their aggregate impact on the market, and raises the relative market strength of the informed. Thus, it is much easier to obtain a situation in which the informed agents set the price and thus reveal their information. The second important point is that the results are somewhat weaker than those in the previous section in that only sufficient conditions are given for a.s. informational efficiency. The question of necessity in this case requires a further analysis of the motion of prices and will be reserved for a later paper.

To verify that indeed risk aversion is responsible for the above results, the last model presents a market in which the informed are risk averse and the uninformed are risk neutral. As a partial justification for this point of view, it might be thought that risk averse investors would be more likely to expend resources (time, energy, money) to become informed. The results of this model are first that only asymptotic efficiency can be expected (in the long run the information may be revealed) and that even in this case, the conditions on arrival rates are more stringent than in the risk neutral case.

5. RISK AVERSE INFORMED, RISK NEUTRAL UNINFORMED

In this model, it is assumed that the information received by the informed is represented by a random variable S. In this case, the informed buying and selling prices are functions of the characteristics, C and the realization of information, s. It is assumed that for a fixed realization of C, say c, the buying price as a function of the information, $s \to F_b(s,c)$ is one to one and the selling price function $s \to F_a(s,c)$ is one to one. Of course, when an uninformed agents seees a price in the market it is a function of characteristics and information, and from this he may try to infer the information. But, in general, the characteristics are not known, and hence any price realization does not reveal exactly the information upon which it was based.

Trades will initially occur either at an informed price or at m_0. As long as trades occur at m_0, no information can be revealed for it is not assumed that the uninformed know that information is in the market, and hence that trades are occurring. As above, assume unit trades and assume that uninformed buy and sell prices arrive according to independent Poisson processes with respective rates r_u^b and r_u^a and informed buy and sell prices arrive according to independent Poisson processes with respective rates $r_I^b(s)$ and $r_I^a(s)$ for the realization of information, s. Let P_a^s and P_b^s be respectively the distribution of informed ask and bid prices and define $v(s)=r_I^a(s)+r_I^b(s)+r_u^a+r_u^b$, $q_i^j(s)=r_i^j(s)/v(s)$, $i=I,u$ $j=a,b$. Analogously to the above models, let $\hat{N}^b(n)$ and $\hat{N}^a(n)$ be the numbers of uninformed buy buy prices and uninformed sell prices respectively in succesive state n. Then, if $(\mathcal{H}_n)$ is the history generated by X(n), on the event $\{\hat{N}^a(n) \geq 1, \hat{N}^b(n) \geq 1\}$, the distributions of changes of $N^b(n)$ and $N^a(n)$ are given by:

$$P\{\hat{N}^b(n+1)-\hat{N}^b(n)=k \mid \mathcal{H}_n\}(w) = \begin{cases} q_u^b(s) & k=1 \\ q_I^a(s)P_a^s(m_o) & k=-1 \\ q_u^a(s)+q_I^b(s)+q_I^a(s)(1-P_a^s(m_o) & k=0 \end{cases}$$

$$P\{\hat{N}^a(n+1)-\hat{N}^a(n)=k \mid \mathcal{H}_n\}(w) = \begin{cases} q_u^a(s) & k=1 \\ q_I^a(s)P_a^s(m_o) & k=-1 \\ q_u^b(s)+q_I^a(s)+q_I^b(s)P_b^s(m_o) & k=0 \end{cases}$$

Theorem 3.

Suppose $\hat{N}^b(0) \geq 1$, $\hat{N}^a(0) \geq 1$ and suppose that for the realization of information s, $q_u^b(s) > q_I^a(s)P_a^s(m_o)$ and $q_u^a(s) > q_I^b(s)(1-P_b^s(m_o))$, then there is a positive probability that no information will be revealed.

Proof

From the random walk arguments already discussed, if the "up jump" probabilities are greater than the "down jump" probabilities there is a positive probability that $N^b(n)$ and $N^a(n)$ will stay positive. Thus, there is a positive probability that all trades will take place at m_o and hence no information can be revealed.

The strength of this theorem lies in the fact that there is a positive probability that the uninformed will receive no information whatsoever, if the conditions on arrival rates are satisfied. Furthermore, these conditions are somewhat weaker than in the risk neutral case. Thus, retaining risk neutrality on the uninformed side and assuming risk aversion on the informed side decreases the chances of achieving informational efficiency.

The question of what information will the uninformed obtain if $q_u^b(s) \leq q_I^a(s)P_a^s(m_o)$ or $q_u^a(s) \leq q_I^b(s)(1-P_b^s(m_o))$ still remains. As discussed above, they cannot infer the value of s from one price observation since characteristics are not known. In effect, one would expect that an infinite number of informed price observations would "average out" the uncertainty of characteristics, and thus reveal the information. Certainly it is true that any finite number of observations will not reveal the information exactly. Let $(P(t))$ be the price process generated by the market process, and let $\mathcal{J}_t = \sigma(P(s); s \leq t)$, i.e., the information available to the uninformed after t units of time. Define $M(t) = E(V \mid \mathcal{J}_t)$, i.e., the uninformed updated mean at time t. It is not unreasonable to suppose that, due to homogeneous information among the uninformed, $r_u^a = r_u^b = r_u$. Furthermore, if the news that the informed receive is favorable (unfavorable) it is not unreasonable to suppose that $r_I^b(s) > r_u > r_I^a(s)$ $(r_I^a(s) > r_u > r_I^b(s))$. In fact, in the following this is assumed and, recognizing that all arguments will be symmetric, without loss of generality we may assume that the realization of the information, s, is favorable and that $r_I^b(s) > r_u > r_I^a(s)$. In this case, if $q_I^b(s)(1-P_b^s(m_o)) \geq q_u(s)$ (where $q_u(s) = r_u/v(s)$). The number of uninformed asking prices on the book will reach zero a.s. for all conditional means. Thus, almost surely one trade will take place at an informed buying price. Assume that this is the case and define $\bar{p}(s)$ to be the solution to $r_u = r_I^b(s)(1-P_b^s(p))$ and note that $\bar{p}(s) > m_o$.

If at any time t, $M(t) > \bar{p}(s)$ then there is a positive probability that all future trades will take place at $M(t)$ and hence no further revision can take place. Without making further distributional assumptions it is impossible to say whether $M(t)$ will stay less than $\bar{p}(s)$ for all s. It is possible to construct examples where $M(t)$ must eventually exceed $\bar{p}(s)$, for all s, but I have found no examples which indicate conditions that will guarantee that $M(t)$ stays less than $\bar{p}(s)$[5]. In order to complete this analysis, however, suppose that the arrival rate of informed buy orders is large enough to guarantee that $M(t)$ will almost surely be bounded away from $\bar{p}(s)$. That is, a.s., an infinite number of trades will occur at informed buy prices.

The price $\bar{p}(s)$ has an interesting interpretation if $M(t)$ stays less than $\bar{p}(s)$. Under the assumption that trade is limited to unit lots, and the informed enter the market only as buyers, the expected excess demand per unit time at the price p is $r_I^b(s)(1-P_b^s(p))-r_u$ if $p > M(t)$.

Thus, $\overline{p}(s)$ is the price at which expected excess demand per unit time is zero.

Suppose for simplicity that $r_I^b(s)$ and $r_I^a(s)$ are independent of s as long as s represents favorable news and that $r_I^b(s)=r_I$ and $r_I^a(s)=0$. Thus, $\overline{p}(s)$ is in fact the price at which expected excess demand per unit time is zero. Furthermore, the times between price changes can convey no information, and hence no information is lost be merely considering the successive prices. In view of this, $\overline{p}(s)$ is the solution to $(r_u/r_I)=1-P_b^s(p)$. The assumption that $s \rightarrow F_b(s,c)$ is one to one for fixed c implies that $\overline{p}(s)$ is a one to one function of s. Assume that a.s. $M(t) \leq \overline{p}(s)-d$ for some $d>0$. Define $N^e(t)$ to be the number of informed buy orders greater than $\overline{p}(s)+e$ at time t and $N_e(t)$ to be the number of informed buy orders greater than $\overline{p}(s)-e$ for $0<e<d$. The next lemma shows that $(N_e(t))$ will hit zero only finitely many times with probability one while $(N^e(t))$ will enter zero infinitely often. In the words of the theory of Markov processes $(N_e(t))$ is transient while $(N^e(t))$ is recurrent (Cinlar 1975,p.125).

Lemma 1

$(N_e(t))$ is a transient Markov process with state space $\{0,1,\ldots\}$ and $(N^e(t))$ is a recurrent Markov process with state space $\{0,1,\ldots\}$.

Proof

It is clear that the assumption of independent arrivals implies that the two processes are Markovian. By definition of $\overline{p}(s)$ the "up jump" probability for $(N_e(t))$ is greater than the "down jump" probability and hence it is transient. Similarly, the "up jump" probability for $(N^e(t))$ is less than the "down jump" probability and hence it is recurrent.

Define $P(u,t)=\inf\{P(s);\ u \leq s \leq t\}$ for $u \leq t$ and note that $P(u,t)$ is nonincreasing int, and a.s. bounded below. Thus, $\lim_t P(u,t)=P^*(u)$ exists a.s. Furthermore, $P^*(u)$ is non-decreasing in u and hence $\lim_u P^*(u)=\overline{P}$ exists a.s. in the finite or infinite sense. The following lemmata and theorem show that $P(u,t)$ converges a.s. to $\overline{p}(s)$ and establish that $M(t)$ converges to $E(V|S)$ a.s.

Lemma 2.

With $P(u,t)$ and $\bar{p}(s)$ as above, $\lim_u \lim_t P\{P(u,t) > \bar{p}(s)+e\} = 0, e > 0.$

Proof

Note that since $P(u,t)$ is non-increasing in t and non-decreasing in u,

$$\lim_u \lim_t P\{P(u,t) > \bar{p}(s)+e\} = \lim_u P\{P(t) > \bar{p}(s)+e,\ t \geq u\} = P\Big(\bigcup_{u \geq 0} \bigcap_{t \geq u} \{P(t) > \bar{p}(s)+e\}\Big)$$

$$= P\{P(t) > \bar{p}(s)+e \text{ for all but a finite amount of time}\}.$$

Since $(N^e(t))$ is recurrent, it must hit zero infinitely often a.s. In order for $P(t)$ to be greater than $\bar{p}(s)+e$ for all but finitely many intervals of time it must be the case that for all but finitely many times after $(N^e(t))$ hits zero an informed buy price greater than $\bar{p}(s)+e$ must arrive before an arrival of an uninformed selling price. But these arrivals are Bernoulli events due to the stationarity of the arrival process and hence such happens with probability zero. Thus, $P(t)$ cannot be greater than $\bar{p}(s)+e$ for all but a finite amount of time.

Lemma 3

With $P(u,t)$ and $\bar{p}(s)$ as above, $\lim_u \lim_t P\{P(u,t) < \bar{p}(s)-e\} = 0,\ e > 0.$

Proof

If $P(u,t) < \bar{p}(s)-e$ it must be that $P(v) < \bar{p}(s)-e$ for some interval of time between u and t. Then it must be the case that $N_e(v)=0$ for some interval of time between u and t. Taking limits,

$\lim_u \lim_t P\{P(u,t) < \bar{p}(s)-e\} \leq P\{N_e(v)=0 \text{ on infinitely many intervals}\}.$

However, $(N_e(t))$ is transient and hence hits zero only finitely many times a.s. Thus, $\lim_u \lim_t P\{P(u,t) < \bar{p}(s)-e\} = 0.$

Theorem 4.

With $M(t)=E(V \mid \mathcal{F}_t)$, $M(t) \to E(V|S)$ a.s.

Proof

The above two lemmata show that $\lim_u \lim_t P\{|P(u,t)-\bar{p}(s)| > e\} = 0$, $0 < e < d$. That is, $P(u,t)$ goes in probability to $\bar{p}(s)$. Furthermore, $\lim_t P(u,t)=P^*(u)$ exists and $\lim_u P^*(u)$ exists in the finite or infinite sense.

Thus, $\lim_{u} \lim_{t} P(u,t)=\overline{p}(s)$ a.s. Now, by definition of $\mathcal{J}_t$, $P(u,t)$ is $\mathcal{J}_t$ measurable and hence $\mathcal{J}_\infty=\sigma(\{P(\cdot\ t)\})$ measurable. Now, $M(t)$ is a square integrable martingale, and thus converges to $E(V \mid \mathcal{J}_\infty)$. The random variable V is independent of the distribution of endowments and risk preferences and is stochastically dependent on arrival rates only through S. Thus, since $\overline{p}(s)$ is $\mathcal{J}_\infty$ measurable, $E(V \mid \mathcal{J}_\infty)=E(V \mid S)$.

The above arguments show that if the informed arrive sufficiently fast, and if the news they receive is important enough to justify the assumption that informed only enter the market to buy, the sequence of prices generated in this market will reveal the information of the informed in the long run. The three lemmata show that if $(P(t))$ is the price sequence then $\lim\inf P(t)=\overline{p}(s)$ a.s. This is a function of observable random variables and since $\overline{p}(s)$ is one to one, the price sequence will reveal s to the uninformed.

Remark

As noted before, the above argument is symmetric with respect to whether the news the informed received was good or bad. For the case of bad news, one would define $\underline{p}(s)$ to be the solution to $P_a^s(p)=r_u/r_I^a$, assume $M(t) \geq \underline{p}(s)-d$ a.s. for some $d>0$, define $P(u,t)=\sup\{P(v);\ u \leq v \leq t\}$ and show that $\lim_{u} \lim_{t} P(u,t)$ exists and is equal to $\underline{p}(s)$. Thus, the same results would hold.

To summarize the results of this model, retaining risk neutrality on the uninformed side and assuming risk aversion on the informed side decreases the likelihood of informational efficiency, and in fact in some instances may prevent it completely no matter what arrival rates are. In this case, risk aversion on the informed side reduces their market impact and thus the uninformed make the price. If arrival rates of the informed are high enough at least some information will be transferred, but in general one cannot expect complete agreement between the informed and uninformed

6. CONCLUSION

The principal implication of the above results is that recognizing disequilibrium trading implies that the informed must have a large enough impact on the trading in order that their information be revealed. This impact is measured not only by the size of the orders arriving from the informed, but on the "urgency" with which orders are placed. That is, the mean order quantity per unit time must be sufficiently high to guarantee that information is to be revealed. Whether or not information is revealed appears to be quite sensitive to the amount and distribution of risk aversion among the investing population. Risk aversion among the uninformed tends to raise the relative impact of the informed by creating a more closely competitive market, while risk aversion on the informed side reduces the impact of the information.

An obvious omission of this report is the case in which all investors are risk averse. The question is, which effect will dominate, the reduction of informed impact or the reduction of uninformed impact through a more competitive market. Introductory results indicate that the latter effect is more important as it appears that risk aversion may lead to assymptotic efficiency. Reporting of results along this line must wait, however, as further properties of the price process must be deduced.

ENDNOTES

[1] The standard reference for informational efficiency is Fama's survey paper, "Efficient Capital Markets: A Review of Theory and Empirical Work" (1970). Another exposition of the theory is given in Rubenstein's paper "Securities Market Eficiency in an Arrow-Debreu Economy."

[2] The extension to several types of information structures is not straightforward and I deal with this issue in my dissertation. Different assumptions about the structure of information must be examined case by case. The results are of the same nature as those presented here, however.

[3] That this is true is a mathematically fortunate but a theoretically unfortunate aspect of assuming an infinite population. The resulting simplification of the market stochastic process is enormous, but it is precisely the transfer of wealth from the uninformed to the informed that could lead to an increased probability of informational efficiency.

[4] Preliminary work on the market process without informational updating indicates that an invariant measure will not exist. Initial results suggest that there will be a piling up of untransacted orders on each side of the book. The price process, however, may have some stability properties.

[5] The conditional mean $M(t)$ must eventually exceed $\overline{p}(s)$ if, for example, borrowing restrictions constrain the distribution of buying prices so that $\overline{p}(s)$ lies below $E(V|S)(w)$. If $M(t)$ were to stay below $\overline{p}(s)$, then the results to be shown imply that $M(t)$ converges to $E(V|S)(w) > \overline{p}(s)$, which is a contradiction.

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