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PRICE ADJUSTMENT SPEED AND DYNAMIC DUOPOLISTIC COMPETITORS

by

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Introduction

Some sixty years ago, G. C. Evans (1924) proposed a dynamic model of monopoly in which a single seller sought to maximize profit over an interval of time and in which demand was a function of current price as well as its first derivative with respect to time. The relationship between the quantity demanded at each instant of time and the product's current price and its first time derivative was supposed to be linear while the cost function was assumed to be quadratic with respect to the quantity produced. There was no discounting of future profits in this formulation. After some rather complicated calculations, in order to avoid explicit use of the calculus of variations which he regarded as a rather exotic methodology, he found that the particular solution to the derived second order linear differential equation in prices was exactly the static "Cournot monopoly price" when the time derivative of price was absent from the demand function. Moreover, he observed that this was the only asymptotically stable solution as the time horizon was extended to infinity. Thus, he had found a dynamic formulation of the monopoly model that yielded the static monopoly price as its stationary state.

Shortly thereafter his colleague Roos (1925) extended Evans' model to the case of duopoly under the same assumptions regarding the demand function, the cost function, and the discount rate. The duopolists were supposed to be identical and to behave according to the Cournot zero conjectural variation assumptions. Roos found too that the symmetric stationary solution to this problem yielded exactly the same price as the counterpart static Cournot price when the time derivative of price did not appear in the demand function. A further generalization of the Roos model in which non-zero conjectural variations were allowed was conducted by Smithies and Savage (1940).

The interesting feature of the Evans-Roos formulation, of course, is that it yields the static Cournot solution as its stationary state. For the stationary-state of a dynamic model describes the persistent characteristics of the economic situation being studied rather than its transitory ones. Now a major justification for the employment of static models of analysis is precisely that they focus on the persistent features of the economic situaion under study and ignore the transitory ones. Indeed, it is implicit in static analysis that transitions are regarded as taking place instantaneously. Thus, it is an important achievement when it can be shown, as in the Evans-Roos model, that the stationary state of a dynamic model coincides with the static analysis of the same economic situation.

The Evans-Roos model was formulated and analyzed long before the development of optimal control theory, dynamic programming, game theory, and the theory of differential games. Analysis of dynamic oligopoly models employing the theory of differential games have been conducted since then--see Simaan and Takayama (1978), Case (1979), and the references cited by him as well as the references provided in the survey papers by Clemhout and Wan (1979), Feichtenger and Jorgensen (1983), and Reynolds (1984). Fudenberg and Tirole (1983) provide a review of recent works on dynamic models of oligopoly involving the use of supergames and games of incomplete information as well as differential games.

In this paper we analyze a dynamic duopoly model similar to the Evans-Roos model. The analysis is carried out by employing a differential game framework that allows us to capture the dynamic structure of the game. A main emphasis of the paper is to investigate the relationship between the equilibrium of the dynamic game and the static Cournot equilibrium.

In discussing dynamic duopoly there are two major strategy sets that can be considered. The first is the open-loop strategies in which the players choose a path of action to which they commit themselves. In this case players cannot condition their actions on state variables that will be observed. In Roos's analysis only the open-loop strategy set was considered. The second possible strategy set is the closed-loop strategies in which each player chooses a decision rule that specifies his actions as a function of the state variables as well as a function of time. Thus, the closed-loop strategies, although more complicated to analyze, are more appropriate to use in the investigation of dynamic interaction among firms. Moreover, the Nash equilibria in open-loop strategies are usually not subgame perfect while the closed-loop strategies we discuss in this paper constitute a subgame perfect Nash equilibrium.

One of our main concerns is to examine the Cournot equilibrium in light of our new finding about the dynamic game. In order to carry out this investigation we incorporate a speed of adjustment term in our formulation of the price equation. When this speed of adjustment is infinity price converges instantaneously to the price indicated by the static demand function and the structural dynamic aspect of the game disappears. In this case the game is similar to the repeated Cournot game in its continuous time version. We denote this game as the "limit game" and investigate the open-and closed-loop equilibria of this game. Our most important result in this investigation is that the Cournot equilibrium price is the limit of the open-loop equilibrium while the closed-loop equilibrium price converges to a price below the Cournot equilibrium price. This result is especially surprising since the open-loop equilibrium is not subgame perfect whereas the closed-loop equilibrium we find is. In order to understand this result and its relationship with the traditional analysis of oligopolistic competition we elaborate on the different meaning of subgames. While in the regular analysis of a repeated Cournot game a subgame is a game that starts at a later period in our formulation it is a game that starts at a different (date, price) pair. Thus, the consistency property of the open-loop equilibrium is sufficent to obtain subgame perfection only under the previous definition of subgame but it is not sufficient for subgame perfection in our formulation. Moreover, the equilibrium strategies that we find are decision rules that specify the output rate as a function of price while the Cournot equilibrium strategies are just output rates and do not specify the players' strategy out of equilibrium, i.e., if the players observe a price different than the equilibirum price.

The intuitive reason for the difference between the open-loop and closed-loop stationary state Nash equilibria is that in the formulation of its closed-loop strategy each rival takes into account the optimal reaction of its rival to a change in the state variable while in the formulation of an open-loop strategy it does not. The stable equilibrium closed-loop strategy, in the problem we analyze, is an increasing linear function of the state variable price. Thus, each firm will decrease its output when price decreases. Let us see what taking the rivals' reaction to a decrease in price into account means in terms of a firm's output decision. If a firm ignores this reaction by its rival and simply makes the Cournot assumption that its rival's output will remain at its present level, then it will make its output decision on the basis of the residual demand curve it faces. If, on the other hand, it takes its rival's reaction into account, it will know that as it expands its output and causes prices to fall, its rival will contract its output. Thus, its movement down its residual demand curve will be offset somewhat by an outward shift of its residual demand curve as its rival contracts its output. This, of course, will cause the firm to optimally expand its output beyond the optimal level when its rival's reaction is ignored. As both rivals will take each others' optimal reaction to a change in price into account in the formulation of their closed-loop strategies, the equilibrium output will be greater than in the equilibrium of the open-loop strategies where rivals' reactions are ignored.

In the first section we pose the duopoly differential game as one in which each firm seeks to maximize the present value of its profits over an infinite time horizon subject to a state equation describing the movement of the price of their common product. We then define the open-loop and the closed-loop Nash equilibria of this game. In the second section we derive the open-loop Nash equilibrium of this game and characterize its price path through time. We also find the unique stationary open-loop equilibrium and prove that it is below the Cournot price and we then show that this equilibrium posses the global asymptotic stability property, i.e., all the equilibrium price paths that start at different initial prices converge to this stationary equilibrium. In the third section we derive the subgame perfect closed-loop Nash equilibrium of this game. We show in fact that there are two equilibria, a stable one and an unstable one. We study only the stable one and characterize its price path through time. In this case too we show the existence of a stable stationary closed-loop equilibrium and prove the existence of the global asymptotic stability property. In the fourth section we discuss the limit game by letting the price adjust instantaneously. It is here that we demonstrate that the Cournot equilibrium price is the limit of the open-loop equilibrium and thus it is not subgame perfect while the limit of the subgame perfect closed-loop equilibrium is below the Cournot price. In the fifth section we consider the finite horizon version of this differential game and derive its closed-loop Nash equilibrium. We then show that the closed-loop Nash equilibrium strategies of this finite horizon game possess a turnpike property, i.e., they approach the infinite horizon closed loop strategies as the finite horizon is extended.

1. Formulation and Notations

Consider a symmetric duopoly in which both firms have the cost function

$$c(u_i) = cu_i + \frac{1}{2} u_i^2, \quad i = 1,2 \tag{1.1}$$

where $u_i \geq 0$ is the i-th firm's output rate.

In the traditional static analysis an inverse demand function is given by $p = a - b(u_1 + u_2)$ where without loss of generality b can be assumed to be equal to one. In this paper we assume that prices do not converge instantaneously to the level suggested by the static demand function but rather change according to the kinematic equation

$$\dot{p} = \frac{dp}{dt} = s[a - (u_1 + u_2) - p]; \; p(0) = p_0 \tag{1.2}$$

where $0 < s \leq \infty$ denotes the speed in which the price converges to its steady state. In other words, a finite s implies that it takes time for the market to react to changes of quantities. When $s = \infty$, the price in the market converges instantaneously to the price indicated by the static demand function $p = a - (u_1 + u_2)$. Thus, the static case can be viewed as a special case of this formulation. Under these assumptions the objective of each firm is to maximize its discounted profits

$$(1.3) \qquad J^i = \int_0^\infty e^{-rt}[pu_i - cu_i - \frac{1}{2} u_i^2]dt, \quad i = 1,2$$

subject to (1.2) and $u_i \geq 0$.

In order to discuss this dynamic game it is essential first to discuss the informational structure, i.e., the information that each player has at the time it makes its output decision. In what follows we will study two solution concepts based on different information sets. First we will study the open-loop Nash equilibrium for the above game, and second, we will study the subgame perfect closed-loop Nash equilibrium.

Definition 1. The open-loop strategy space for player i is

$$S_i^{o\ell} = \{u_i(t) | u_i(t) \text{ is piecewise continuous and } u_i(t) \in [0,\bar{u}_i] \text{ for every } t\}.$$

Thus, an open-loop strategy is a path that the player commits to. The player's actions in this case depend only on time and not on the state variable p.

Definition 2. An open-loop Nash equilibrium for the game described by (1.2) and (1.3) is a pair of open-loop strategies (u_1^*,u_2^*) such that

$$J^i(u_1^*,u_2^*) \geq J^i(u_i,u_j^*), \quad i = 1,2$$

for every $u_i \in S_i^{o\ell}$.

Definition 3. A stationary open-loop Nash equilibrium is a pair of stationary strategies $u_1^*(t) = u_1^*$ and $u_2^*(t) = u_2^*$ and a price p^* such that u_1^*,u_2^* constitutes an open-loop Nash equilibrium for the game that starts at

$p(0) = p^* \text{ and } p^* - (a - (u_1^* + u_2^*)) = 0.$

<u>Definition 4.</u> The closed-loop strategy space for player i is

$$S_i = \{u_i(t,p) \,|\, u_i(t,p) \in [0,\bar{u}_i],\ u_i(t,p) \text{ is continuous in } (t,p) \text{ and } |u_i(t,p) - u_i(t,p')| \leqslant m(t)|p - p'| \text{ for some integrable } m(t) \geqslant 0\}.$$

The closed-loop strategy space is a set of <u>decision rules</u>. At every t each firm observes the price that exists in the market and choose its output rate accordingly.

<u>Definition 5.</u> A closed-loop Nash equilibrium is a pair of closed-loop strategies $(u_1^*,u_2^*) \in S_1 \times S_2$ such that

$$J^i(u_i^*,u_j^*) \geqslant J^i(u_i,u_j^*),\ \forall u_i \in S_i,\quad i = 1,2,\ j \neq i$$

for every possible initial condition (p_0,t_0).

<u>Definition 6.</u> A stationary closed-loop Nash equilibrium is a pair of closed-loop strategies $u_1^*(t,p) = u_1^*$, $u_2^*(t,p) = u_2^*$ and a price p^* such that (u_1^*,u_2^*) constitutes a closed-loop Nash equilibrium and $p^* - (a - (u_1^* + u_2^*)) = 0$.

From the kinematic equation (1.2) we can learn that the game under consideration has a dynamic structure. The price in this formulation is a state variable and changes over time according to a differential equation. Thus, the game does not repeat itself every period since at different periods the game can start at different p. An important question that can be asked here is how the equilibrium of this game behaves as s, the speed of price adjustment, approaches infinity. In this case there is no delay in price adjustment and the price adjusts instantaneously to $p = a - (u_1 + u_2)$. This description is somewhat similar to the static game that is associated with the above dynamic game. In this static game the demand function is given by $p = a - (u_1 + u_2)$ and the price at the Cournot duopoly equilibrium is given by

$$(1.4) \qquad p_D^* = \frac{a + c}{2}$$

The cooperative solution in which the firms maximize their profits jointly is

$$(1.5) \qquad p_{\text{cooperative}}^* = \frac{3a + 2c}{5}$$

For the sake of later comparison we define the competitive solution of the static game as the outcome in a market in which both firms behave according to the rule marginal cost equals price.

$$(1.6) \qquad p_C^* = \frac{a + 2c}{3}$$

However, since the game we investigate in this paper takes place through time it is appropriate to compare it with the repeated game and not with the one shot static game. This investigation is carried out in section 4.

2. Open-Loop Nash Equilibrium

The open-loop strategies (Definition 1) can be characterized as path strategies. Each player chooses a path of action $u_i(t)$ to which he commits himself. Neither has the option to reconsider its strategy and change it. A Nash equilibrium in such strategies (Definition 2) is a pair of paths (or n-tuple in the case of n players) such that each player's path is the best reponse to its rival's path. An important characteristic of the open-loop equilibrium is that a pair of open-loop strategies that constitutes an equilibrium for the game starting at some initial price p_0 do not necessarily constitute an equilibrium for a game that starts at a different price. Thus, we can conclude that the open-loop equilibrium is in general not subgame perfect. However, it is important to emphasize that the open-loop Nash equilibrium has the consistency property, which means that if $(u_1^*(t),u_2^*(t))$ constitutes an open-loop Nash equilibrium for a game that starts at p_0, then for every t_1, the truncated paths constitute an open-loop Nash equilibrium for the game that starts at t_1 and the price that is reached at t_1, i.e., p_{t_1}. Nevertheless, it is clear that the use of the closed-loop strategies is much more realistic since players have the opportunity to observe prices and to condition their action on the price that they observe. Moreover, the closed-loop Nash equilibrium that will be discussed in the next section has the subgame perfectness property. We chose to solve also the open-loop equilibrium to demonstrate the differences between the open-loop and the closed-loop equilibria. In particular, when we discuss the limit game in section 4, it will be shown that the static Cournot equilibrium is the limit of the open-loop equilibrium while the limit of the closed-loop equilibrium yields a different output rate and different equilibrium price. This result is surprising especially if we note that the open-loop equilibrium is not subgame perfect.

Theorem 1. There is a unique stationary open-loop Nash equilibrium for the above game. The price at this equilibrium is

$$p^* = \frac{as + (a + 2c)(s + r)}{s + 3(s + r)} \tag{2.1}$$

and the firms' strategies are given by

$$u_i^* = \frac{(a - c)(s + r)}{s + 3(s + r)}, \quad i=1,2 \tag{2.2}$$

Proof. For every given path $u_j(t)$ of firm j, firm i faces the problem of maximizing (1.3) subject to (1.2) and given $u_j(t)$. The current value Hamiltonian of this problem is given by

$$H_i = pu_i - cu_i - u_i^2/2 - \lambda_i s[a - (u_i + u_j) - p] \tag{2.3}$$

A similar problem of course faces player j. An equilibrium in the market is a pair of open-loop strategies that solves the two optimization problems simultaneously. Thus, the necessary conditions for open-loop equilibrium are

$$\partial H_i/\partial u_i = p - c - u_i - \lambda_i s = 0, \quad i=1,2 \tag{2.4}$$

and

$$-\dot{\lambda}_i = \partial H_i/\partial p = u_i - \lambda_i(s + r), \quad i=1,2 \tag{2.5}$$

and the transversality condition for control problems with infinite horizon implies that the discounted Hamiltonian vanishes as t approaches infinity (see Michel (1982)). From (2.4) we obtain

$$\lambda_i = \frac{p - c - u_i}{s} \tag{2.6}$$

Differentiating (2.6) yields

$$\dot{\lambda}_i = \frac{\dot{p} - \dot{u}_i}{s} = \frac{s(a - (u_i + u_j) - p) - \dot{u}_i}{s} \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) yields

$$\frac{s(a - (u_i + u_j) - p) - \dot{u}_i}{s} + u_i - \frac{p - c - u_i}{s}(s + r) = 0 \tag{2.8}$$

The conditions that must prevail at the stationary point are that $\dot{p} = \dot{u}_1 = \dot{u}_2 = 0$. Moreover, since we consider the symmetric case, at the equilibrium $u_1 = u_2 = u$, these conditions and equation (2.8) imply that at the stationary open-loop equilibrium the conditions that must hold are

$$u^* = \frac{p^* - c - u^*}{s}(s + r) = 0 \tag{2.9}$$

and

$$a - 2u^* - p^* = 0 \tag{2.10}$$

Substituting p^* from (2.10) into (2.9) and rearranging yields (2.1) and (2.2) as claimed. Q.E.D.

The above theorem implies that if player j plays the strategy $u_j(t) = u^*$ the optimal response of player i to this strategy is to play $u_i(t) = u^*$. In this case the price that will exist in the market is p^*, which will remain fixed over time.

Remark . Notice that equation (2.4) implies that at every instance each player will follow the policy

$$u_i + c = p - \lambda_i s \tag{2.11}$$

This rule is the well-known MC = MR, but in this case the marginal revenue consists of two elements. The price is the instantaneous marginal revenue and

$-\lambda_i s$ is the long run effect of an incremental change in the output rate. Thus, this condition equates the marginal cost with the long run marginal revenue.

Proposition 1. For every $r > 0$, the stationary open-loop equilibrium price is below the static Cournot price. Only for $r = 0$ do the two prices coincide. Moreover, as the interest rate increases, the equilibrium price declines. As r approaches infinity the stationary open-loop equilibrium price converges to the static competitive price.

Proof. Substituting $r = 0$ into (2.1) yields that the stationary open-loop equilibrium price is

$$p^* = \frac{a + c}{2} \tag{2.12}$$

which is exactly the Cournot price. Differentiating (2.1) yields

$$\frac{dp^*}{dr} = \frac{2s(c - a)}{(s + 3(s + r))^2} < 0 \tag{2.13}$$

For $r \to \infty$ it is straightforward to check that the stationary equilibrium price converges to $p^* = (a + 2c)/3$, which is the competitive price (1.6) at which firms produce at price equal marginal cost. Q.E.D.

The above necessary conditions can be used to find the open-loop equilibrium price trajectory for a game that starts at $p_0 \neq p^*$.

Theorem 2. The open-loop Nash equilibrium price trajectory is given by

$$p^e(t) = p^* + (p_0 - p^*)e^{k_1 t} \tag{2.14}$$

where p^* is the stationary equilibrium price given by (2.1), p_0 is the initial price at $t = 0$, and k_1 is a negative constant that will be specified below.

Before proving the above theorem, note that an immediate corollary from this theorem is that the game has the global asymptotic stability property. The equilibrium price trajectory converges to the stationary equilibrium price, which does not depend on the initial price p_0.[1]

Proof. A detailed proof is long but standard. A proof will merely be outlined here. Differentiating the kinematic equation (1.2) with respect to t and substituting in it λ, $\dot{\lambda}$ and u from equations (2.4), (2.5), and (1.2), respectively, yield that the equilibrium price trajectory must satisfy the following second order linear differential equation

$$(2.15) \qquad \ddot{P} + A\dot{P} + BP = R$$

when $\ddot{P} = \frac{d^2p}{dt^2}$, and

$$(2.16.1) \qquad A = s - r$$

$$(2.16.2) \qquad B = -s^2 - 3s(s + r)$$

$$(2.16.3) \qquad R = -[s^2a + s(2c + a)(s + r)]$$

Clearly, a particular solution of (2.15) is $p(t) = R/B$ which is exactly the stationary equilibrium price (2.1). Note that the roots of the characteristic equation associated with the homogeneous part of (2.15) are both real--one

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[1]See Fershtman and Muller (1984) for a discussion on the global asymptotic stability property in open loop capital accumulation games.

positive and the other negative. This is true since B is negative. Then, if we take the stable solution and use the initial price $p(0) = p_0$, the following trajectory

$$p^e(t) = p^* + (p_0 - p^*)e^{k_1 t} \tag{2.17}$$

is the open-loop Nash equilibrium price trajectory where

$$k_1 = -1/2[A + (A^2 - 4B)^{1/2}] < 0$$

Q.E.D.

3. The Infinite Horizon Closed-Loop Subgame Perfect Nash Equilibrium

The open-loop strategy space assumes that players cannot condition their actions on the observed price. This restriction is very severe since in a realistic situation price can be observed and strategies can be contingent upon the observed price. This is not to say that the open loop-equilibrium is not consistent. However, if from some reason there is a deviation from the equilibrium price path the truncated open-loop equilibrium strategies are not necessarily (and usually not) equilibrium for the game that starts at the new price. In order to overcome these two problems we discuss in this section the equilibrium in closed-loop (feedback) strategies. The closed-loop strategies (see definition 4) describe decision rules such that each player chooses a function describing his actions as a function of time and the state variables he observes. Thus, in this case, players do not commit themselves to some path and can respond to the different prices they observe. Morever, the way we define these decision rules makes it clear that they do not depend on the initial conditions of the game, i.e., on p_0. Thus, these decision rules constitute a Nash equilibrium for every set of initial conditions (p_0, t_0), and the equilibrium is subgame perfect. In what follows we find and discuss the closed-loop (feedback) Nash equilibrium of our game. We show the existence of a unique stable stationary closed loop equilibrium such that for every p_0 the equilibrium price path converges to this stationary equilibrium price.

Theorem 3. Let

(3.1) $$u_i^*(p) = \begin{cases} 0 & p \leq \hat{p} \\ (1 - sK)p + (sE - c) & p > \hat{p} \end{cases}, \quad i = 1,2$$

where

(3.2) $$K = \frac{r + 6s - \sqrt{(r + 6s)^2 - 12s^2}}{6s^2}$$

(3.3) $$E = \frac{-asK + c - 2sKc}{r - 3s^2K + 3s}$$

(3.4) $$\hat{p} = \frac{c - sE}{1 - sK}$$

Then $(u_1^*(p), u_2^*(p))$ constitutes a stable closed-loop subgame perfect Nash equilibirum for the infinite horizon dynamic game under consideration.

Proof. The proof will be carried out in two steps. First we consider the case in which $p_o \geq \hat{p}$. In this case we have an interior solution. Then we will consider the case $p_o < \hat{p}$.

Case 1. Assume that $p_o \geq \hat{p}$. Using the value functions approach the closed-loop equilibrium strategy (u_1^*, u_2^*) must satisfy the following Hamiltonian-Jacobi-Bellman equation (see Starr and Ho (1969)).

(3.5) $$rV^i(p) = \underset{u_i}{\text{Max}}\{(p - c)u_i - \tfrac{1}{2}u_i^2 + sV_p^i(p)[a - p - (u_i + u_j)]\}, \quad i = 1,2$$

where $V^i(p)$ is the value for player i of the game that starts at price p.

Note that although in the general case V^i is also a function of t and not just of the state variable under the current formulation the two value functions do not depend on t. This, of course, implies, as it is shown later, that the equilibrium strategies are stationary and do not depend on t. The firms' actions depend only on the observed price. This result can be established by carrying out the entire proof with $V^i(t,p)$ rather than $V^i(p)$ or by making use of the discussion which appears in Kamien and Schwartz (1981, p. 238).

Since the right side of the above equation is conave, the u_i that maximizes it is given by

$$(3.6) \qquad u_i^* = p - c - sV_p^i(p), \quad i = 1,2$$

Substituting (3.6) into (3.5) yields

$$(3.7) \qquad rV_i(p) = (p - c)(p - c - V_p^i s) - \frac{1}{2}(p - c - V_p^i s)^2$$

$$+ V_p^i s[a - p - (2p - 2c - sV_p^i - sV_p^j)], \quad i = 1,2.$$

Equation (3.7) presents a system of two partial differential equations. By solving this system and finding the value functions $(V^1(p),V^2(p))$, we can use (3.6) to find the equilibrium strategies. For every $p \geq \hat{p}$ we propose the quadratic value function

$$(3.8) \qquad V^i(p) = \frac{1}{2} K_i p^2 - E_i p + g_i, \quad i = 1,2$$

which implies that

(3.9) $$V_p^i(p) = K_i p - E_i, \quad i = 1,2$$

Substituting (3.8) and (3.9) into (3.7) yields

(3.10) $$\frac{1}{2} rK_i p^2 - rE_i p + rg_i = (p - c)(p - c - sK_i p + sE_i) - \frac{1}{2}(p - c - sK_i p + sE_i)^2 +$$

$$+ (sK_i p - sE_i)[a - p - (2p - 2c - sK_i p - sK_j p + sE_i + sE_j)], \quad i = 1,2, \; j \neq i$$

Since the game under consideration is completely symmetric (i.e., the two firms are identical) we will discuss the symmetric solution. Thus, from now on we assume that $K_i = K_j = K$, $E_i = E_j = E$, and $g_i = g_j = g$. Condition (3.10) must hold for every p, thus K, E, and g must be the solution of the following system of equations.

(3.11.1) $$-\frac{1}{2} rK + 1 - sK - \frac{1}{2}(1 - sK)^2 + 2s^2K^2 - 3sK = 0$$

(3.11.2) $$rE - 2c + sE + 3sKc - (1 - sK)(sE - c) + asK - 4s^2KE + 3sE = 0$$

(3.11.3) $$-rg + c^2 - sEc - \frac{1}{2}(sE - c)^2 - saE - 2scE + 2s^2E^2 = 0.$$

Equation (3.11.1) can be rewritten as

(3.12) $$3s^2K^2 - (r + 6s)K + 1 = 0$$

In a similar way equation (3.11.2) can be rewritten as

(3.13) $$E(r - 3s^2K + 3s) + asK - c + 2sKc = 0$$

and thus

$$E = \frac{-asK + c - 2sKc}{r - 3s^2K + 3s} \tag{3.14}$$

Now substitute (3.9) into (3.6) to yield that the strategies

$$u_i^* = (1 - sK)p + (sE - c), \quad i = 1,2 \tag{3.15}$$

where K and E as defined by (3.12) and (3.14) respectively constitute a subgame perfect closed-loop Nash equilibrium for the dynamic game under consideration. However, since (3.12) is a quadratic equation it has two possible solutions for K. These two solutions define two possible equilibrium strategies. In what follows we will show that just one of these solutions is stable.

Substituting (3.15) into the kinematic equation (1.2) and rearranging yields

$$\dot{p} = s[(2(sK - 1) - 1)p + a + 2(c - sE)] \tag{3.16}$$

In order to solve the above first order linear differential equation we will find a particular solution and then solve the homogeneous part of the equation.

A particular solution to (3.16) is

$$\bar{p} = \frac{a + 2(c - sE)}{2(1 - sK) + 1} \tag{3.17}$$

which is the stationary solution of (3.16).

The homogeneous part of (3.16) is

$$\dot{p} + s(2(1 - sK) + 1)p = 0 \tag{3.18}$$

Solving (3.18) yields

$$p(t) = Ce^{Dt} \tag{3.19}$$

where $D = s[2(sK - 1) - 1]$ and C is the constant of integration. Thus, the complete solution of (3.16) is

$$p(t) = \bar{p} + Ce^{Dt} \tag{3.20}$$

Substituting the initial conditions $p(0) = p_0$ into (3.20) yields that $p_o = \bar{p} + C$ and C is equal to $p_o - \bar{p}$. Thus, the equilibrium path, which is the solution of (3.16), is

$$p(t) = \bar{p} + (p_o - \bar{p})e^{Dt} \tag{3.21}$$

and can be rewritten as

$$p(t) = \bar{p}(1 - e^{Dt}) + p_o e^{Dt} \tag{3.22}$$

Thus, $\lim_{t \to \infty} p(t) = \bar{p}$ iff $D < 0$. This means that $s[2(sK - 1) - 1] < 0$, or that a necessary condition for asymptotic stability is that

(3.23) $$K < \frac{3}{2s}$$

Note that the explicit solution to (3.12) is

(3.24) $$K = \frac{r + 6s \pm \sqrt{(r + 6s)^2 - 12s^2}}{6s^2}$$

Taking first the positive sign in (3.24) implies that

(3.25) $$K > \frac{5}{3s} > \frac{3}{2s}$$

which contradicts (3.23).

Taking the negative sign in (3.24) implies that

(3.26) $$K < \frac{1}{3s} < \frac{3}{2s}$$

In this case K is sufficiently small to satisfy condition (3.23). Thus, let K be given by (3.24) with the negative sign. Since the choke off price a is above p_o we can conclude that $a > \hat{p}$. This condition is sufficient for $\bar{p}$ (given by (3.17)) to be above $\hat{p}$. Using (3.22) we can conclude now that $p(t) > \hat{p}$ for every t. Thus, the equilibrium strategy u_i^* is always positive.

<u>Case 2.</u> $P_o < \hat{p}$. In this case the constraint $u_i \geq 0$ is binding and thus we need to make some modification in the proof. In maximizing the right hand side of (3.5) for $p < \hat{p}$ no interior solution can be reached and the optimal output policy in this case is $u_i^* = 0$, $i = 1,2$. However, as (1.2) indicates when $u_1 = u_2 = 0$ the price goes up. If $a < \hat{p}$, price will go up until it will be equal to a, but since $a < \hat{p}$ no production will take place. If $a > \hat{p}$, price will go up until $p \geq \hat{p}$ and then the equilibrium is the one discussed in

Case 1. In order to establish the subgame perfectness of this equilibrium we need to define a value function also for prices below $\hat{p}$ and then to show that condition (3.5) is satisfied for this value function. When $u_i = 0$, $i = 1,2$, the price path $p(t)$ can be found directly from the kinematic equation

$$(3.27) \qquad p(t) = p_o e^{-st} + a(1 - e^{-st})$$

Thus, if at time t the price p(t) is below $\hat{p}$ the equilibrium strategies imply $u_i^* = 0$, $i = 1,2$, and the price in the market changes according to (3.27). Let $\hat{t}(p)$ denote the time that it takes for the price to reach the level $\hat{p}$ from the level p. Now for every $p < \hat{p}$ let

$$(3.28) \qquad V^i(p) = e^{-r\hat{t}(p)} V^i(\hat{p}),$$

where $V^i(\hat{p})$ is defined by (3.8), be the value function for every $p < \hat{p}$. The economic explanation of this value function is straightforward. For every $p < \hat{p}$ the optimal output is zero and thus profits are zero. The first time that the firms deviate from zero production level is when price reaches $\hat{p}$, the value of a game strating at $\hat{p}$ is already discussed and defined by (3.8). Thus, the value of the game that starts at $p < \hat{p}$ is the discounted value of $V^i(\hat{p})$.

Using (3.27), the condition that $\hat{t}(p)$ must satisfy is

$$(3.29) \qquad \hat{p} = pe^{-s\hat{t}} + a(1 - e^{s\hat{t}})$$

Differentiating (3.29) yields

(3.30) $$\frac{d\hat{t}}{dp} = \frac{-1}{a - p}$$

Thus, differentiating the value function with respect to p yields

(3.31) $$V_p^i(p) = \frac{r}{a - p} V^i(p), \quad i = 1,2$$

Thus, the suggested value function satisfies (3.5) since for every $p < \hat{p}$ $u_i^* = 0$. Q.E.D.

Corollary 1. From the above proof it is evident that there is a stationary closed-loop Nash equilibrium price given by

(3.32) $$p^* = \frac{a + 2(c - sE)}{2(1 - sK) + 1}$$

and the appropriate strategies defined by (3.15). This price is actually the stationary solution to (3.16) given by (3.17). Thus, if the game starts at the initial condition $p(0) = p^*$, the closed-loop subgame perfect equilibrium strategy defined by (3.15) is such that we will not observe any deviation from this price, i.e., the equilibrium price path is $p(t) = p^*$. Moreover, the equilibrium price path (3.22) implies that the game has the global asymptotic stability property, for if a game starts at $p_0 \neq p^*$ the closed-loop subgame perfect equilibrium price path converges to the stationary price given by (3.32).

Corollary 2. As the price in the market increases the firms increase their output rate.

Proof. The equilibrium strategies are given by (3.1) which are linear functions of price. The coefficient that multiplies p is given by $1 - sK$.

Using (3.26) we can conclude that $1 - sK > 0$. Q.E.D.

An immediate corollary of the above discussion is that as $r \to \infty$ the equilibrium price converges to the competitive price. This is true since (3.2) and (3.3) imply that $\lim_{r\to\infty} K = 0$ and $\lim_{r\to\infty} E = 0$ which implies that $V_p^i = 0$, $i = 1,2$. The equilibrium strategy in this case (see (3.1)) is $u_i^* = p - c$, $i = 1,2$. This policy implies that $u^* + c = p$ which is identical to the well-known rule MC = MR, when MR is taken to be the instantaneous marginal revenue and not the long run marginal revenue. The stationary equilibrium price can be calculated in this case from (3.32) and is given by

$$p^* = \frac{a + 2c}{3}$$

which is exactly the "competitive" equilibrium of the static game in which firms charge price equal to marginal cost. This result is intuitively appealing. A very high interest rate implies that the importance of the future declines. As r approaches infinity firms stop taking into consideration the future effects of their current actions. Thus, the policy that they follow is marginal cost equal short run marginal revenue which is equal, under the assumption of this model, to the price. The resultant equilibrium price in this case is, of course, the "competitive" price.

4. The "Limit Game": The Dynamic Game With Instantaneous Price Adjustment

In previous sections we based our dynamic game on the assumption that the speed of adjustment is finite. It is this assumption that makes the game we consider different from the regular repeated Cournot game even in its continuous time version. The assumption of finite speed of price adjustment introduces the dynamic structure to this game. In this section we let s, the speed of price adjustment, go to infinity and examine what we define as the limit game. In this case the price jumps instantaneously to the price indicated by the static demand functions $p = a - (u_1 + u_2)$ (see (2.17) and (3.21)), and firms cannot take advantage of the delay in price adjustment. Thus, under this assumption the limit game is somewhat similar to the repeated static game--in its continuous time version.

Theorem 4. The Cournot equilibrium of the static game, given by (1.4) is the limit of the stationary open-loop equilibrium.

Proof. The stationary open-loop Nash equilibrium of the dynamic game is given by (2.1) and therefore when $s \to \infty$ yields that

$$\lim_{s\to\infty} p^* = \frac{a + c}{2} \tag{4.1}$$

which is identical to the Cournot equilibrium of the static game, given by (1.4). Q.E.D.

Theorem 5. The stationary closed-loop equilibrium price converges to a price which is a convex combination of the Cournot duopolistic equilibrium price (1.4) and the competitive price (1.6).

Proof. The stationary closed-loop equilibrium price is given by

$$p^* = \frac{a + 2(c - sE)}{2(1 - sK) + 1} \tag{4.2}$$

From (3.2) it is evident that $K \to 0$ as $s \to \infty$. Similarly, from (3.3) $E \to 0$ as $s \to \infty$. However, in the expression for p^*, sK and sE appear. Let us denote $\beta = \lim_{s\to\infty} sK$ and $\gamma = \lim_{s\to\infty} sE$. Now, using equation (3.2) yields that

(4.3) $$\beta = 1 - \sqrt{2/3} > 0$$

Similarly, from (3.3) we obtain that

(4.4) $$\gamma = \frac{c - a\beta - 2c\beta}{3 - 3\beta}$$

Note also from (3.2) and (3.3) that $\lim_{r\to 0} sK = \beta$ and $\lim_{r\to 0} sE = \gamma$. Using (3.22) it is evident that $p(t) = \bar{p} + (p_o - \bar{p})e^{DT}$ where D is a negative constant which depends on the parameters of the model. Thus, the equilibrium price trajectory converges to the stationary equilibrium price. Moreover, as $s \to \infty$ or $r \to 0$, the equilibrium price approaches

(4.5) $$p^* = \frac{a + 2(c - \gamma)}{3 - 2\beta}$$

Substituting (4.4) for γ in equation (4.5) and rearranging yields

(4.6) $$p^* = \frac{(a + 2c)(1 - \beta) + 2(a + c)}{3(3 - 2\beta)(1 - \beta)}$$

From (1.4) and (1.6) it is evident that $p_D = (a + c)/2$ and $p_C = (a + 2c)/3$. Thus, the equilibrium price p^* can be written as a convex combination of p_D and p_C such that

(4.7) $$p^* = \frac{p_C + 2\sqrt{2/3}\ p_D}{1 + 2\sqrt{2/3}}$$ Q.E.D.

Remark. Since $p_C < p_D$ and p^* is a convex combination of the two it is evident that $p^* < p_D$. Moreover, using Theorem 4 we can conclude that the closed-loop equilibrium price of the limit game is below the open-loop equilibrium price and, since the two prices are below the cooperative price the closed-loop subgame perfect equilibrium yields lower profits than the open-loop. Since the open-loop equilibrium presents the case in which each player commits himself to some output path and does not condition his output rate on the observed price it is clear from the above result that the players can benefit from such commitments.

Remark. Besides the stable equilibrium, discussed above, there is another equilibrium which is the limit of the unstable equilibrium. This equilibrium yields a price higher than the Cournot price and the equilibrium strategies are decreasing linear functions of price.

Before discussing the new equilibrium note that the closed-loop equilibrium strategies are given by

$$u_i^* = \sqrt{2/3}\, p + (\gamma - c), \quad i = 1,2 \tag{4.8}$$

which describe a pair of decision rules that prescribe output rates for every price while the open-loop (Cournot) equilibrium strategies are given by

$$u_i^* = \frac{a - c}{4}, \quad i = 1,2 \tag{4.9}$$

which describe a pair of output rates independent of p.

It is the difference between the equilibrium strategies that can lead to a better understanding of the limit game and the repeated static game. While discussing the dynamic game with a finite speed of adjustment it was clear that the appropriate strategy space that should be used is the closed-loop strategy space. Firms do have the opportunity to condition their output on the observed price and unless commitment is somehow enforced, firms can reconsider their strategy and change it. The open-loop strategy set describes just such path strategies in which each firm commits itself to a path and does not have the opportunity to change it. In general we can say that the open-loop equilibrium, although satisfying a consistency property, is not subgame perfect. The closed-loop strategy space describes decision rules, not commitment to some output path. The firms' output depend on the price they observed. The closed-loop equilibrium that we discuss above is also subgame perfect. Thus, it is clear that the closed-loop strategies are a more appropriate description of reality. In discussing the limit game it is therefore obvious that we should choose the limit of the closed-loop equilibrium rather than the limit of the open-loop equilibrium. However, the limit of the closed-loop equilibrium price, as given by (4.6), is different than the Cournot price. If the game starts at this price, i.e., $p(0) = p^*$, players will play the equilibrium strategies (4.8) and the price in the market will continue to be p^*. If the game starts at any different price, including the Cournot equilibrium price, the equilibrium price path will converge to p^*. From the above discussion it is evident that the Cournot equilibrium as a limit of the open-loop equilibrium is <u>not</u> a subgame perfect equilibrium for the limit game. This result is very surprising especially if we confront it with the well-known results in the supergames literature in which the Cournot equilibrium is subgame perfect. However, in order to understand this result we need first to elaborate on the meaning of a subgame in the two games that we discuss here, i.e., limit game and supergame. In the supergame framework the meaning of a subgame is a game that starts at a later period. Under this definition of subgame it is clear that the Cournot equilibrium is subgame perfect since in our discussion about open-loop equilibrium we noted that the open-loop equilibrium is consistent, which means that for every $t_1 > t_0$ the truncated output paths constitute an equilibrium for the game that starts at t_1. In the model we consider in this paper a subgame is a game that starts at (p,t) different than (p_o,t_o). From this definition the consistency property is not sufficient for obtaining subgame perfectness. The equilibrium strategies must also constitute an equilibrium for the games that start at different prices. The strategies given by (4.6) satisfy this condition.

5. The Finite Horizon Subgame Perfect Closed-Loop Nash Equilibrium

In this section we consider the dynamic game under the assumption of finite planning horizon. Our objectives are twofold: first, we want to demonstrate that although in the finite horizon case the equilibrium path does not converge to a stationary equilibrium the technique used in section 3 can be applied under some modifications to the finite horizon case. Second, we will prove that as T approaches infinity, the equilibrium path converges to the equilbrium path of the infinite horizon game. This property was denoted by Friedman (1981) as a turnpike property in games. For a discussion of turnpike properties in dynamic games, see Fershtman and Muller (1984b).

Using the value function approach the closed-loop equilibrium strategies (u_1^*,u_2^*) must satisfy the Hamiltonian-Jacobi-Bellman equations.

$$(5.1) \quad -V_t^i(t,p) + rV^i(t,p) = \max_{u_i}\{(p - c_i)u_i - \frac{1}{2}u_i^2 + sV_p^i(t,p)[a - p - (u_i + u_j)]\},\ i=1,2,\ j \neq i$$

Notice that the right side of the above expressions is concave. For simplicity we assume that p_o is high enough so that there is $u_i^* > 0$ that maximizes the right hand side of (5.1). In the infinite horizon case we discuss in detail the relationship between p_o and the value function. In this section we analyze only the interior solution case in which both firms produce throughout the horizon. Thus u_i^* that maximizes this expression is given by

$$(5.2) \qquad u_i^* = p - c - sV_p^i, \quad i = 1,2$$

Substituting (5.2) into (5.1) we obtain that

$$(5.3) \qquad V_t^i(t,p) - rV^i(t,p) + (p - c)(p - c - sV_p^i) - \frac{1}{2}(p - c - sV_p^i)^2$$

$$+ V_p^i s[a - (2p - 2c - sV_p^i - sV_p^j) - p] = 0, \text{ for } i = 1,2,\ j \neq i.$$

As in the infinite horizon case we consider the quadratic value function

$$(5.4) \qquad V^i(t,p) = \frac{1}{2} K_i(t)p^2 - E_i(t)\, p + g_i(t), \quad i = 1,2$$

Differentiating with respect to t and p yield

$$(5.5) \qquad V_t^i(t,p) = \frac{1}{2} \dot{K}_i(t)p^2 - \dot{E}_i(t)p + \dot{g}_i(t), \quad i = 1,2$$

$$(5.6) \qquad V_p^i(t,p) = K_i(t)p - E_i(t)$$

where $\dot{K}_i = dK_i/dt$ and $\dot{E}_i = dE_i/dt$.

Substituting (5.5) and (5.6) into (5.3) yields

$$(5.7) \qquad \frac{1}{2}\dot{K}_i p^2 - \dot{E}_i p + \dot{g}_i - rK_i p^2/2 + rE_i p - rg_i + (p - c)(p - c - sK_i p + sE_i)$$

$$- \frac{1}{2}(p - c - sK_i p + sE_i)^2 + s(K_i p - E_i)[a - 3p + 2c + sK_i p - sE_i +$$

$+ sK_j p - sE_j] = 0$, all $i = 1,2$.

Since the game under consideration is completely symmetric (i.e., the two firms are identical) we will discuss the symmetric solution. Thus, from now on we assume that $K_i = K_j = K$, $E_i = E_j = E$ and $g_i = g_j = g$.

Theorem 6. For the game described above, the subgame perfect closed-loop equilibrium strategies are

$$u_i^* = (1 - sK(t))p - c + sE(t), \quad i = 1,2$$

where K(t) and E(t) are the solutions of the following system of Riccati equations.

$$\dot{K} = -3s^2K^2 + (6s + r)K - 1 \tag{5.8}$$

$$\dot{E} = (r + 3s - 3s^2K)E + (sKa + 2sKc - c) \tag{5.9}$$

The solution to this system of Riccati equations will be specified after proving the above theorem.

Proof. The proof follows the same method used in proving Theorem 2. Let g(t) be a solution of the differential equation

$$\dot{g} - rg + c^2 - sEc - \frac{1}{2}(sE - c)^2 - sEa - 2sEc + 2s^2E^2 = 0 \tag{5.10}$$

where E is defined by (5.9). Thus, equations (5.8), (5.9), and (5.10) define a triple of functions (K(t), E(t), g(t)) that solve equation (5.7). By making use of equations (5.2) and (5.6) it is evident that the above strategies constitute a closed-loop subgame perfect equilibrium. Q.E.D.

Let α_1 and α_2 be the two solutions of the quadratic equation $3s^2K^2 - (6s + r)K + 1 = 0$. Then $K(t) = \alpha_1$ and $K(t) = \alpha_2$ are both solutions of (5.8).

Thus, the general solution of (5.8) is given by (see Ford (1955))

$$\text{(5.11)} \qquad \frac{K(t) - \alpha_1}{K(t) - \alpha_2} = Ae^{-(\alpha_1-\alpha_2)3s^2t}$$

Note that by switching α_1 and α_2 in (5.11) we obtain another formula for general solution. Since $V^i_p(T,p) = 0$ for every p, this boundary condition implies that at time T, $K(T) = E(T) = g(T) = 0$. Thus, evaluating (5.11) at $t = T$ yields

$$\text{(5.12)} \qquad \frac{\alpha_1}{\alpha_2} = Ae^{-(\alpha_1-\alpha_2)3s^2T}$$

which implies that

$$\text{(5.13)} \qquad A = \frac{\alpha_1}{\alpha_2}\, e^{(\alpha_1-\alpha_2)3s^2T}$$

Substituting (5.13) into (5.11) yields that

$$K(t) - \alpha_1 = \frac{\alpha_1}{\alpha_2}\, e^{(\alpha_1-\alpha_2)3s^2(T-t)}(K(t) - \alpha_2)$$

which implies that

$$(5.14) \qquad K(t) = \frac{\alpha_1(1 - e^{(\alpha_1-\alpha_2)3s^2(T-t)})}{1 - \frac{\alpha_1}{\alpha_2} e^{(\alpha_1-\alpha_2)3s^2(T-t)}}$$

After K(t) is known we can solve (5.9) and find E(t). The general solution of (5.9) is

$$(5.15) \qquad E(t) = e^{-\int_0^t [3s^2K(\xi)-r-3s]d\xi} [C + \int_0^t e^{-\int_0^\tau [r+3s-3s^2K(\xi)]d\xi} [saK(\tau) + 2scK(\tau) - c]d\tau$$

Using the boundary condition E(T) = 0, we can find C, the constant of integration

$$(5.16) \qquad C = -\int_0^T e^{\int_0^\tau [3s^2K(\xi)-r-3s]d\xi} [saK(\tau) + 2scK(\tau) - c]d\tau$$

Substituting (5.16) into (5.15) yields

$$(5.17) \qquad E(t) = -e^{-\int_0^t [3s^2K(\xi)-r-3s]d\xi} \cdot$$

$$\cdot \int_t^T e^{\int_0^\tau [3s^2K(\xi)-r-3s]d\xi} [saK(\tau) + 2scK(\tau) - c]d\tau$$

which can be rewritten as

$$(5.18) \qquad E(t) = -\int_t^T e^{\int_t^\tau [3s^2K(\xi)-3s-r]d\xi} [saK(\tau) + 2scK(\tau) - c]d\tau$$

Thus, the closed-loop equilibrium strategies are given by

$$u_i^*(t) = (1 - sK(t))p + (sE(t) - c), \; i = 1,2.$$

where K(t) and E(t) are given by (5.14) and (5.18), respectively.

Given the above equilibrium strategies for the finite horizon game we want to find out how they relate to the infinite horizon equilibrium strategies. Specifically, we want to show that the game satisfies the following turnpike property: the closed-loop equilibrium strategies for the finite horizon game tend to the equilibrium strategies of the infinite horizon game as T approaches infinity. However, note that the closed-loop strategies are actually functions of prices and not output rates. Thus, we need to clarify in what norm the finite equilibrium strategies converge to the infinite equilibrium strategies. Clearly we cannot use the supremum norm because for every finite horizon problem, end game considerations imply that the finite horizon equilibrium will deviate from the infinite horizon equilibrium. Nevertheless, the following turnpike property can be proven.

Theorem 7 (Turnpike Property). For every ε and T_1 there is T_2 such that for every $T > T_2$ the equilibrium strategies for the finite horizon game are in ε neighborhoods of the infinite horizon equilibrium strategies for every $0 < t < T_1$.

Proof. The equilibrium strategies of the finite horizon game are given by

$$u_i^*(t) = (1 - sK_T(t))p + (sE_T(t) - c), \; i = 1,2$$

Note that we deviate here from our previous notations and we write $K_T(t)$ and $E_T(t)$ to emphasize that these two functions depend on the horizon of the game. For the infinite horizon game the equilibrium strategies are given by the same expression when K_∞ and E_∞ are specified by (3.2) and (3.3), respectively. The proof then will be carried out by comparing K_T and E_T with K_∞ and E_∞. Observe that α_1 and α_2 are actually the two solutions of the quadratic equation (3.12) which are explicity described by (3.24). Let α_1 be the solution with the negative sign and α_2 be the solution with the positive sign. Thus, $\alpha_1 < \alpha_2$. From the discussion in section 3 it is evident that $K_\infty = \alpha_1$. By investigating (5.14) it is clear that for every ε_1 and T_1 there is $T_2 > T_1$ such that for every $t < T_1$ $|K_T(t) - \alpha_1| < \varepsilon_1$ for every $T > T_2$. This is true because for any given $t < T_1$ $K_T(t) \to \alpha_1$ as $T \to \infty$. Now compare equations (5.9) and (3.3). Clearly, for the same value of K, (3.3) describes a particular solution of the differential equation (5.9). From the continuity of these equations and from the above discussion it is long but rather straightforward to prove that for every ε_2 and T_1 there is $T_2 > T_1$ such that for $t < T_1$, $|E_T(t) - E_\infty| < \varepsilon_2$ for every $T > T_2$. Clearly, by choosing ε_1 and ε_2 to be sufficiently small, we can find such T_2 that the equilibrium strategies for the finite horizon game will be in the ε neighborhood of the infinite horizon equilibrium strategies. Q.E.D.

An immediate corollary of the above theorem is that the same turnpike property holds for the price equilibrium path, i.e., the price equilibrium path of the finite horizon game tends to the price equilibrium path of the infinite horizon game as the horizon approaches infinity.

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