Optimal Auctions with Financially Constrained Bidders *

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Abstract

We consider an environment where potential buyers of an indivisible good have liquidity constraints, in that they cannot pay more than their 'budget' regardless of their valuation. A buyer's valuation for the good as well as her budget are her private information. We derive constrained-efficient and revenue maximizing auctions for this setting. In general, the optimal auction requires 'pooling' both at the top and in the middle despite the maintained assumption of a monotone hazard rate. Further, the auctioneer will never find it desirable to subsidize bidders with low budgets.

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1 Introduction

Auction theory revolves around the design and analysis of auctions when a seller with goods for sale is confronted with buyers whose willingness to pay he knows little about. A standard assumption in this literature has been to conflate a buyer's willingness to pay with her ability to pay- an unpalatable assumption in a variety of situations.[1] For instance, in government auctions (privatization, license sales etc.), the sale price may well exceed a buyers' liquid assets, and she may need to rely on an imperfect (i.e. costly) capital market to raise funds. These frictions limit her ability to pay, but not her valuation (how much she would pay if she had the money). In some sense, these financial constraints are more palpable than valuations, which are relatively amorphous. There has been some applied and empirical work suggesting that these considerations play a role both in the design of, and bidder behavior in, real world auctions. However, there has been a small amount of theoretical work investigating the (optimal) design of auctions when bidders are liquidity constrained.[2]

In this paper, we are agnostic about the source of this liquidity constraint- an interested reader should refer to Che and Gale [8] for a discussion on possible sources of these constraints. Here we assume that there is a 'hard budget constraint', in the sense that no buyer can pay more than her budget regardless of her valuation. Assumptions of a similar flavor have been made in the monetary search literature, see for example, Galenianos and Kircher [11] and the references therein. The key difference is that in their models agents choose their monetary holdings *a priori*, whereas here they are given exogenously.

We derive the revenue maximizing and constrained efficient auctions in this setting, when both valuations and budgets are bidders' private information. We implicitly disallow mechanisms that require bidders to 'prove' their budgets by posting a bond equal to their budget up front.[3]

[1] Not every potential buyer of a David painting who values it at a million dollars has access to a million dollars to make the bid.

[2] There has been more progress analyzing various 'standard' auction formats when bidders are financially constrained.

[3] This prevents bidders from overstating their budgets since they would not have the

For a seller, budget constraints mean that low budget bidders cannot put competitive pressure on high budget bidders. For this reason it has been suggested the seller should subsidize some bidders to foster competition. We give three examples. In the FCC spectrum auctions, Ayres and Cramton [3] argued that subsidizing women and minority bidders actually increased revenues since it induced other bidders to bid more aggressively.[4] In a procurement context, Rothkopf et al [22] find that subsidizing inefficient competitors can be desirable. Zheng [24] studies a stylized setting where liquidity constrained bidders may be able to get additional funds from the market at some cost. He considers a specific auction format, and shows that if the auctioneer in this setting has access to cheaper funds, he may wish to subsidize some bidders.

A subsidy is not the only instrument for encouraging competition nor is it necessarily the best. For this reason an analysis of the optimal auction will be useful. It may suggest other instruments that are more effective. Our main finding is that if the seller were running an optimal auction, he would never find it it beneficial to subsidize bidders. Rather he should favor budget constrained bidders with a higher probability of winning.

Subsidizing bidders has two effects. The positive effect has been described. However, to preserve incentive compatibility, one may be forced to offer a subsidy to other bidders, thus diluting the positive effect. Our analysis shows that the negative effect dominates.

The technical contribution of this paper is to the literature on mechanism design when agents' types are multidimensional. In general, mechanism design when agents' types are multidimensional is known to be hard (see for example Rochet and Choné [20]). Solved cases, in the sense of mechanisms that have simple descriptions, are rare. Intuitively, this is because when types are multidimensional, there are 'too many' incentive compatibility constraints. Further, several of these papers use the structure of the problem they consider to 'reduce' the type of the agent to a single dimension, something we are unable to do here. Armstrong [2], Wilson [23] and Manelli and Vincent

cash to post a larger bond. In practice however, posting a bond equal to one's budget may be expensive, and regardless, our methods apply to this case as well.

[4]Their argument was based on the assumption that minority bidders would typically assign lower valuations to the asset than large bidders.

[17] are examples of the difficulties encountered in this class of problems, and Rochet and Stole [21] survey solved cases. Malakhov and Vohra [16], use a discrete types approach and the tools of linear programming to solve some other cases (see Iyengar and Kumar [13] for the continuous version).

Budget constraints render the associated incentive compatibility constraints non-differentiable, despite the standard assumption of quasi-linear utility. Therefore the Kuhn-Tucker-Karush first order conditions have no bite in this setting. We skirt this difficulty by considering a model of discrete types, i.e there are only a finite (if large) number of possible valuations and budgets.[5] This makes the problem of optimal design amenable to the use of tools from linear programming, which is less involved than its continuum of types counterpart. In our opinion, the arguments used are significantly more transparent, and the intuition cleaner and easier to grasp.

1.1 Related Literature

The literature on auctions with budget constraints can be divided into two groups. The first analyzes the impact of budget constraints on standard auction forms. Che and Gale [8] consider the revenue ranking of standard auction formats (first price, second price and all pay) under financial constraints. Benoit and Krishna [4] look into the effects of budget constraints in multi-good auctions, and they compare sequential to simultaneous auctions. Brusco and Lopomo [7] study strategic demand reduction in simultaneous ascending auctions and show that inefficiencies can emerge even if the probability of bidders having budget constraints is arbitrarily small. Several other works too numerous to enumerate here study the effects of financial constraints in a variety of settings.

The second group considers the problem of designing an 'optimal' auction. Maskin [18] proposed the 'constrained efficient' auction, i.e. the auction that maximized efficiency when bidders had common knowledge budget constraints. Laffont and Robert [14] proposed a revenue maximizing auction for this setting, with the added restriction that all bidders had the same budget constraint. Both of the aforementioned papers imposed Bayesian incentive

[5] Readers with long memories will recall that the 'original' optimal auction paper by Harris and Raviv [12] also assumed discrete types.

compatibility. Malakhov and Vohra [15] design the dominant strategy revenue maximizing auction when there are 2 bidders, only one of whom is liquidity constrained. None of these papers considers the problem of design when both budget and valuation are private information. Che and Gale [9] compute the revenue maximizing pricing scheme when there is a single buyer whose budget constraint and valuation are both his private information.[6] Borgs et al [6] study a multi-unit auction and design an auction that maximizes worst case revenue when the number of bidders is large. Nisan et al [10] show in a closely related setting that no dominant strategy incentive compatible auction can be Pareto-efficient when bidders are budget constrained.

1.2 Discussion of Main Results

In this section we describe the main qualitative features of the revenue maximizing auction subject to budget constraints.[7] In particular, we draw a contrast with the features of the classic optimal auction of Myerson [19]. First, some notation. Denote a generic type by t, and a profile of types, one for each agent, by t^n. An auction must specify how the good is allotted at each profile t^n, and each agent's payment at this profile. Given this allotment rule, let $a(t)$ be an agent's interim probability of being allocated the good when he reports type t.

When bidders are not budget constrained, the type of an agent is just her valuation v, and Myerson [19] applies. Suppose Myerson's regularity condition on the distribution of valuations, the monotone hazard rate condition, is met. In this case we know that at each realized profile of types, the optimal allocation rule allots to the highest valuation subject to it being above a reserve $\underline{v}$, where the reserve is the lowest type with a non-negative 'virtual valuation'. Assuming 2 bidders and valuations to be uniform in $[0, 1]$, the resulting interim allocation probabilities are as graphed in Figure 1.

Now suppose all bidders have the same (common knowledge) budget constraint. The type of an agent is still just her valuation. Laffont and Robert

[6]Their definition of a financial constraint is more general than ours, at the expense of tractability.

[7]The constrained efficient auction shares many of the same properties.

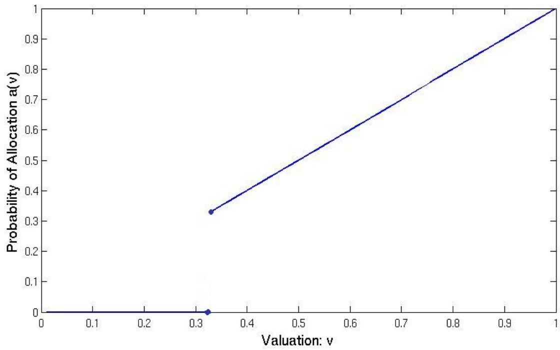

FIGURE 1: Optimal Allocation Rule

showed that the revenue maximizing auction will 'pool' some types at the top. In other words, all types above some $\bar{v}$ will be treated as if they had valuation exactly $\bar{v}$, and the budget constraint will bind for precisely these types. Laffont and Robert argued that the allocation rule will allot the good to the highest valuation subject to this 'pooling', and subject to it being higher than an appropriately chosen reserve $\underline{v}$. Further, this reserve will be lower than the one in Myerson. The resulting interim allocation probabilities are as graphed in Figure 2. The constrained efficient auction according to Maskin is similar except there is no reserve $\underline{v}$.

A consequence of our analysis is that the claims of Laffont and Robert, and Maskin are not quite correct.[8] A condition on the distribution of valuations in addition to the monotone hazard rate is needed. Specifically, the density function of valuations must be decreasing. If this condition fails, our analysis shows that there can be pooling in the middle as displayed in Figure 3.

Finally, suppose bidders have one of 2 budgets $b_H > b_L$. Here, the type of a bidder is 2 dimensional- his valuation, and his budget. As in Laffont and Robert, there will be pooling at the top, however there will be two cutoffs,

[8]Appendix A provides counter-examples.

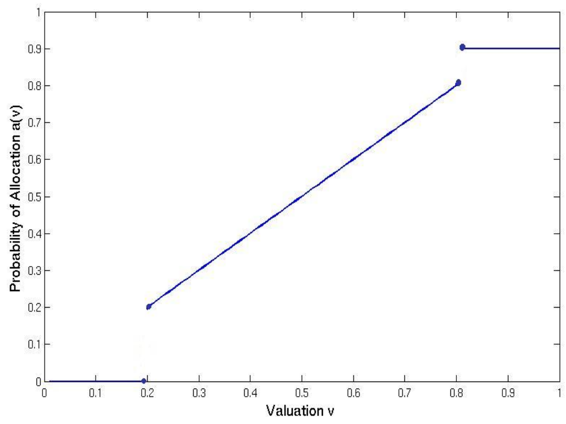

FIGURE 2: Common Knowledge Common Budget, Decreasing Density

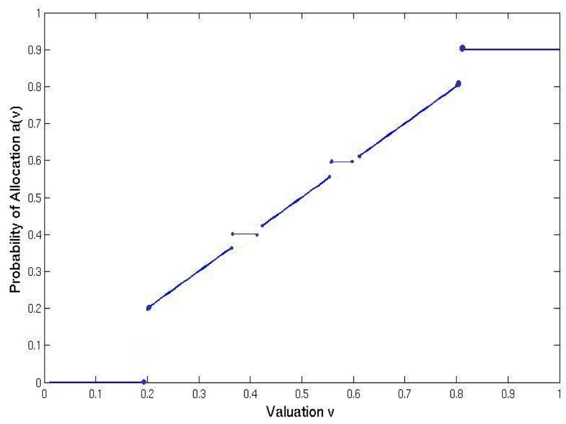

FIGURE 3: Optimal Allocation Rule: Pooling in the middle

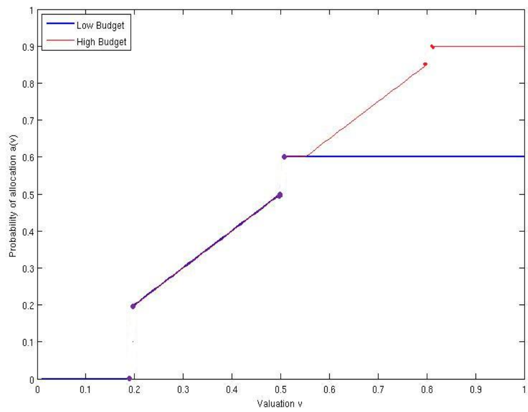

FIGURE 4: Optimal Allocation Rule

$\bar{v}_H \geq \bar{v}_L$, such that all high budget bidders with valuation at least $\bar{v}_H$ will be pooled and all low budget bidders with valuation at least $\bar{v}_L$ will be pooled. A bidder with valuation $v < \bar{v}_L$ will get the same allocation whether he is of a high budget or low budget type. Finally, the auction will require 'ironing' in the middle, around the cutoff $\bar{v}_L$. High budget bidders whose valuations are slightly higher than $\bar{v}_L$ will be treated as if they had a lower valuation. The resulting interim allocation probabilities are graphed in Figure 4.[9]

The last of these properties merits attention. The worry with budget constrained bidders is that bidders with 'low' budgets are unable to compete, effectively reducing competition in the auction, and thus revenue. This property says that the optimal auction compensates for this by shading down the valuations of high budget bidders. Surprisingly, this property is present in the constrained efficient auction, where it is clearly inefficient.

The method of analysis yields another insight regarding the design of auctions in such settings. Where prior work suggested there may be gains to subsidizing low budget bidders (see Section 1.1 above) our analysis shows that the auctioneer would decline to subsidize bidders if he was running the optimal auction. Thus, arguments in favor of subsidies depend on the

[9]The constrained efficient auction is structurally similar to the above auction, except that there is no reserve price.

analysis of specific (i.e. sub-optimal) auction mechanisms.

1.3 Organization of this paper

In Section 2 we describe the model. In Section 3 we examine the special case when all bidders have the same common knowledge budget constraint. This helps build intuition for the more involved private information case. In Section 4 we examine the case when bidders' budgets are private information. In Section 5 we discuss the (im)-possibility of profitably subsidizing bidders as well as the implementation of this auction, and concludes.

2 A Discrete Formulation

There are N risk neutral bidders interested in a single indivisible good. Each has a private valuation for the good v in $V = \{\epsilon, 2\epsilon, \ldots, m\epsilon\}$. For notational convenience we take $\epsilon = 1$. Further, each bidder has a privately known budget constraint b in $B = \{b_1, b_2, \ldots, b_k\}$, wlog $b_1 < b_2 < \ldots < b_k$. The type of a bidder is a 2-tuple consisting of his valuation and his budget $t = (v, b)$; and the space of types is $T = V \times B$. An agent of type $t = (v, b)$ who is given the good with probability a and asked to make a payment p derives utility:

$$u(a, p|(v, b)) = \begin{cases} va - p & \text{if } p \leq b, \\ -\infty & \text{if } p > b. \end{cases}$$

In other words an agent has a standard quasi-linear utility up to his budget constraint, but cannot pay more than his budget constraint under any circumstances.

We assume that bidders' types are i.i.d. draws from a commonly known distribution π over T. We require that π satisfy a generalization of the monotone hazard rate condition. Define $f_b(v) = \pi(v|b) > 0$, i.e. the probability a bidder has valuation v conditional on her budget being b. Further, define $F_b(v) = \sum_1^v f_b(v)$. We require that:

$$(v, b) \geq (v', b') \Rightarrow \frac{1 - F_b(v)}{f_b(v)} \geq \frac{1 - F_{b'}(v')}{f_{b'}(v')}$$

For notational simplicity only we assume that the valuation and budget components of a bidder's type are independent, and that all budgets are equally likely:[10]

$$\mathbb{P}\big(t = (v, b)\big) = \pi(t) = \frac{1}{k} f(v). \tag{1}$$

By the Revelation Principle, we confine ourselves without loss of generality to direct revelation mechanisms. The seller must specify an allocation rule and a payment rule. The former determines how the good is to be allocated as a function of the profile of reported types and the latter the payments each agent must make as a function of the reported types. We denote the implied interim expected allocation and payment for a bidder of type t as $a(t)$ and $p(t)$ respectively.

To ensure participation of all agents we require that:

$$\forall t \in T,\, t = (v, b): \quad va(t) - p(t) \geq 0. \tag{2}$$

The budget constraint and individual rationality require that no type's payments exceed their budget:

$$\forall t \in T,\, t = (v, b): \quad p(t) \leq b. \tag{3}$$

To ensure that agents truthfully report their types we require that Bayesian incentive compatibility hold. However, due to the budget constraint, the incentive constraints will only require that a type $t = (v, b)$ has no incentive to misreport as types t' such that $p(t') \leq b$. We can write this as:

$$\forall t, t' \in T,\, t = (v, b): \quad va(t) - p(t) \geq \chi\{p(t') \leq b\}\, va(t') - p(t'), \tag{4}$$

where χ is the characteristic function. Note that the presence of this characteristic function renders the incentive compatibility constraints non-differentiable, and thus the standard KTK first order conditions do not apply.

A key prior result we use in this paper is from Border [5]. Border provides a set of linear inequalities that given the distribution over types, characterize the space of feasible interim allocation probabilities. In other words, they

[10] It will be clear from the proofs that these assumptions are not necessary.

characterize which interim allocation probabilities can be achieved by some feasible allocation rule. These inequalities simplify our problem significantly, since we now search over the (lower dimensional) space of interim allocation probabilities, rather than concerning ourselves with the allocation rule profile by profile. The Border inequalities state that a set of interim allocation probabilities $\{a(t)\}_{t \in T}$ is feasible if and only if the $a(t)$'s are non-negative:

$$\forall t \in T : a(t) \geq 0, \tag{5}$$

and:

$$\forall T' \subseteq T : \sum_{t \in T'} \pi(t)a(t) \leq \frac{1 - \left(\sum_{t \notin T'} \pi(t) \right)^N}{N}. \tag{6}$$

The left hand side of (6) is the expected probability the good is allocated to an agent with a type in T', which must be less than the probability that at least one agent has a type in T'.

Therefore, the problem of finding the revenue maximizing auction can be written as:

$$\max_{\{a(t),p(t)\}_{t \in T}} \sum_t \pi(t)p(t) \tag{RevOpt}$$

Subject to: (2-6).

Similarly, the problem of finding the constrained efficient auction can be written as:

$$\max_{\{a(t),p(t)\}_{t \in T}} \sum_t \pi(t)va(t) \tag{ConsEff}$$

Subject to: (2-6).

To orient the reader, we give an overview of the approach taken. First, by using a discrete type space, we are able to formulate the problem of finding the revenue maximizing auction as a linear program. At a high level, it has the following form:

$$
\begin{aligned}
Z = \max cx \\
\text{s.t.} \quad Cx \le d \\
Ax \le b \\
x \ge 0
\end{aligned}
$$

The first set of constraints, $Cx \le d$, corresponding to (2 - 4), are 'complicated'. The second set, $Ax \le b$, correspond to (6). We show that this set is 'easy' in the sense that A is an upper triangular matrix. Let

$$
\begin{aligned}
Z(\lambda) = \max cx + \lambda(d - Cx) \\
\text{s.t.} \quad Ax \le b \\
x \ge 0
\end{aligned}
$$

For each $\lambda \ge 0$, $Z(\lambda)$ is easy to compute because A is upper triangular. By the duality theorem of linear programming,

$$Z = \min_{\lambda \ge 0} Z(\lambda).$$

Thus our task reduces to identifying the non-negative λ that minimizes $Z(\lambda)$. Now, $Z(\lambda)$ is a piecewise linear function of λ with a finite number of breakpoints. We find an indirect way to enumerate the breakpoints without explicitly listing them. In this way we compute the value Z.

In the auction context, the coefficients of the x variables in the function $cx + \lambda(d - Ax)$ have an interpretation as 'virtual values'.

3 The Common Knowledge Budget Case

In this section, we analyze the case where all bidders have the same, commonly known budget. This helps us build intuition and familiarity with the proof methods used subsequently to analyze the general case. We examine the case of revenue maximization.

Since all bidders have the same budget constraint b, a bidder's type is just her valuation. Further, we can drop the characteristic function in the IC constraints since, by individual rationality, all types must have a payment of at most b. Given these simplifications, problem(RevOpt) becomes:

$$\max_{\{a(v),p(v)\}_{v\in V}} \sum f(v)p(v) \tag{RevOptCK}$$
$$\text{s.t. } p(v) \leq b \ \forall v$$
$$va(v) - p(v) \geq va(v') - p(v') \ \forall v, v'$$
$$va(v) - p(v) \geq 0 \ \forall v$$
$$\sum_{v\in V'} f(v)a(v) \leq \frac{1 - \left(\sum_{v\notin V'} f(v)\right)^N}{N} \ \forall V' \subseteq V$$
$$a(v) \geq 0 \ \forall v$$

First, add a 'dummy' type 0 to the space of types, and define $a(0) = p(0) = 0$. We can subsume the IR constraint, by requiring IC over the extended type space $V' = V \bigcup \{0\}$. Standard arguments imply that an allocation rule $a(\cdot)$ can be part of an incentive compatible mechanism if and only if $a(v)$ is non-decreasing in v. Further, the payment rule that maximizes revenue associated with this allocation rule is:

$$p(v) = va(v) - \sum_{1}^{v-1} a(v'). \tag{7}$$

Substituting (7) back into (RevOptCK), we can rewrite it as:

$$\max_{\{a(v)\}_{v\in V}} \sum f(v)\nu(v)a(v) \tag{OPT}$$
$$\text{s.t. } va(v) - \sum_{v'=1}^{v-1} a(v') \leq b \ \forall v \tag{8}$$
$$\sum_{v\in V'} f(v)a(v) \leq \frac{1 - \left(\sum_{v\notin V'} f(v)\right)^N}{N} \ \forall V' \subseteq V \tag{9}$$
$$a(v) - a(v+1) \leq 0 \ \forall v$$
$$a(v) \geq 0 \ \forall v$$

where $\nu(v) = v - \frac{1-F(v)}{f(v)}$ is type v's 'virtual valuation', as in Myerson [19].

Monotonicity of the allocation rule makes many of the constraints in (9) redundant.

Lemma 1 *If $a(\cdot)$ is monotonic, it is feasible if and only if, $\forall v \in V$:*

$$\sum_{v}^{m} f(v')a(v') \leq \frac{1 - F^N(v-1)}{N} \tag{10}$$

Proof: See Appendix B. □

For convenience we set $c_v = \frac{1-F^N(v-1)}{N}$.

The utility of the Border formulation follows from this simplification. The constraint matrix in (10) is upper triangular, which makes determining the structure of an optimal solution easy. In addition, a straightforward calculation shows that if $a(t)$ is the efficient allocation then all of the inequalities in (10) bind.

By inspection, $a(v+1) > a(v) \Rightarrow p(v+1) > p(v)$; $a(v+1) = a(v) \Rightarrow p(v+1) = p(v)$. Therefore, if the budget constraint (8) binds for some valuation $\bar{v}$, it must bind for all valuations $v > \bar{v}$. If the budget constraint does not bind in the optimal solution, the solution must be the same as Myerson's. Hence we assume the budget constraint binds in the optimal solution. We summarize this in the following observation.

Observation 1 If a^* is an optimal solution to (RevOptCK), the budget constraint must bind for some types $\{\bar{v}, \bar{v}+1, \ldots, m\}$. Further,

$$a^*(v) = a^*(\bar{v}) \qquad \forall v \geq \bar{v}.$$

Suppose the lowest type for which the budget constraint binds in the optimal solution a^* is $\bar{v}$. Substituting into program (OPT); and dropping the redundant Border constraints by Lemma 1, we conclude that a^* must be a solution to problem (RevOptCK):

$$\max_{\{a(v)\}_{v\in V}} \left(\sum_{1}^{\bar v-1} f(v)\nu(v)a(v)\right) + (1-F(\bar v-1))\bar v a(\bar v)$$

$$\text{s.t.} \quad -\sum_{1}^{\bar v-1} a(v') + \bar v a(\bar v) = b$$

$$\sum_{v}^{\bar v-1} f(v')a(v') + (1-F(\bar v-1))a(\bar v) \le c_v \;\; 1\le v\le \bar v$$

$$a(v) - a(v+1) \le 0 \;\forall v$$

$$a(v) \ge 0 \;\forall v$$

Denote the dual variable for the budget constraint by η, the dual variable for the Border constraint corresponding to type v by β_v and the dual variable for the monotonicity constraint corresponding to type v by μ_v. The dual program is:[11]

$$\min_{\eta,\{\beta_v\}_1^{\bar v},\{\mu_v\}_1^{\bar v-1}} b\eta + \sum_{1}^{\bar v} c_v\beta_v \tag{DOPT}$$

$$\bar v\eta + (1-F(\bar v-1))\sum_{1}^{\bar v}\beta_v - \mu_{\bar v-1} \ge (1-F(\bar v-1))\bar v \tag{$a(\bar v)$}$$

$$-\eta + f(v)\sum_{1}^{v}\beta_{v'} + \mu_v - \mu_{v-1} \ge f(v)\nu(v) \quad \forall v\le \bar v-1 \tag{$a(v)$}$$

$$\beta_v, \mu_v \ge 0$$

Let $\underline{v}$ be the lowest valuation for which $a^*(\underline{v}) > 0$. Complementary slackness

[11] The primal variables associated with each dual constraint is displayed in brackets next to the constraint.

implies that:

$$\bar{v}\eta + (1 - F(\bar{v}-1))\sum_{1}^{\bar{v}} \beta_v - \mu_{\bar{v}-1} = (1 - F(\bar{v}-1))\bar{v} \tag{11}$$

$$-\eta + f(v)\sum_{1}^{v} \beta_{v'} + \mu_v - \mu_{v-1} = f(v)\nu(v) \quad \forall \underline{v} \le v \le \bar{v}-1 \tag{12}$$

Re-writing (11-12) yields:

$$\sum_{1}^{\bar{v}} \beta_v - \frac{\mu_{\bar{v}-1}}{1 - F(\bar{v}-1)} = \bar{v} - \bar{v}\frac{\eta}{1 - F(\bar{v}-1)},$$

$$\sum_{1}^{v} \beta_{v'} + \frac{\mu_v}{f(v)} - \frac{\mu_{v-1}}{f(v)} = \nu(v) + \frac{\eta}{f(v)} \ : \forall \underline{v} \le v \le \bar{v}-1$$

Intuitively, these equations tell us that the 'correct' virtual valuation of a type v is $\nu(v) + \frac{\eta}{f(v)}$, where $\nu(v)$ is the Myersonian virtual valuation, and $\frac{\eta}{f(v)}$ corrects for the budget constraint: allocating to lower types reduces the payment of the high types, and hence 'relaxes' the budget constraint. As in Myerson, we require that the adjusted virtual valuation $\nu(v) + \frac{\eta}{f(v)}$ be increasing in v. A sufficient condition for this is that $f(v)$ is decreasing and satisfies the monotone hazard rate condition. By analogy with Myerson, the lowest type that will be allotted is the lowest type ($\underline{v}$) whose adjusted virtual valuation is non-negative. Finally, the optimal allocation rule will be efficient between types $\bar{v}-1$ and $\underline{v}$.

PROPOSITION 1 *Suppose $f(v)$ is decreasing in v, and $f(\cdot)$ satisfies the monotone hazard rate condition, i.e. $\frac{1-F(v)}{f(v)}$ is decreasing in v. Then the solution of (RevOptCK) can be described as follows: there will exist two cutoffs $\bar{v}$ and $\underline{v}$. No valuation less than $\underline{v}$ will be allotted. All types $\bar{v}$ and above will receive the same interim allocation probability, and the budget constraint will bind for exactly those types. The allocation rule will be efficient between types $\bar{v}-1$ and $\underline{v}$. Finally, $\underline{v}$ is the lowest type such that*

$$\nu(\underline{v}) + \frac{\eta}{f(\underline{v})} \ge 0,$$

where

$$\eta = \frac{(1 - F(\bar{v} - 1))(1 - F(\bar{v} - 2))}{\bar{v}f(\bar{v} - 1) + (1 - F(\bar{v} - 1))}.$$

If the sufficient conditions are not met, the optimal solution may require pooling in the middle.

PROOF: The proof proceeds by constructing dual variables that complement the primal solution described in the statement of the proposition.

Since $a^*(v) = 0$ for $v < \underline{v}$ and $f(v) > 0$ for all v, the corresponding Border constraints (9) do not bind at optimality. Therefore $\beta_v = 0$ for all $v < \underline{v}$. Further $0 = a^*(\underline{v} - 1) < a^*(\underline{v})$ by definition of $\underline{v}$, and so, by complementary slackness, $\mu_{\underline{v}-1} = 0$. Similarly, since $\bar{v}$ is the lowest type for which the budget constraint binds, $a^*(\bar{v}) > a^*(\bar{v} - 1)$, implying that $\mu_{\bar{v}-1} = 0$.

Subtracting the dual constraints corresponding to types $\bar{v}$ and $\bar{v} - 1$ and using the fact that $\mu_{\underline{v}-1} = 0$, we have:[12]

$$\beta_{\bar{v}} + \frac{\mu_{\bar{v}-2}}{f(\bar{v} - 1)} = \bar{v} - \bar{v}\frac{\eta}{1 - F(\bar{v} - 1)} - \nu(\bar{v} - 1) - \frac{\eta}{f(\bar{v} - 1)} \tag{13}$$

Subtracting the dual constraints corresponding to v and $v-1$, where $\underline{v}+1 \leq v \leq \bar{v} - 1$, we have:

$$\begin{aligned} \beta_v + \frac{\mu_v}{f(v)} - \frac{\mu_{v-1}}{f(v)} - \frac{\mu_{v-1}}{f(v-1)} + \frac{\mu_{v-2}}{f(v-1)} &= \nu(v) + \frac{\eta}{f(v)} \\ &\quad -\nu(v-1) - \frac{\eta}{f(v-1)} \end{aligned} \tag{14}$$

Finally, the dual constraint corresponding to type $\underline{v}$ reduces to:

$$\beta_{\underline{v}} + \frac{\mu_{\underline{v}}}{f(\underline{v})} = \nu(\underline{v}) + \frac{\eta}{f(\underline{v})} \tag{15}$$

It suffices to identify a non-negative solution to the system (13-15) such that $\beta_v = 0$ for all $v < \underline{v}$ and $\mu_{\underline{v}-1} = 0$.

[12]This step is where the upper triangular constraint matrix is helpful.

Consider the following solution.

$$\beta_{\bar{v}} = 0 \tag{16}$$

$$\beta_v = \nu(v) - \nu(v-1) + \eta\Big(\frac{1}{f(v)} - \frac{1}{f(v-1)}\Big) \quad \underline{v}+1 \leq v \leq \bar{v}-1 \tag{17}$$

$$\mu_{v-1} = 0 \quad \underline{v}+1 \leq v \leq \bar{v}-1 \tag{18}$$

$$\eta = \frac{(1-F(\bar{v}-1))(1-F(\bar{v}-2))}{\bar{v}f(\bar{v}-1)+(1-F(\bar{v}-1))} \tag{19}$$

Direct computation verifies that the given solution satisfies (13-15). In fact it is the unique solution to (13-15) with all μ's equal to zero. All variables are non-negative. In particular, β_v for $\underline{v}+1 \leq v \leq \bar{v}-1$ is positive. This is because $f(\cdot)$ satisfies the monotone hazard rate and decreasing density conditions, for any v, $\nu(v)-\nu(v-1)+\eta\big(\frac{1}{f(v)} - \frac{1}{f(v-1)}\big) > 0$. Furthermore, it complements the primal solution described in the statement of the proposition. This concludes the case where our regularity condition on the distribution of types (monotone hazard rate, decreasing density) are met.

Now suppose our sufficient condition is violated, i.e. $\nu(v) - \nu(v-1) + \eta\big(\frac{1}{f(v)} - \frac{1}{f(v-1)}\big) < 0$ for some v. The dual solution identified above will be infeasible since $\beta_v < 0$. More generally, there can be no dual solution that satisfies (13-15) with all $\mu_v = 0$. Hence, there must be at least one v between $\underline{v}$ and $\bar{v}-1$ such that $\mu_v > 0$. This implies, by complementary slackness, that the corresponding primal constraint, $a(v)-a(v+1) \leq 0$ binds at optimality, implying pooling. □

In fact one can further restrict the set of optimal dual solutions.

Lemma 2 *In any solution to the primal problem (OPT), at most one of the Border constraint (9) corresponding to type v, and the monotonicity constraint corresponding to type $v-1$ can bind. Further, by complementary slackness:*

$$\forall v: \qquad \beta_v \mu_{v-1} = 0 \tag{20}$$

Proof: See Appendix B. □

The solution to the system of equations (13- 15), (20) constitutes the optimal dual solution. It is easily seen that this solution is unique- therefore even in the case where ironing is required, there is a unique solution. Further the μ's in the solution are the 'ironing' multipliers a la Myerson.

We are also in a position to describe the constrained efficient auction for this setting. The proof is very similar to that of Proposition 1, and therefore omitted.

PROPOSITION 2 *Suppose $f(v)$ is decreasing in v. Then the constrained efficient auction in this settin can be described as follows: there will exist a cutoff $\bar{v}$. All types $\bar{v}$ and above will receive the same interim allocation probability, and the budget constraint will bind for exactly those types. The allocation rule will be efficient for types below $\bar{v}-1$. If the sufficient conditions are not met, the optimal solution may require pooling in the middle.*

4 The General Case

Recall the original program (RevOpt). We deal with the case where a bidder's valuation and budget are determined independently, and all budgets are equally likely, i.e. for any type $t = (v, b)$,

$$\pi(v, b) = \frac{1}{k} f(v).$$

$$\max \sum_{j=1}^{k} \sum_{v=1}^{m} \frac{1}{k} f(v) p(v, b_j) \tag{RevOpt}$$
$$\text{s.t.}\, va(v, b) - p(v, b) \geq 0$$
$$va(v, b) - p(v, b) \geq \chi\{p(v', b') \leq b\}[va(v', b') - p(v', b')]$$
$$p(v, b) \leq b$$
$$\sum_{t \in T'} \pi(t) a(t) \leq \frac{1 - (\sum_{t \notin T'} \pi(t))^N}{N}$$
$$a(t) \geq 0$$

The incentive compatibility constraints can be separated into 3 types:

1. Misreport of value only:

$$va(v,b) - p(v,b) \geq va(v',b) - p(v',b). \tag{21}$$

2. Misreport of budget only:

$$va(v,b) - p(v,b) \geq \chi\{p(v,b') \leq b\}[va(v,b') - p(v,b')]. \tag{22}$$

3. Misreport of both:

$$va(v,b) - p(v,b) \geq \chi\{p(v',b') \leq b\}[va(v',b') - p(v',b')]. \tag{23}$$

Standard arguments imply that the IC constraints corresponding to a misreport of value, (21), can be satisfied by some pricing rule if and only if $v \geq v'$ implies that $a(v,b) \geq a(v',b)$. Incentive compatibility and individual rationality imply

$$p(v,b) \leq va(v,b) - \sum_{1}^{v-1} a(v',b).$$

The difficulty stems from the IC constraints relating to misreport of budget, (22) and (23). In particular, we need (further) constraints on the allocation rule such that there exists an incentive compatible pricing rule. The following lemmata identify the space of interim allocations such that each type's payment is the maximum possible, i.e.

$$p(v,b) = va(v,b) - \sum_{1}^{v-1} a(v',b). \tag{24}$$

Lemma 3 *For any budget b, individual rationality can be satisfied if and only if:*

$$p(v,b) = b \quad \Rightarrow \quad a(v',b) = a(v,b) \quad \forall v' \geq v. \tag{25}$$

Proof: It is easy to see that for any v, b:

$$a(v+1,b) \geq a(v,b) \Rightarrow p(v+1,b) \geq p(v,b).$$

Further, by observation, (24) implies that:

$$\begin{aligned} a(v+1,b) > a(v,b) \quad &\Rightarrow \quad p(v+1,b) > p(v,b), \\ a(v+1,b) = a(v,b) \quad &\Rightarrow \quad p(v+1,b) = p(v,b). \end{aligned}$$

Equation (25) follows. □

Lemma 4 *Fix an allocation rule a such that a is incentive compatible and individually rational with pricing rule (24). Fix two budgets $b' > b$. Let $\underline{v}_b$ be the largest v such that $p(\underline{v}_b, b') \leq b$. Then*

$$a(v,b') = a(v,b) \quad \forall v \leq \underline{v}_b.$$

Further, $a(\underline{v}_b + 1, b') > a(m,b)$.

Proof: By assumption, $p(v,b') \leq b$ for any $v \leq \underline{v}_b$. By individual rationality, $p(v,b) \leq b$ for any v. Therefore the incentive compatibility constraints (22) corresponding to type (v,b) misreporting as (v,b') and type (v,b') misreporting as (v,b) for any $v \leq v_b$ imply that:

$$\begin{aligned} va(v,b) - p(v,b) = va(v,b') - p(v,b') \quad \forall v \leq \underline{v}_b \\ \Rightarrow \sum_{1}^{v-1} a(v',b) = \sum_{1}^{v-1} a(v',b') \quad \forall v \leq \underline{v}_b \\ \Rightarrow a(v,b) = a(v,b') \quad \forall v \leq \underline{v}_b - 1 \end{aligned}$$

To see that $a(\underline{v}_b, b) = a(\underline{v}_b, b')$, first consider the IC constraint corresponding to type $(\underline{v}_b + 1, b)$ misreporting as type $(\underline{v}_b, b')$. By assumption $p(\underline{v}_b, b') \leq b$, therefore we can drop the characteristic function and write the IC constraint as:

$$\begin{aligned} (\underline{v}_b + 1)a(\underline{v}_b + 1, b) - p(\underline{v}_b + 1, b) \geq (\underline{v}_b + 1)a(\underline{v}_b, b') - p(\underline{v}_b, b') \\ \Rightarrow \sum_{1}^{\underline{v}_b} a(v,b) \geq \sum_{1}^{\underline{v}_b} a(v,b') \\ \Rightarrow a(\underline{v}_b, b) \geq a(\underline{v}_b, b'). \end{aligned}$$

The last inequality follows since $\sum_1^{\underline{v}_b-1} a(v,b) = \sum_1^{\underline{v}_b-1} a(v,b')$. Similarly one can show that $a(\underline{v}_b, b) \leq a(\underline{v}_b, b')$.

Finally, we need to show that $a(\underline{v}_b+1, b') > a(m,b)$. By assumption,

$$\begin{aligned}
p(\underline{v}_b+1, b') &> b \geq p(m,b) \\
\Rightarrow (\underline{v}_b+1)a(\underline{v}_b+1,b') - \sum_1^{\underline{v}_b} a(v,b') &> ma(m,b') - \sum_1^{m-1} a(v,b) \\
\Rightarrow (\underline{v}_b+1)a(\underline{v}_b+1,b') &> ma(m,b') - \sum_{\underline{v}_b+1}^{m-1} a(v,b) \\
&> (\underline{v}_b+1)a(m,b),
\end{aligned}$$

where the last inequality follows from the fact that for any v, $a(v+1,b) \geq a(v,b)$. □

Lemma 3 shows that for each b_i there is a cutoff $\bar{v}_i \in V$, the lowest valuation such that $p(\bar{v}_i, b_i) = b_i$, and $a(v,b_i) = a(\bar{v}_i, b_i)$ for all $v \geq \bar{v}_i$. Lemma 4 shows that for each b_i there exists a cutoff $\underline{v}_i$, the highest valuation such that $p(\underline{v}_i, b_{i+1}) \leq b_i$; and that $a(v,b_i) = a(v,b_{i+1})$ for all $v \leq \underline{v}_i$. We summarize this in the following definition:

Definition 1 *Given an allocation rule a that is incentive compatible and individually rational with pricing rule (24), define cutoffs:*

$$\begin{aligned}
\bar{v}_i &= \arg\min\{v : p(v,b_i) = b_i\} \qquad \forall i \leq k, \\
\underline{v}_i &= \arg\max\{v : p(v,b_{i+1}) \leq b_i\} \qquad \forall i \leq k-1.
\end{aligned}$$

Note that it must be the case that $\underline{v}_i < \bar{v}_{i+1}$.

Further, define :

$$\begin{aligned}
\bar{V} &= \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}, \\
\underline{V} &= \{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_{k-1}\}.
\end{aligned}$$

Lemmas 3 and 4 imply:

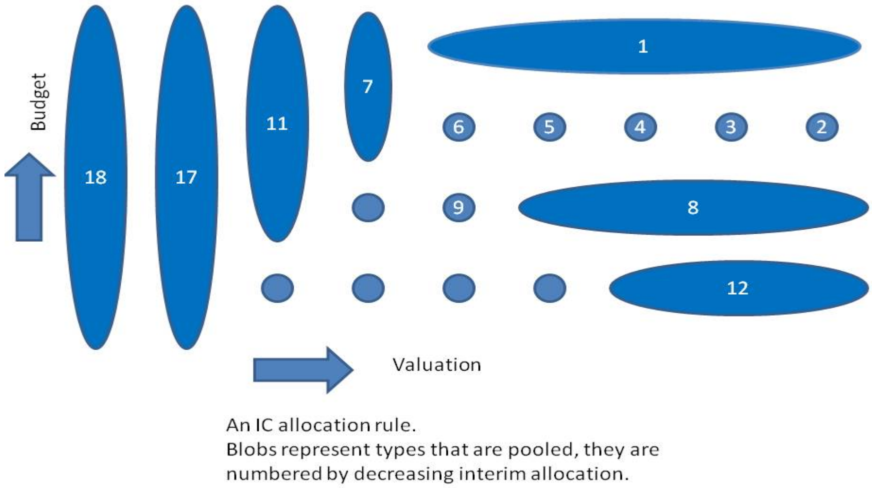

Figure 5

Observation 2 An allocation rule $a : T \to [0,1]$ is consistent with cutoffs $\bar{V} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ and $\underline{V} = \{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_{k-1}\}$, where $\underline{v}_i \leq \bar{v}_{i+1}$ for all i, and pricing rule (24), incentive compatible and individually rational if and only if:

$$a(v,b) \leq a(v+1,b) \quad \forall v, b \tag{26}$$

$$a(\bar{v}_i - 1, b_i) < a(\bar{v}_i, b_i) \quad \forall i \tag{27}$$

$$p(\bar{v}_i, b_i) = b_i \quad \forall i \tag{28}$$

$$a(v, b_i) = a(\bar{v}_i, b_i) \quad \forall i, \forall v \geq \bar{v}_i \tag{29}$$

$$a(v, b_i) = a(v, b_{i+1}) \quad \forall i, \forall v \leq \underline{v}_i \tag{30}$$

$$p(\underline{v}_i + 1, b_i + 1) > b_i \quad \forall i \tag{31}$$

$$a(\underline{v}_i + 1, b_i + 1) > a(m, b_i) \quad \forall i \tag{32}$$

Figure 5 depicts an incentive compatible, individually rational allocation rule for a type space with 10 possible valuations and 4 possible budgets.

Given a collection of cut-offs we describe how to find an allocation rule consistent with those cutoffs that maximizes revenue. By Observation 2 we can drop the individual rationality, budget, and incentive compatibility constraints in (RevOpt) and substitute instead (26-32). Therefore, we have:

$$\max_a \sum_{i=1}^{k} \sum_{v=1}^{m} \frac{f(v)}{k} \nu(v) a(v, b_i) \tag{REVOPT}$$

Subject to: (26-32), (5),(6).

To ensure a well defined program the strict inequalities in (27) and (32) have to be relaxed to a weak inequality. If for a given set of cutoffs, the optimal solution to (REVOPT) binds at inequality (27) or (32), we know that the set of cutoffs being considered cannot be feasible. Hence we can restrict attention to cut-offs where (the weak version of) the inequalities do not bind at optimality.

Recall that by Border [5] we know:

PROPOSITION 3 (BORDER) *Let $a : T \to [0, 1]$ be the interim probability of allocation for a type space T. For each $\alpha \in [0, 1]$, set*

$$E_\alpha = \{t : a(t) \geq \alpha\}.$$

Then a is feasible if and only if:

$$\sum_{t \in E_\alpha} a(t) f(t) \leq \frac{1 - \left(\sum_{T - E_\alpha} f(t) \right)^N}{N}. \tag{33}$$

Therefore, having fixed the cutoffs $\underline{V}$, by (26) and (32), most of the Border constraints are rendered redundant. In particular consider type (v, b_i); $v \leq \bar{v}_i$. Then, by Observation 2, $E_{(v,b_i)}$, the set of all types t such that $a(t) \geq a(v, b_i)$ is:

$$E_{(v,b_i)} = \bigcup_{j=i+1}^{k} \{(v', b_j) : v' \geq \min(v, \underline{v}_i + 1, \ldots, \underline{v}_{j-1} + 1)\} \bigcup \{(v', b_i) : v' \geq v\}.$$

It follows from Proposition 3 that the relevant Border constraints to be considered are:

$$\sum_{t \in E_{(v,b_i)}} a(t)f(t) \leq \frac{1 - \left(\sum_{T - E_{(v,b_i)}} f(t) \right)^N}{N}. \quad \forall i, v \leq \bar{v}_i \tag{34}$$

The next lemma further restricts the configurations of cutoffs in a revenue maximizing rule.

Lemma 5 *Let a^* solve (REVOPT). Then the cutoffs $\bar{V}, \underline{V}$ as defined in Definition 1 must satisfy:*

$$\underline{v}_i \geq \bar{v}_i - 1, \quad \forall i \leq k - 1.$$

A complete proof of Lemma 5 is in Appendix B.2. Here we give a sketch.

Suppose instead that in some profit maximizing allocation rule a; for some i, $\underline{v}_i < \bar{v}_i - 1$. We outline how to construct a rule a' with cutoff $\underline{v}'_i = \underline{v}_i + 1$ that achieves weakly more revenue. Since $\underline{v}_i < \bar{v}_i - 1$, $a(\underline{v}_i + 1, b_{i+1}) > a(v, b_i)$ for all v. Consider decreasing $a(\underline{v}_i + 1, b_{i+1})$ by δ and increasing each $a(v, b_i), v \geq \bar{v}_i$ by δ'. If $\delta f(\underline{v}_i + 1) = \delta'(1 - F(\underline{v}_i))$, we will maintain feasibility with respect to the Border conditions. Pick δ such that $a(\underline{v}_i + 1, b_i) = a(\underline{v}_i + 1, b_{i+1}) - \delta$. This modified allocation rule corresponds to the cutoff $\underline{v}'_i = \underline{v}_i + 1$. The net change in revenue is $(\bar{v}_i - \nu(\underline{v}_i + 1))\delta \frac{f(\underline{v}_i + 1)}{k}$, which is clearly non-negative. However this simple procedure will violate the budget constraints. Appendix B.2 shows that there exists a similar revenue increasing construction such that the resulting rule is feasible in the optimization program (REVOPT).

With this added restriction on cutoffs; the set of incentive compatible and individually rational rules are summarized in Observation 3. Since $\underline{v}_i \geq \bar{v}_i - 1$, (31) and (32) are satisfied automatically; $\underline{v}_i \equiv \arg\min\{v : a(v + 1, b_{i+1}) > a(\bar{v}_i, b_i)\}$.

Lemma 5 further implies that the budget constraint corresponding to budget b_i can bind in an optimal solution only if the budget constraints corresponding to each $b_j < b_i$ bind. Therefore, in any optimal solution, there must be a largest budget b_i such that the budget constraints corresponding to $b \leq b_i$ bind, and the budget constraints corresponding to $b > b_i$ are slack.

For notational simplicity we assume that in the optimal solution, all budget constraints bind.

Observation 3 An allocation rule $a : T \to [0,1]$, consistent with the cutoffs $\bar{V} = \{\bar{v}_1 \le \bar{v}_2 \le \ldots \le \bar{v}_k\}$ and pricing rule (24) is incentive compatible and individually rational if and only if there exist, $x : V \to [0,1]$ and $y : \bar{V} \to [0,1]$ such that:

$$a(v, b_i) = y(\bar{v}_i) \quad \forall i \le k, v \ge \bar{v}_i, \tag{35}$$

$$a(v, b_j) = x(v) \quad \forall i \le k, \bar{v}_{i-1} \le v \le \bar{v}_i - 1, j \ge i, \tag{36}$$

$$\bar{v}_i y(\bar{v}_i) - \sum_{1}^{\bar{v}_i - 1} x(v) = b_i, \quad \forall i \le k, \tag{37}$$

$$x(v) \le x(v+1) \quad \forall v, \tag{38}$$

$$x(\bar{v}_i - 1) < y(\bar{v}_i) \quad \forall i \le k, \tag{39}$$

$$y(\bar{v}_i) \le x(\bar{v}_i) \quad \forall i < k. \tag{40}$$

Figure 6 displays an incentive compatible, individually rational allocation rule whose cutoffs satisfy Lemma 5.

Substituting (35) and (36) into (34), the Border constraints to be considered are:

$$\sum_{v'=v}^{\bar{v}_i - 1} \frac{k-i+1}{k} f(v')x(v') + \sum_{j=i+1}^{k} \sum_{\bar{v}_{j-1}}^{\bar{v}_j - 1} \frac{k-j+1}{k} f(v')x(v') + \sum_{j=i}^{k} \frac{(1 - F(\bar{v}_j - 1)}{k} y(\bar{v}_j) \le c_v \quad \forall i \le k, \bar{v}_{i-1} + 1 \le v \le \bar{v}_i, \tag{41}$$

$$\sum_{j=i+1}^{k} \frac{(1 - F(\bar{v}_j - 1)}{k} y(\bar{v}_j) + \sum_{j=i+1}^{k} \sum_{\bar{v}_{j-1}}^{\bar{v}_j - 1} \frac{k-j+1}{k} f(v)x(v) \le \frac{1 - (1 - \frac{k-i}{k}(1 - F(\bar{v}_i - 1)))^N}{N} \quad \forall i \le k. \tag{42}$$

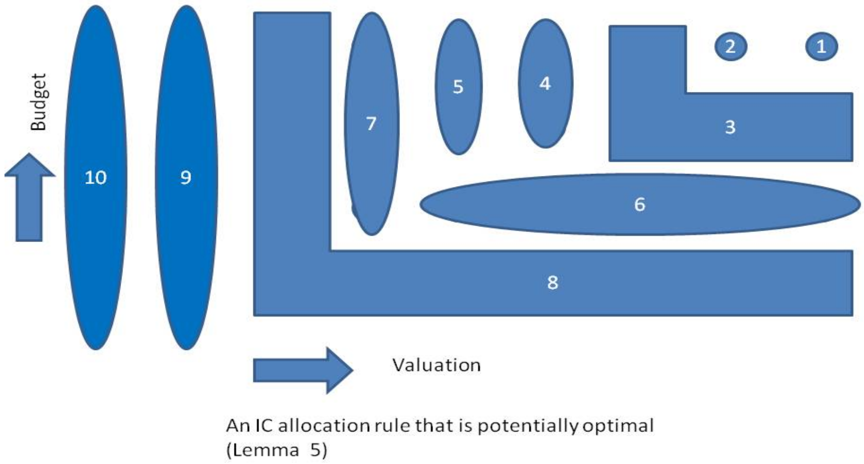

FIGURE 6

where the c's are the right hand side of the appropriate Border inequality (34), i.e.

$$c_v = \frac{1 - (1 - \frac{k-i}{k}(1 - F(\bar{v}_i - 1)) - \frac{1}{k}(1 - F(v-1)))^N}{N} \quad (\bar{v}_{i-1} + 1) \leq v \leq \bar{v}_i.$$

Making the appropriate substitutions, (REVOPT) becomes:

$$\max_{x,y} \sum_{j=1}^{k} \sum_{\bar{v}_{j-1}}^{\bar{v}_j - 1} \frac{k-j+1}{k} f(v)\nu(v)x(v) + \frac{\sum_{i=1}^{k} \bar{v}_i(1 - F(\bar{v}_i - 1))y(\bar{v}_i)}{k} \tag{REVOPT2}$$

Subject to: (5), (37-42).

As before, we conjecture an optimal solution and verify optimality with a suitably chosen dual solution. Hence we flip to the dual and examine its properties.

Let η_i be the dual variable associated with the budget constraint (37). Since we assume constraint (39) does not bind at optimality, the corresponding dual variable will be 0, and therefore is dropped. Let μ_v be the dual variable associated with the monotonicity constraint (38), and $\bar{\mu}_{\bar{v}_i}$ the dual variable associated with the constraint (40). Denote by β_v the dual variable associated with (41), and $\bar{\beta}_{\bar{v}_i}$, the dual variable associated with (42). The dual program is:

$$\min_{\eta,\mu,\beta} \sum_{i=1}^{k} b_i \eta_i + \sum_{v=1}^{\bar{v}_k} c_v \beta_v + \sum_{i=1}^{k} \bar{c}_{\bar{v}_i} \bar{\beta}_{\bar{v}_i} \tag{DOPT2}$$

$$-\sum_{j=i}^{k} \eta_j + \frac{k-i+1}{k} f(v)(\sum_{1}^{v} \beta_v + \sum_{j=1}^{i-1} \bar{\beta}_{\bar{v}_j}) \geq \frac{k-i+1}{k} f(v)\nu(v), \tag{43}$$

$$+\mu_v - \mu_{v-1} \qquad \forall i \leq k, (\bar{v}_{i-1}+1) \leq v \leq (\bar{v}_i - 1)$$

$$-\sum_{j=i+1}^{k} \eta_j + \frac{k-i}{k} f(v)(\sum_{1}^{\bar{v}_i} \beta_v + \sum_{j=1}^{i} \bar{\beta}_{\bar{v}_j}) \geq \frac{k-i}{k} f(\bar{v}_i)\nu(\bar{v}_i), \tag{44}$$

$$+\mu_{\bar{v}_i} - \bar{\mu}_{\bar{v}_i} \qquad \forall i \leq k$$

$$\bar{v}_i \eta_i + \frac{(1-F(\bar{v}_i - 1))}{k}((\sum_{1}^{\bar{v}_i - 1} \beta_v + \sum_{j=1}^{i} \bar{\beta}_{\bar{v}_j}) \geq \frac{(1-F(\bar{v}_i - 1))}{k} \bar{v}_i, \tag{45}$$

$$+\bar{\mu}_{\bar{v}_i} \qquad \forall i \leq k$$

$$\eta, \beta, \mu \geq 0,$$

Here, (43) is the dual inequality corresponding to primal variable $x(v)$ where $(\bar{v}_{i-1}+1) \leq v \leq (\bar{v}_i - 1)$, (44) the dual inequality corresponding to $x(\bar{v}_i)$ and (45) the dual inequality corresponding to $y(\bar{v}_i)$. Fix an optimal primal solution(x^*, y^*) and let $\underline{v}$ be the lowest valuation which gets allotted in that solution. Therefore any type (v, b) where $v \geq \underline{v}$ gets allotted. It is easy to see that $\underline{v} \leq \bar{v}_1$. Complementary slackness implies that the inequalities (43) bind for all $v \geq \underline{v}$, as do (44, 45) for all i. Rewriting (43-45) as in the common knowledge case:

$$(\sum_{1}^{v}\beta_v+\sum_{j=1}^{i-1}\bar{\beta}_{\bar{v}_j})+\frac{k(\mu_v-\mu_{v-1})}{(k-i+1)f(v)}=\nu(v)+\frac{k\sum_{j=i}^{k}\eta_j}{(k-i+1)f(v)}, \tag{46}$$

$$(\sum_{1}^{\bar{v}_i}\beta_v+\sum_{j=1}^{i}\bar{\beta}_{\bar{v}_j})+\frac{k(\mu_{\bar{v}_i}-\bar{\mu}_{\bar{v}_i})}{(k-i)f(\bar{v}_i)}=\nu(\bar{v}_i)+\frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(\bar{v}_i)}, \tag{47}$$

$$(\sum_{1}^{\bar{v}_i-1}\beta_v+\sum_{j=1}^{i}\bar{\beta}_{\bar{v}_j})+\frac{k\bar{\mu}_{\bar{v}_i}}{(1-F(\bar{v}_i-1))}=\bar{v}_i-\frac{k\bar{v}_i\eta_i}{(1-F(\bar{v}_i-1))} \tag{48}$$

Subtracting the equation (46) corresponding to $v-1$ from the equation corresponding to v for $\bar{v}_{i-1}+2\leq v\leq\bar{v}_i-1$, we have:

$$\beta_v+\frac{k(\mu_v-\mu_{v-1})}{(k-i+1)f(v)}-\frac{k(\mu_{v-1}-\mu_{v-2})}{(k-i+1)f(v-1)}=\nu(v)-\nu(v-1)+\frac{k\sum_{j=i}^{k}\eta_j}{k-i+1}(\frac{1}{f(v)}-\frac{1}{f(v-1)}). \tag{49}$$

Subtracting the equation (47) corresponding to $\bar{v}_i$ from equation (46) corresponding to $\bar{v}_i+1$, we have:

$$\beta_{\bar{v}_i+1}+\frac{k(\mu_{\bar{v}_i+1}-\mu_{\bar{v}_i})}{(k-i+)f(\bar{v}_i+1)}-\frac{k(\mu_{\bar{v}_i}-\bar{\mu}_{\bar{v}_i})}{(k-i)f(\bar{v}_i)}=\nu(\bar{v}_i+1)-\nu(\bar{v}_i)+\frac{k\sum_{j=i}^{k}\eta_j}{k-i}(\frac{1}{f(\bar{v}_i+1)}-\frac{1}{f(\bar{v}_i)}).$$

Similarly, subtracting the equation (48) corresponding to $\bar{v}_i$ from (47) corresponding to $\bar{v}_i$ we have:

$$\beta_{\bar{v}_i}+\frac{k(\mu_{\bar{v}_i}-\bar{\mu}_{\bar{v}_i})}{(k-i)f(\bar{v}_i)}-\frac{k\bar{\mu}_{\bar{v}_i}}{(1-F(\bar{v}_i-1))}=\nu(\bar{v}_i)-\bar{v}_i+\frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(\bar{v}_i)}+\frac{k\bar{v}_i\eta_i}{(1-F(\bar{v}_i-1))} \tag{50}$$

Finally, subtracting (46) corresponding to $\bar{v}_i-1$ from (48) corresponding to $\bar{v}_i$, we have:

$$\bar{\beta}_{\bar{v}_i}+\frac{k\bar{\mu}_{\bar{v}_i}}{(1-F(\bar{v}_i-1))}-\frac{k\mu_{\bar{v}_i-2}}{(k-i)f(\bar{v}_i-1)}=\bar{v}_i-\nu(\bar{v}_i-1)-\frac{k\bar{v}_i\eta_i}{(1-F(\bar{v}_i-1))}-\frac{k\sum_{j=i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i-1)} \tag{51}$$

If the optimal solution a^* is strictly monotone, the inequalities (38-40) do not bind. Complementary slackness implies all the μ's are 0. As in the common knowledge budget case we set $\bar{\beta}_{\bar{v}_i} = 0$ for all i since this will satisfy complementary slackness. Therefore, from (51), we have that:

$$\eta_k = \frac{1}{k} \frac{(1 - F(\bar{v}_k - 1))(1 - F(\bar{v}_k - 2))}{\bar{v}_k f(\bar{v}_k - 1) + (1 - F(\bar{v}_k - 1))},$$
$$\eta_i = \frac{1}{k} \frac{(1 - F(\bar{v}_i - 1))(1 - F(\bar{v}_i - 2))}{(k - i + 1)\bar{v}_i f(\bar{v}_i - 1) + (1 - F(\bar{v}_i - 1))} - \frac{(1 - F(\bar{v}_i - 1)) \sum_{i+1}^{k} \eta_j}{(k - i + 1)\bar{v}_i f(\bar{v}_i - 1) + (1 - F(\bar{v}_i - 1))}$$

It is easily verified that the η's as specified are non-negative and therefore dual feasible. Further, one can show that $i \leq j \Rightarrow \eta_i \geq \eta_j$, in other words, as one would suspect, smaller budgets have larger shadow prices. Substituting into (49) we have, $\forall v : \bar{v}_{i-1} < v < \bar{v}_i$,

$$\beta_v = \nu(v) - \nu(v - 1) + \frac{k \sum_{j=i}^{k} \eta_j}{k - i + 1} \left(\frac{1}{f(v)} - \frac{1}{f(v - 1)}\right).$$

Note that β_v for all v such that $\bar{v}_{i-1} < v < \bar{v}_i$ will be positive if f is decreasing. Finally, substituting the η's into (50), we have:

$$\beta_{\bar{v}_i} = \nu(\bar{v}_i) - \nu(\bar{v}_i - 1) + \frac{k \sum_{i+1}^{k} \eta_j}{(k - i) f(\bar{v}_i)} - \frac{k \sum_{i}^{k} \eta_j}{(k - i + 1) f(\bar{v}_i - 1)}.$$

Observe that $\beta_{\bar{v}_i}$ can be negative. The adjusted virtual value of valuation $\bar{v}_i$ is $\nu(\bar{v}_i) + \frac{k \sum_{i+1}^{k} \eta_j}{(k-i) f(\bar{v}_i)}$ which may be larger than the adjusted virtual valuation of $\nu(\bar{v}_i - 1) + \frac{k \sum_{i}^{k} \eta_j}{(k-i+1) f(\bar{v}_i - 1)}$ even if f is decreasing and satisfies the monotone hazard rate. This is because allocating to valuation $\bar{v}_i - 1$ also 'relaxes' the budget constraint corresponding to b_i (in addition to the budget constraints for larger budgets), which allocating to $\bar{v}_i$ does not.

In this instance, therefore, the allocation rule for the revenue maximizing rule will require ironing. As described in the introduction, for each budget b_i there will be an additional cutoff $\underline{v}_i$. Types (v, b) where $\bar{v}_i \leq v \leq \underline{v}_i$ and $b > b_i$ will be pooled with the types (v, b_i), $v \geq \bar{v}_i$ (i.e. the types for whom the budget constraint binds). The allocation rule will be efficient between $\underline{v}_i$

and $\bar{v}_{i+1}$.

Finally, the lowest valuation to be allotted will be $\underline{v}_0$, which is the lowest valuation whose adjusted virtual valuation is non-negative. To summarize:

Proposition 4 *Suppose $f(v)$ is decreasing in v, and $\frac{1-F(v)}{f(v)}$ is increasing in v. Then, there is an optimal solution $a^*(v,b)$ to (RevOpt) that can be described as follows: there will exist cutoffs $\bar{v}_1 \leq \underline{v}_1 \leq \bar{v}_2 \leq \ldots \underline{v}_{k-1} \leq \bar{v}_k$ and $\underline{v}_0$. No valuation less than $\underline{v}_0$ will be allotted. The allocation rule will satisfy (35-40). The allocation will be efficient between each $\underline{v}_i$ and $\bar{v}_{i+1}$. Further, for all $b > b_i$ and $\bar{v}_i \leq v \leq \underline{v}_i$, $a^*(v,b) = y(\bar{v}_i)$. If the sufficient conditions are not met, the optimal solution may require additional pooling in the middle.*

Proof: As before, our proof proceeds by constructing a dual solution that complements the primal solution described in the statement of the proposition. Since $a^*(v,b) = 0$ for all $v < \underline{v}_0$, the corresponding Border constraints must be slack, and therefore $\beta_v = 0$ for all $v < \underline{v}_0$. Since $x^*(\underline{v}_i + 1) > y^*(\bar{v}_i)$, $\mu_{\underline{v}} = 0$.

The β_v for $\underline{v}_i + 2 \leq v \leq \bar{v}_{i+1} - 1$ is as specified in (49), with the corresponding μ's set to 0. By Lemma 2, β_v for $\bar{v}_i \leq v \leq \underline{v}_i$ is 0 since, by the statement of the proposition, $a^*(v,b) = y(\bar{v}_i)$ for all $b > b_i$. The relevant μ's can be calculated from the relevant equations.

Instead of computing these μ's, we can instead suppose that the types which have been ironed, $\{(v,b_i) \text{ for } v \geq \bar{v}_i\} \bigcup \{(v,b_j) \text{ for } j > i, \bar{v}_i \leq v \leq \underline{v}_i\}$, all correspond to one 'artificial' type, t_i. The probability of t_i is

$$\pi(t_i) = \frac{(k-i)}{k}(F(\underline{v}_i) - F(\bar{v}_i - 1)) + \frac{1}{k}(1 - F(\bar{v}_i - 1)).$$

Further, its adjusted virtual valuation is:

$$\nu(t_i) = \bar{v}_i - \frac{(k-i)(\underline{v}_i - \bar{v}_i + 1)(1 - F(\underline{v}_i))}{k\pi(t_i)} + \frac{(\underline{v}_i - \bar{v}_i + 1)\sum_{j=i+1}^{k}\eta_j}{\pi(t_i)} - \frac{\bar{v}_i\eta_i}{\pi(t_i)}$$

Since the budget constraint for budget b_i binds at $\bar{v}_i$, analogous to the proof of Proposition 1, $\bar{\beta}_{\bar{v}_i}$ is 0, and therefore we can solve for η_i from:

$$\nu(t_i) - \nu(\bar{v}_i - 1) - \frac{k\sum_{j=i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)} = 0. \tag{52}$$

Note that the adjusted virtual valuation of $\bar{v}_i - 1$ can be written as:

$$\nu(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)} + \frac{k\eta_i}{(k-i+1)f(\bar{v}_i - 1)}.$$

To see that the η_i that solves (52) is positive, we need to show that:

$$\nu(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)} < \bar{v}_i - \frac{(k-i)(1-F(\underline{v}_i))}{k\pi(t_i)} + \frac{(\underline{v}_i - \bar{v}_i + 1)\sum_{j=i+1}^{k}\eta_j}{\pi(t_i)} \tag{53}$$

$$= \frac{1}{\pi(t_i)}(\bar{v}_i\frac{(1-F(\bar{v}_i - 1))}{k} + \sum_{v=\bar{v}_i}^{\underline{v}_i}(\frac{(k-i)f(v)}{k}\nu(v) + \sum_{j=i+1}^{k}\lambda_j))$$

However,

$$\nu(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)} < \bar{v}_i,$$

since $\sum_{j=i+1}^{k}\eta_j$ is appropriately small (see Proposition 6). Further,

$$\nu(\bar{v}_i - 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)} < \nu(v) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(v)},$$

$\bar{v}_i \leq v \leq \underline{v}_i$, follows from the monotone hazard rate and decreasing density conditions. (53) follows since the right hand side of (53) is a weighted average of the right hand side of the latter two inequalities.

Further, we have that:

$$\beta_{\underline{v}_i+1} = \nu(\underline{v}_i + 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(\underline{v}_i + 1)} - \nu(t_i). \tag{54}$$

To ensure that $\beta_{\underline{v}_i+1} \geq 0$ it suffices by inequality (52) that cutoffs $\bar{v}_i$ and $\underline{v}_i$

satisfy:

$$\nu(\underline{v}_i + 1) + \frac{k\sum_{j=i+1}^{k}\eta_j}{(k-i)f(\underline{v}_i + 1)} \geq \nu(\bar{v}_i - 1) + \frac{k\sum_{i}^{k}\eta_j}{(k-i+1)f(\bar{v}_i - 1)}.$$

Finally, note that $\underline{v}_0$ will be the lowest valuation such that $\nu(v) + \frac{\sum_1^k \eta_j}{f(v)} \geq 0$; and

$$\beta_{\underline{v}_0} = \nu(v) + \frac{\sum_1^k \eta_j}{f(v)}.$$

The partial solution identified above, with all other dual variables set to 0, is an optimal dual solution. Since $\beta_v > 0$ for all $\underline{v}_i + 1 \leq v \leq \bar{v}_{i+1} - 1$, by complementary slackness, the corresponding Border constraints (9) bind. This concludes the case where our regularity condition on the distribution of types (monotone hazard rate, decreasing density) are met. If the monotone hazard rate or decreasing density assumptions are not satisfied then the dual solution identified may be infeasible, and therefore additional pooling will be required due to Lemma 2. □

We can also describe the constrained efficient auction for this setting. The proof is similar, and omitted.

PROPOSITION 5 *Suppose $f(v)$ is decreasing in v. Then the solution of (RevOpt) can be described as follows: there will exist cutoffs $\bar{v}_1 \leq \underline{v}_1 \leq \bar{v}_2 \leq \ldots \underline{v}_{k-1} \leq \bar{v}_k$ and $\underline{v}_0 = 0$. The allocation rule will satisfy (35-40).The allocation will be efficient between each $\underline{v}_i$ and $\bar{v}_{i+1}$. Further, for all $b > b_i$ and $\bar{v}_i \leq v \leq \underline{v}_i$, $a(v, b) = y(\bar{v}_i)$. If the sufficient conditions are not met, the optimal solution may require additional pooling in the middle.*

5 Subsidies

Since budget constrained bidders are unable to effectively compete in the auction, this will depress auction revenues. To get around this problem, prior work has examined various kinds of subsidies (lump sum transfer, discounts) and their effect in a particular auction setting.

In our setting, there is only one possible (incentive compatible) means of subsidy- a lump sum transfer from the auctioneer to the agents. This is because, given an allocation rule, incentive compatibility determines prices up to a constant:

$$p(v,b) = va(v,b) - \sum_{1}^{v-1} a(v',b) + c.$$

Let us consider a subsidy via lump sum payment of some amount ϵ. This costs the auctioneer ϵ per agent. The effect of this subsidy is to relax the budget constraints by ϵ. Therefore the gain in revenue is (at most) $\epsilon \sum_i \eta_i$.[13] We show below that $\sum_i \eta_i \leq 1$, and thus $\epsilon \sum_i \eta_i \leq \epsilon$. As a result, if the auctioneer were running the optimal auction, he should not offer subsidies. This result remains true even when bidders' budgets are common knowledge.

PROPOSITION 6 *For all i,*

$$\sum_{i}^{k} \eta_j \leq \frac{(k-i+1)}{k}(1 - F(\bar{v}_i - 1)). \tag{55}$$

PROOF: We prove by induction on i. For $i = k$, we know that

$$\begin{aligned}
\eta_k &= \frac{1}{k} \frac{(1 - F(\bar{v}_k - 1))(1 - F(\bar{v}_k - 2))}{\bar{v}_k f(\bar{v}_k - 1) + (1 - F(\bar{v}_k - 1))} \\
&= \frac{(1 - F(\bar{v}_k - 1))}{k} \frac{(1 - F(\bar{v}_k - 2))}{(\bar{v}_k - 1) f(\bar{v}_k - 1) + (1 - F(\bar{v}_k - 2))} \\
&\leq \frac{(1 - F(\bar{v}_k - 1))}{k}.
\end{aligned}$$

For the induction hypothesis, assume that

$$\sum_{i+1}^{k} \eta_j \leq \frac{(k-i)}{k}(1 - F(\bar{v}_{i+1} - 1)).$$

Therefore we are left to show (55).

[13] Recall that η_i is the shadow price of the budget constraint.

Recall from the proof of Proposition 4 that at optimality, η_i, the dual variable corresponding to the budget constraint corresponding to b_i, solves

$$\nu(t_i) - \nu(\bar{v}_i - 1) - \frac{k \sum_{j=i}^{k} \eta_j}{(k - i + 1) f(\bar{v}_i - 1)} = 0,$$

where

$$\nu(t_i) = \bar{v}_i - \frac{(\underline{v}_i - \bar{v}_i + 1)}{\pi(t_i)} \left(\frac{k - i}{k} (1 - F(\underline{v}_i)) - \sum_{i+1}^{k} \eta_j \right) - \frac{\bar{v}_i}{\pi(t_i)} \eta_i,$$

and,

$$\pi(t_i) = \frac{1}{k}(1 - F(\bar{v}_i - 1)) + \frac{k - 1}{k}(F(\underline{v}_i) - F(\bar{v}_i - 1)).$$

By the induction hypothesis,

$$\nu(t_i) \geq \bar{v}_i - \frac{\bar{v}_i}{\pi(t_i)} \eta_i,$$

and therefore

$$\begin{aligned} \bar{v}_i - \nu(\bar{v}_i - 1) &\geq \frac{\bar{v}_i}{\pi(t_i)} \eta_i + \frac{k \sum_{j=i}^{k} \eta_j}{(k - i + 1) f(\bar{v}_i - 1)} \\ \Rightarrow \frac{1 - F(\bar{v}_i - 2)}{f(\bar{v}_i - 1)} &\geq \frac{\bar{v}_i}{\pi(t_i)} \eta_i + \frac{k \sum_{j=i}^{k} \eta_j}{(k - i + 1) f(\bar{v}_i - 1)}. \end{aligned}$$

Rearranging terms, we have

$$\frac{k - i + 1}{k} \frac{(1 - F(\bar{v}_i - 2))\pi(t_i) + \bar{v}_i f(\bar{v}_i - 1) \sum_{i+1}^{k} \eta_j}{\frac{k-i+1}{k} \bar{v}_i f(\bar{v}_i - 1) + \pi(t_i)} \geq \sum_{i}^{k} \eta_j. \tag{56}$$

Once again, by the induction hypothesis,

$$\sum_{i+1}^{k} \eta_j \leq \frac{k - i}{k}(1 - F(\underline{v}_i)) = \frac{k - i + 1}{k}\big(1 - F(\bar{v}_i - 1)\big) - \pi(t_i). \tag{57}$$

Substituting (57) into (56),

$$\sum_i^k \eta_j \leq \left(\frac{k-i+1}{k}\right) \frac{(1-F(\bar{v}_i-2))\pi(t_i) + \bar{v}_i f(\bar{v}_i-1)\left(\frac{k-i+1}{k}(1-F(\bar{v}_i-1)) - \pi(t_i)\right)}{\frac{k-i+1}{k}\bar{v}_i f(\bar{v}_i-1) + \pi(t_i)}$$
$$\equiv \phi(\bar{v}_i, \pi(t_i)).$$

Observation 4 in Appendix B shows that $\phi(\cdot)$ is decreasing in its second argument. Given $\bar{v}_i$, the lowest possible value for $\pi(t_i)$ is $\frac{1}{k}(1-F(\bar{v}_i-1))$, at which the left hand side of the bound will be maximized. Therefore, substituting $\pi(t_i) = \frac{1}{k}(1-F(\bar{v}_i-1))$,

$$\sum_i^k \eta_j \leq \frac{k-i+1}{k} \frac{(1-F(\bar{v}_i-2))(1-F(\bar{v}_i-1)) + (k-i)\bar{v}_i f(\bar{v}_i-1)(1-F(\bar{v}_i-1))}{(k-i+1)\bar{v}_i f(\bar{v}_i-1) + (1-F(\bar{v}_i-1))}$$
$$\leq \frac{k-i+1}{k}(1-F(\bar{v}_i-1))$$

□

How then does the optimal auction encourage competition? Recall that for each i, types $\{(v, b_i) \text{ for } v \geq \bar{v}_i\} \bigcup \{(v, b_j) \text{ for } j > i, \bar{v}_i \leq v \leq \underline{v}_i\}$ are pooled. The pooling serves to allot the good to disadvantaged bidder types (v, b_i), $v \geq \bar{v}_i$ even in profiles where there are bidders with higher valuations and budgets present. Intuitively, favoring bidders in this way is better than lump-sum transfers because there are more degrees of freedom: a lump-sum transfer must be given to a bidder regardless of his type in order to maintain incentive compatibility.

Further, it should be clear that the revenue achieved by the optimal auction cannot be matched by a simple auction with subsidies provided via lump sum transfers and discounts studied in the literature. The optimal auction can be implemented as an all pay auction with a modified rule to select the winner. In a standard all pay auction, the highest bidder wins the good, subject to this bid being larger than the reserve price. Here, there will be thresholds, i.e. the highest bidder may need to out-bid the next highest bidder by a margin in order to win the good outright. In the event that she does not, the auctioneer selects the winner randomly.

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A Counterexamples

A.1 Laffont and Robert's Solution

In this section, we examine the classical formulation, with a continuum of types. We show by counterexample that the Laffont and Robert solution is not optimal for all distributions that meet the monotone hazard rate condition.

Suppose that, as in the original Laffont and Robert paper (herein LR), we have valuations belonging to the continuum, say interval $[0,1]$; distributed with density $f(v)$, $F(v) = \int_0^v f(v)dv$. Their virtual valuation is defined as $\nu(v) = v - \frac{1-F(v)}{f(v)}$.

Further, suppose we have (as per their solution) 2 cutoffs, v_1,v_2. The allocation rule does not allot types below v_1; and pools all types above v_2. This will make the allocation rule:

$$a(v) = \begin{cases} \frac{1-F^N(v_2)}{N(1-F(v_2))} & v \geq v_2 \\ F^{N-1}(v) & v \in [v_1, v_2] \\ 0 & o.w. \end{cases} \tag{58}$$

At the optimal solution, the budget constraint must bind for all types v_2 and above, [14]

$$v_2 a(v_2) - \int_{v_1}^{v_2} a(v)dv = b. \tag{59}$$

Choose v_1 and v_2 to solve:

$$\begin{aligned} \max_{v_1,v_2} a(v_2)(1 - F(v_2))v_2 + \int_{v_1}^{v_2} \nu(v)f(v)a(v)dv \\ \text{s.t. } v_2 a(v_2) - \int_{v_1}^{v^2} a(v)dv &= b \end{aligned}$$

[14] If not, the solution of the overall program would be the same as Myerson's solution [19].

The first order conditions for optimality imply

$$f(v_1)\nu(v_1) + \frac{(1-F(v_2))^2}{(1-F(v_2)) + v_2 f(v_2)} = 0. \tag{60}$$

Therefore v_1 and v_2 are the solutions to (60) and the budget equation (59). Further note that this is the L-R solution. We are now in a position to state without proof the 'correct' version of Laffont-Robert's theorem.

THEOREM 1 *Suppose the distribution on types is such that the density is decreasing. Further, suppose that the monotone hazard rate condition is met. The allocation described by (58), where v_1, v_2 jointly satisfy (59) and (60), and the associated pricing rule*

$$p(v) = v.a(v) - \int_{\underline{v}}^{v} a(v)dv,$$

constitute the expected revenue maximizing mechanism.

A proof of this theorem requires taking the dual of an infinite dimensional linear program (see for example Anderson and Nash [1]), and defining the appropriate measure on the dual space. We can now use the intuition gleaned from Section 3 to identify a flaw in the L-R solution in the event that densities are not decreasing. Pick a $v_3 \in (v_1, v_2)$; and 'iron' some small interval of types $[v_3, v_3+\epsilon]$. The new allotment rule is therefore:

$$a'(v) = \begin{cases} \frac{1-F^N(v_2)}{N(1-F(v_2))} & v \geq v_2 \\ \frac{F^N(v_3+\epsilon)-F^N(v_3)}{N(F(v_3+\epsilon)-F(v_3))} & v \in [v_3, v_3+\epsilon] \\ F^{N-1}(v) & v \in [v_1, v_3) \bigcup (v_3+\epsilon, v_2] \\ 0 & o.w. \end{cases} \tag{61}$$

By Lemma 6 (see Appendix A.3), if $f(v)$ is increasing in the interval; then

$$g_\epsilon \equiv \epsilon \frac{F^N(v_3+\epsilon) - F^N(v_3)}{N(F(v_3+\epsilon) - F(v_3))} - \int_{v_3}^{v_3+\epsilon} F^{N-1}(v)dv > 0.$$

Let us assume that $f(v)$ is increasing in the range $[0,1]$. As a result, the budget constraint is now slack. We can now potentially improve on the revenue by 'un-pooling' v_2 to $v_2 + \delta$.

First, δ solves the implicit equation:

$$\delta \frac{F^N(v_2+\epsilon) - F^N(v_2)}{N(F(v_2+\epsilon) - F(v_2))} - \int_{v_2}^{v_2+\delta} F^{N-1}(v)dv = g_\epsilon. \tag{62}$$

The change in revenue from ironing types $[v_3, v_3 + \epsilon]$ is:

$$\Delta(v_3,\epsilon) \equiv \int_{v_3}^{v_3+\epsilon} \nu(v)f(v)\left[\frac{F^N(v_3+\epsilon) - F^N(v_3)}{N(F(v_3+\epsilon) - F(v_3))} - F^{N-1}(v)\right]dv.$$

Similarly, the change in revenue from 'un-pooling' $[v_2, v_2 + \delta]$ is:

$$\Delta(v_2,\epsilon) \equiv -\int_{v_2}^{v_2+\delta} \nu(v)f(v)\left[\frac{F^N(v_2+\delta) - F^N(v_2)}{N(F(v_2+\delta) - F(v_2))} - F^{N-1}(v)\right]dv.$$

Therefore the total change in revenue is:

$$\Delta = \Delta(v_3,\epsilon) + \Delta(v_2,\epsilon)$$

Since $\nu(\cdot)$ and $f(\cdot)$ are both increasing; $\Delta(v_2,\epsilon) \geq 0 \geq \Delta(v_3,\epsilon)$. Potentially, $\Delta \geq 0$ for some suitable parameter choices. In other words, our perturbation of the L-R solution can increase expected revenue, therefore the L-R solution is not optimal. We flesh out a numerical example below.

A.1.1 An Example

There are 2 bidders, i.e. $N = 2$. Both have valuations in the interval $[0, 1]$ which are drawn i.i.d. with density $f(v) = 2v$; $F(v) = v^2$. Both have a common budget constraint $b = 0.5$. The 'virtual value' of a bidder of valuation v, $\nu(v) = \frac{3v^2-1}{2v}$, which is increasing on the interval $[0, 1]$. If there was no budget constraint, the optimal auction would be a second price auction with reserve price $v_0 = \frac{1}{\sqrt{3}}$, i.e. $\nu(v_0) = 0$.

Recall that the L-R solution would require us to compute v_1 and v_2 which jointly solve (59) and (60). Making appropriate substitutions, v_1 and v_2 solve:

$$\begin{aligned} \frac{v_2}{2} + \frac{v_2^3}{6} + \frac{v_1^3}{3} &= 0.5 \\ 3v_1^2 + \frac{(1-v_2^2)^2}{1+v_2^2} &= 1 \end{aligned}$$

Solving, we get $v_1 = 0.5415$ and $v_2 = .7523$. Therefore; $v_1 < v_0 < v_2$. For the perturbation we outlined above, select $v_3 = \frac{1}{\sqrt{3}}(= v_0)$; and $\epsilon = 10^-4$. Our functional forms lend themselves to easy analytic calculation. It can be shown that:

$$\begin{aligned} g_\epsilon &= \frac{\epsilon^3}{6} \\ \delta &= \epsilon \\ \Delta(v_3, \epsilon) &= -\frac{(3v_3^2+1)\epsilon^3}{6} - \frac{v_3\epsilon^4}{2} + \epsilon^5 \\ \Delta(v_2, \epsilon) &= +\frac{(3v_2^2+1)\epsilon^3}{6} + \frac{v_2\epsilon^4}{2} - \epsilon^5 \end{aligned}$$

Substituting we see that net change in revenue

$$\begin{aligned} \Delta &\approx \frac{(v_2^2 - v_3^2)\epsilon^3}{2} \\ &> 0 \end{aligned}$$

where the final inequality follows from the fact that $v_2 > v_3$.

A.2 Maskin

Recall that Maskin [18] considered the same environment as Laffont and Robert, the only difference being he was interested in specifying the constrained efficient auction for this setting. Analogous to our analysis for Laffont and Robert, we can state the correct version of Maskin's main theorem:

Theorem 2 *Suppose the distribution on types is such that the density is* *decreasing. The allocation described by (58), where $v_1 = 0$ and v_2 satisfies (59), and the associated pricing rule*

$$p(v) = v.a(v) - \int_{\underline{v}}^{v} a(v)dv,$$

constitute the expected revenue maximizing mechanism.

A.2.1 A Counter-example

There are 2 bidders, i.e. $N = 2$. Both have valuations in the interval $[0, 1]$ which are drawn i.i.d. with density $f(v) = 2v$; $F(v) = v^2$. Both have a common budget constraint $b = 0.5$. The Maskin solution would require us to pick a cutoff $\bar{v}$ to solve:

$$\frac{\bar{v}}{2} + \frac{\bar{v}^3}{6} = 0.5,$$

i.e. $\bar{v} = 0.8177$. Let us now pick $\epsilon \ll 1$, and iron $[0, \epsilon]$. It can be shown that the budget constraint for type $\bar{v}$ is relaxed by $\frac{\epsilon^3}{6}$. Therefore we can now have the efficient allocation for types $[\bar{v}, \bar{v} + \epsilon)$ and still satisfy the budget constraint. Further, one can show the expected loss of efficiency from ironing the interval $[0, \epsilon]$ is $O(\epsilon^5)$, while the gain in efficiency from unpooling the types $[\bar{v}, \bar{v} + \epsilon)$ is roughly $\frac{1}{3}v_1^2\epsilon^3$.

A.3 Ironing

Let $f(.)$ be the density function for some distribution on $\Re$, and let $F(.)$ be the associated cumulative distibution function.

Lemma 6 *If $f(.)$ is (strictly) increasing on some interval $[v_1, v_2]$, then for any $N > 1$, we have:*

$$(v_2 - v_1)\frac{F^N(v_2) - F^N(v_1)}{N(F(v_2) - F(v_1))} > \int_{v_1}^{v_2} F^{N-1}(v)dv$$

Proof: Rewriting, we have to show that

$$\frac{\int_{v_1}^{v_2} f(v)F^{n-1}(v)dv}{\int_{v_1}^{v_2} f(v)dv} > \frac{\int_{v_1}^{v_2} F^{n-1}(v)dv}{\int_{v_1}^{v_2} dv}$$

This is true if and only if

$$\int_{v_1}^{v_2} dv \int_{v_1}^{v_2} f(v)F^{n-1}(v)dv > \int_{v_1}^{v_2} F^{n-1}(v)dv \int_{v_1}^{v_2} f(v)dv$$

Note that both sides are equal (to zero) at $v_2 = v_1$. Therefore we have the desired inequality if the derivative w.r.t v_2 of the left hand side is greater than the right hand side. Differentiating both sides w.r.t. v_2 and rearranging we have that this is true if and only if:

$$F^{N-1}(v_2)\big[f(v_2)(v_2 - v_1) - \int_{v_1}^{v_2} f(v)dv\big] + \int_{v_1}^{v_2} (f(v) - f(v_2))F^{N-1}(v)dv > 0$$

The inequality now follows by observing that $f(v)$ is increasing in v, therefore

$$\int_{v_1}^{v_2} (f(v) - f(v_2))F^{N-1}(v)dv > F^{N-1}(v_2) \int_{v_1}^{v_2} (f(v) - f(v_2))dv$$

□

B Miscellaneous Proofs

B.1 Border Inequalities

The proofs of Lemmas 1 and 2 follow (almost trivially) from Proposition 3.2 of Border [5]. We reproduce it here (adapted to our notation):

Proposition 7 (Border) *Let* $a : T \to [0, 1]$ *be the interim probability of allocation for a type space* T*. For each* $\alpha \in [0, 1]$*, set*

$$E_\alpha = \{t : a(t) \geq \alpha\}.$$

Then a is feasible if and only if:

$$\sum_{E_\alpha} a(t)f(t) \leq \frac{1 - \left(\sum_{T-E_\alpha} f(t)\right)^N}{N}.$$

Lemma 1 follows since if a is monotonic, then the sets E_α must be sets of the form $\{v, v+1, \ldots, m\}$.

Lemma 2 follows from Lemma 1; and the fact that if for some v, $a(v) = a(v-1) = \alpha'$; then $E_{\alpha'} = \{v-1, v, \ldots, m\}$. Further $E_\alpha \subseteq \{v+1, v+2, \ldots, m\}$ if $\alpha > \alpha'$ while $E_\alpha \supseteq \{v-2, v-1, \ldots m\}$ if $\alpha < \alpha'$. Thus the inequality corresponding to the set of types $\{v, v+1, \ldots, m\}$ can also be dropped by Proposition 7. Equation (20) follows.

B.2 Cutoffs

This section provides a proof of Lemma 5. The Lemma states that in any solution to (REVOPT), the cutoffs as defined in Definition 1 are such that

$$\underline{v}_i \geq \bar{v}_i - 1 \quad \forall i \leq k-1. \tag{63}$$

As the intuition outlined in the main body points out, this result is not surprising- if this condition is violated for some i, roughly speaking, decrease the allocation of types $(\underline{v}_i + 1, b_{i+1})$ (and types pooled with it); and increase the allocation of types $\{(\bar{v}_i, b_i), \ldots, (m, b_i)\}$. This will clearly increase revenue since the virtual valuation of the latter is $\bar{v}_i > \nu(\bar{v}_i) \geq \nu(\underline{v}_i)$ which is the virtual valuation of the former. The trouble is that this simple change can violate the budget constraints.

Below we show how to perturb allocation rules not satisfying (63). The construction relies critically on the assumption that the distribution over types is such that valuation and budget are independent- the assumption that all budgets are equally likely is however only for notational convenience.

Proof: Suppose not, i.e. suppose that allocation rule a solves (REVOPT), with cutoffs $\bar{V}$ and $\underline{V}$ such that $\underline{v}_i < \bar{v}_i - 1$ for some i.

To this end, let $j \equiv \max\{i : \underline{v}_i < \bar{v}_i - 1\}$. Therefore $\underline{v}_i \geq \bar{v}_i - 1$ for all $i > j$. We show how to construct an allocation rule a' is feasible in (REVOPT) that achieves weakly more revenue, such that $\underline{v}'_i \geq \bar{v}'_i - 1$ for all $i > j-1$.

For ease of notation assume that $\underline{v}_{j-1} \leq \underline{v}_i$ and define $\hat{v} \equiv \underline{v}_j + 1$. Consider the following perturbation of a^*:

1. Reduce the allocation of all types $(\hat{v}, b_{j+1}), \ldots, (\hat{v}, b_k)$ by ϵ each.
2. Reduce the allocation of all types in $\{(v, b) : v > \hat{v}, b \geq b_{j+1}\}$ by $\epsilon/(\hat{v}+1)$ each.
3. Increase the allocation of type $(\hat{v}, b_j)$ by $(k-j)\epsilon$.
4. Increase the allocation of types in $\{(v, b_j) : v > \hat{v}\}$ by $(k-j)\epsilon/(\hat{v}+1)$.

Firstly note that this perturbation is revenue neutral. Next we show that the resulting allocation is feasible in the optimization program. Feasibility with respect to the Border constraints is clear by construction. Next note that the payment of type (v, b), $v > \hat{v}$, $b > b_{j+1}$ changes by

$$-v\frac{\epsilon}{\hat{v}+1} + \sum_{\hat{v}+1}^{v-1} \frac{\epsilon}{\hat{v}+1} + \epsilon = 0.$$

Similarly the payment of type (v, b_j), $v > \hat{v}$ changes by

$$v\frac{(k-j)\epsilon}{\hat{v}+1} - \sum_{\hat{v}+1}^{v-1} \frac{(k-j)\epsilon}{\hat{v}+1} + (k-j)\epsilon = 0.$$

Therefore the budget constraints for all types are still satisfied. Further, the payment of type $(\hat{v}+1, b_{j+1})$,

$$\begin{aligned} p'(\hat{v}+1, b_{j+1}) &= p(\hat{v}+1, b_{j+1}) \quad \text{(By Construction)} \\ &> b_j \quad \text{(By definition of } \hat{v}\text{)} \\ &\geq p'(m, b_j) \quad \text{(By budget constraint)} \end{aligned}$$

Finally, set ϵ such that

$$a(\hat{v}, b_{j+1}) - \epsilon = a(\hat{v}, b_j) + (k-j)\epsilon,$$

.

Let us assume that $a'(\hat{v}, b_j) \leq a'(\hat{v}+1, b_j)$. We show that a' is incentive compatible and individually rational. By Observation 2 it is enough to show that a' satisfies (26- 31) (with $\underline{v}_i' = \underline{v}_i + 1$).

Recall that a would have satisfied (26- 31). Verifying that (27- 31) are satisfied with the new cutoff is straightforward. Inequality (26), i.e. that $a'(v, b) \geq a'(v-1, b)$ for all v, b, for $b < b_j$ follows from the fact that $a(v, b) \geq a(v-1, b)$. For $b = b_j$, it follows from our assumption that $a'(\hat{v}, b_j) \leq a'(\hat{v}, b_j)$. For $b > b_j$ we are done if $a'(\hat{v}, b) \geq a'(\hat{v}-1, b)$. But note that $a'(\hat{v}, b) = a'(\hat{v}, b_j) \geq a'(\hat{v}-1, b_j) = a'(\hat{v}-1, b)$ (here the first equality follows from our choice of ϵ, the second by construction, and the third since $a(\hat{v}-1, b) = a'(\hat{v}-1, b)$ for any b).

Now suppose instead that $a'(\hat{v}, b_j) > a'(\hat{v}+1, b_j)$. In this case our perturbation of a proceeds in two steps: the first step is the same as before with ϵ such that

$$a(\hat{v}, b_j) + (k-j)\epsilon = a(\hat{v}+1, b_j) + (k-j)\frac{\epsilon}{\hat{v}+1}.$$

Call the resulting allocation rule a''. Clearly, this perturbation will be revenue neutral; and will satisfy (27- 30) with the same cutoffs as a. Further $a''(\hat{v}, b_{j+1}) > a''(\hat{v}, b_j) = a''(\hat{v}+1, b_j)$. Next consider the following perturbation of a'' :

1. Reduce the allocation of all types $(\hat{v}, b_{j+1}), \ldots, (\hat{v}, b_k)$ by ϵ each.
2. Reduce the allocation of all types in $\{(v, b) : v > \hat{v}, b \geq b_{j+1}\}$ by $\epsilon/(\hat{v}+1)$ each.
3. Increase the allocation of type $(\hat{v}, b_j)$ and $(\hat{v}+1, b_j)$ by $(k-j)\epsilon'$.
4. Increase the allocation of types in $\{(v, b_j) : v > \hat{v}\}$ by $(k-j)\epsilon/(\hat{v}+1)$.

Pick ϵ, ϵ' to jointly solve:

$$\epsilon'(f(\hat{v}) + f(\hat{v})) = (k-j)\epsilon f(\hat{v})$$
$$a''(\hat{v}, b_{j+1}) - \epsilon = a''(\hat{v}, b_j) + (k-j)\epsilon'$$

Denote the resulting allocation rule a'. By construction, a' feasible with respect to the Border conditions and (weakly) revenue increasing. Further, given the decreasing density assumption; as long as $a'(\hat{v}+1, b_j) \leq a'(\hat{v}+2, b_j)$,

a' will satisfy (26- 31) with cutoff $\underline{v}'_j = \underline{v}_j + 1$. If $a'(\hat{v}+1, b_j) > a'(\hat{v}+2, b_j)$, this second perturbation will have to be analogously modified- it should be clear how this can be done.

Note that this construction will increase $\underline{v}_j$, and (weakly) decrease $\bar{v}_j$. Therefore it can be continued until $\underline{v}_j \geq \bar{v}_j - 1$, and therefore $\underline{v}_i \geq \bar{v}_i - 1$ for all $i > j - 1$. □

B.3 Subsidies

This section proves a technical result needed in the proof of Proposition 6

Observation 4 The function

$$\phi(\pi) = \frac{(1 - F(v-2))\pi + vf(v-1)(\frac{k-i+1}{k}(1 - F(v-1)) - \pi)}{\frac{k-i+1}{k}vf(v-1) + \pi}$$

is decreasing in π.

Proof: We are done if we can show that $\phi'(\pi) \leq 0$. Writing $\phi(\pi) = \frac{N(\pi)}{D(\pi)}$ with $N(\cdot), D(\cdot)$ appropriately defined,

$$\phi'(\pi) = \frac{N'(\pi)D(\pi) - D'(\pi)N(\pi)}{D^2(\pi)}.$$

Therefore we are done if we can show that $N'(\pi)D(\pi) - D'(\pi)N(\pi) < 0$. Note that

$$D'(\pi) = 1,$$
$$N'(\pi) = (1 - F(v-2)) - vf(v-1).$$

Therefore

$$
\begin{aligned}
&N'(\pi)D(\pi) - D'(\pi)N(\pi) \\
=&((1 - F(v-2)) - vf(v-1))\left(\frac{k-i+1}{k}vf(v-1) + \pi\right) \\
&- \left((1 - F(v-2))\pi + vf(v-1)\left(\frac{k-i+1}{k}(1 - F(v-1)) - \pi\right)\right) \\
=&((1 - F(v-2)) - vf(v-1))\left(\frac{k-i+1}{k}vf(v-1)\right) - \frac{k-i+1}{k}vf(v-1)(1 - F(v-1)) \\
=&(-(v-1)f(v-1))\big(\frac{k-i+1}{k}vf(v-1)\big) \\
\leq&0.
\end{aligned}
$$

□