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An Algorithm for Finding Almost All of the Medians of a Network

by

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Abstract

All of the medians of a weighted network are shown to be extreme points on the same polyhedron P. An algorithm is presented which makes a tour of P, passing through most of these special extreme points and through very few others. A problem of optimally locating facilities in a network is treated and computational results are given.

I. Introduction

In a recent paper T. C. Hu [1971] has called attention to several important problems in discrete optimization for which breakthroughs are needed. One of these is the p-center problem. In Hu's own words:

> The sixth problem may be called the p-center problem. One example of how this problem may occur is as follows. Assume that the population distribution of the state of Wisconsin is known. It is decided to build p theaters throughout the state such that the maximal number of people can enjoy the shows. A reasonable objective function would be to minimize the total distance travelled by all the people to the p theaters. Another objective function might be to minimize the maximal distance travelled by any one person. The plant-location or warehouse problem also belongs to this class. More abstractly, we will have a network with n nodes, each having a positive weight, and p centers are to be located in the network so as to minimize the total weighted distances between the centers and their assigned nodes. Various other choices of objective functions are possible, some choices making the problem more difficult than the others........ Many seemingly unrelated problems belong to this class....... Very little is known about the p-center problem.

The present paper addresses the case where the objective function is the usual one of minimizing the total weighted distance. This problem is also referred to in the literature as the "p-median" problem. See, for example, Hakimi (1964, 1965), Teitz and Bart (1968), and Diehr (1972). Hakimi has shown that even if we allow the placement of centers on the arcs of the network, there will always be an optimal solution that has every center at a node. Thus it suffices to restrict our attention to the n nodes as the potential sites for the p centers.

ReVelle and Swain [1970] have taken advantage of this fact to formulate the problem as follows. Let

w_i = the non-negative weight assigned to node i (e.g. population, demand, etc.)

d_{ij} = the length of the shortest path from i to j, through the network

$c_{ij} = w_i d_{ij}$

$$y_{ij} = \begin{cases} 1 & \text{if node } i \text{ is assigned to a center at node } j \\ 0 & \text{otherwise.} \end{cases}$$

Note that $y_{jj} = 1$ if and only if a center is placed at node j. The problem is then to

$$\text{(RS)} \qquad \text{minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} y_{ij}$$

subject to

$$(1.1) \qquad \sum_{j=1}^{n} y_{jj} = p$$

$$(1.2) \qquad \sum_{j=1}^{n} y_{ij} = 1 \qquad \text{for } i = 1, \ldots, n$$

$$(1.3) \qquad y_{ij} \le y_{jj} \qquad \text{for all } i \ne j$$

$$(1.4) \qquad y_{ij} = 0 \text{ or } 1.$$

A feasible solution of this problem is a set of p clusters, each cluster containing one center (self-assigned node) and the other nodes assigned to that center. An optimal solution is called a p-median of the network.

Problem (RS) is an integer linear program. It can be relaxed to a linear program by replacing the binary condition (1.4) with

$$(1.5) \qquad y_{ij} \ge 0.$$

It is not necessary to impose an upper bound, as this is insured by the constraints (1.2). ReVelle and Swain have discovered that this linear program, which we denote $(\overline{RS})$, has in most cases a natural integer solution. This is indeed very fortunate. The only difficulty is that $(\overline{RS})$ is a very large linear program even for moderate values of n. $(\overline{RS})$ has n^2 variables and $n^2 + 1$ constraints.

In this paper, it will be shown that every p-median of the network $(1 \le p \le n)$ is an extreme point of the same polyhedron P, and that it is possible to take a tour around P that passes through most of these special extreme points and through very few others. A typical step on this tour is from an extreme point corresponding to, say, a 12-median to an extreme point corresponding to an 11-median. The tour may, however, pass through extreme points corresponding to "p-medians" for fractional values of p. Finally, the tour may not encounter a p-median for certain values of p, say p = 9. Except for such missing values, then, the tour will produce a complete set of medians of the network. An example of such a tour will be given below. An efficient way of performing the necessary linear programming calculations will also be given.

II. The Grand Tour

We begin by defining the polyhedron P. Although not necessary, it will be convenient to eliminate the y_{jj} variables by means of the equations (1.2). Thus

$$(2.1) \qquad y_{jj} = 1 - \sum_{k \neq j} y_{jk} \qquad \text{for } j = 1, \ldots, n.$$

Carrying out this elimination gives an equivalent form of $(\overline{RS})$, namely

$$(\overline{RS}) \qquad \text{minimize} \quad \sum_{j} \sum_{k \neq j} c_{jk} y_{jk}$$

subject to

$$(2.2) \qquad \sum_{j} \sum_{k \neq j} y_{jk} = n - p$$

$$(2.3) \qquad y_{ij} + \sum_{k \neq j} y_{jk} \leq 1 \qquad \text{for all } i \neq j$$

$$(2.4) \qquad y_{ij} \geq 0 \qquad \text{for all } i \neq j.$$

P is now defined as

$$(2.5) \qquad P = \left\{ y \in E^{n(n-1)} \middle| y \geq 0 \text{ and } y \text{ satisfies (2.3)} \right\} .$$

It is clear that any <u>integer</u> solution of $(\overline{RS})$ is an extreme point of P, and that this is true regardless of the value of p.

Now dualize $(\overline{RS})$ with respect to the single constraint (2.2), using λ as the dual variable.

$$(2.6) \qquad \underset{y \in P}{\text{minimize}} \quad \sum_{j} \sum_{k \neq j} (c_{jk} - \lambda) y_{jk} + (n - p)\lambda$$

Let us define

$$(2.7) \qquad v(\lambda) = \min_{y \in P} \sum_j \sum_{k \neq j} (c_{jk} - \lambda) y_{jk}$$

and

$$(2.8) \qquad Z_p(\lambda) = (n - p)\lambda + v(\lambda).$$

The dual problem is then

$$(2.9) \qquad \max_{\lambda} \quad Z_p(\lambda).$$

If $m = n(n - 1)$ and e_m is a vector of m ones, then (2.7) can be written more compactly as

$$(2.10) \qquad v(\lambda) = \min_{y \in P} \ (c - \lambda e_m)y.$$

Equation (2.10) leads to the following observations.

(a) $v(\lambda)$ does not depend on p.

(b) $v(\lambda)$ is the optimal value of a linear program parameterized in its objective function. (Note that λ is a scalar.) Thus $v(\lambda)$ is a piecewise-linear concave function.

(c) $Z_p(\lambda)$ is just the linear function $(n - p)\lambda$ plus $v(\lambda)$. Therefore $v(\lambda)$ gives us the whole family of $Z_p(\lambda)$ functions for $p = 1, \ldots, n$. Note that each $Z_p(\lambda)$ is also piecewise-linear and concave.

(d) Let $Y(\lambda)$ denote the set of optimal solutions for fixed λ,

$$(2.11) \qquad Y(\lambda) = \left\{ y \in P \mid v(\lambda) = (c - \lambda e_m)y \right\}.$$

Inspection of the objective function reveals that $v(\lambda) \leq 0$ for all λ and that y_{jk} cannot participate in an optimal solution as long as $\lambda < c_{jk}$. That is,

$\lambda < c_{jk}$ implies $y_{jk} = 0$ for all $y \in Y(\lambda)$.

In fact, $y = 0$ is an optimal solution as long as

$$\lambda \le c^* = \min_{j \ne k} \quad c_{jk}$$

i.e., as long as λ is less than or equal to the smallest number in the weighted distance matrix. So for $\lambda \le c^*$ we have

$$v(\lambda) = 0$$

and

$$Z_p(\lambda) = (n - p)\lambda \ .$$

The tour of the polyhedron P mentioned in the previous section will be made in the course of constructing the $v(\lambda)$ function. The construction of $v(\lambda)$ is a straightforward application of parametric linear programming. Let $y^o \in Y(\lambda^o)$. Then

$$(2.12) \qquad v(\lambda^o + \delta) = [c - (\lambda^o + \delta)e_m]y^o$$

as long as

$$(2.13) \qquad y^o \in Y(\lambda^o + \delta).$$

Equation (2.12) can be written as

$$(2.14) \qquad v(\lambda^o + \delta) = v(\lambda^o) - \delta \ e_m y^o$$

which reveals that $v(\lambda)$ has slope $-e_m y^o$ as long as (2.13) holds.

If $y^o \in P$ is a vector of zeros and ones, then $e_m y^o$ is simply the number of ones, hence the number of assignments. When $y = 0$ there are no assignments and therefore n centers. (Every node is a center.) Each assignment reduces the number of centers by one. An integer solution y^o must therefore have $n - e_m y^o$ centers. This is illustrated in Figure 1, where $n = 5$ and $y^o_{13} = y^o_{23} = y^o_{53} = 1$. Here there are $e_{20} y^o = 3$ assignments

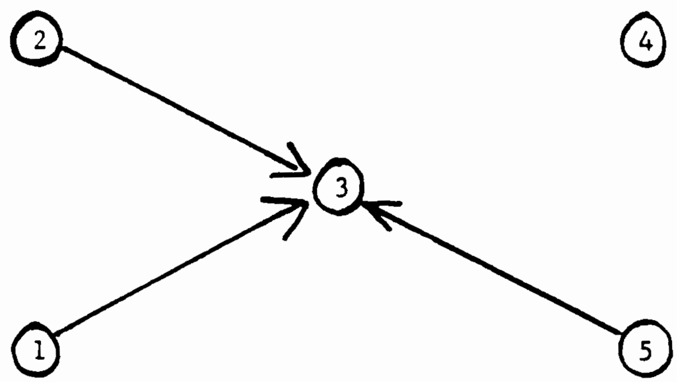

Figure 1. Three assignments, two centers.

and 5 - 3 = 2 centers (nodes 3 and 4). Finally, taking $p = n - e_m y^o$ and using (2.8) and (2.14) we get

(2.15)
$$\begin{aligned} Z_p(\lambda^o + \delta) &= (n - p)(\lambda^o + \delta) + v(\lambda^o + \delta) \\ &= [n - (n - e_m y^o)](\lambda^o + \delta) \\ &\quad + v(\lambda^o) - \delta\, e_m y^o \\ &= v(\lambda^o) + \lambda^o\, e_m y^o \end{aligned}$$

as long as (2.13) holds. Subject to (2.13), then, $Z_p(\lambda)$ has slope zero for $p = n - e_m y^o$. This means that λ^o maximizes the dual objective function $Z_p(\lambda)$, and since $y^o \in Y(\lambda^o)$ it follows that y^o is an optimal solution of the primal problem $(\overline{RS})$. Therefore y^o is a p-median of the network for $p = n - e_m y^o$.

The argument just given proves that every integer extreme point y^* of P that belongs to $Y(\lambda^*)$ for some value of λ^* is a p-median of the network for $p = n - e_m y^*$. Note that it is possible that none of the, say, 9-medians belong to $Y(\lambda)$ for any value of λ. Fractional extreme points of P that appear in some $Y(\lambda)$ can be thought of as "generalized" p-medians.

The procedure for generating the entire $v(\lambda)$ function is to start at $y = 0$ and $\lambda = c^*$ and increase λ, moving from one extreme point of P to another so that we are always at a member of $Y(\lambda)$ for the current value of λ. Every integer extreme point we pass through is a p-median of the network. Note that these medians are encountered in decreasing order ($p = n, n - 1, \ldots.$) since $v(\lambda)$ is concave with slope $-e_m y^o$ for $y^o \in Y(\lambda)$. Thus $y = 0$ belongs to $Y(c^*)$ and we start with a $p = n - e_m 0 = n$ median. The pivoting mechanism will be discussed in more detail in the next section, where the special structure of P will be exploited. We conclude this section with Figure 2, a sketch of a portion of a typical $v(\lambda)$ curve and the corresponding portion of $Z_3(\lambda)$.

III. Implementation

3.1 The coefficient matrix.

Let A be the coefficient matrix for the constraints (2.3) that determine P. A is a square matrix of dimension $m = n(n - 1)$. For each element of the index set

(3.1) $$S = \{(i, j) | 1 \le i, j \le n \text{ and } i \ne j\}$$

there is a row A_{ij} and a column A^{ij} of the matrix A. The entries are determined by the following rules:

(Rule 1) Row A_{ij} has a one in column A^{ij} and in every column A^{jk}. It has zeros elsewhere.

(Rule 2) Column A^{ij} has a one in row A_{ij} and in every row A_{ki}. It has zeros elsewhere.

Thus the matrix A does not have to be stored in explicit form. Its elements can be generated whenever needed by application of the two rules given above.

3.2 Determining the entering basic variable.

Suppose that we have a basic feasible solution of the linear program (2.10) for some fixed value of λ, say $\lambda = \lambda^{o}$. Let s_{ij} be the slack variable for row A_{ij}. Define

(3.2) $$I = \{(i, j) \in S | y_{ij} \text{ is basic}\}$$

(3.3) $$II = \{(i, j) \in S | s_{ij} \text{ is basic}\}$$

and let I' and II' denote their complements with respect to S. The constraints can then be written as

$$(3.4)\qquad \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} y_I + \begin{bmatrix} 0 \\ I_2 \end{bmatrix} s_{II} + \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} y_{I'} + \begin{bmatrix} I_1 \\ 0 \end{bmatrix} s_{II'} = e_m$$

where I_1 and I_2 are identity matrices of appropriate dimensions. The rows and columns of A have been permuted and partitioned accordingly. Thus

$$(3.5)\qquad A = \begin{array}{c} \begin{array}{cc} I & I' \end{array} \\ \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix} \end{array} \begin{array}{l} II' \\ II \end{array}$$

The basis matrix is then

$$(3.6)\qquad B = \begin{bmatrix} A_1 & 0 \\ A_2 & I_2 \end{bmatrix}$$

where A_1, to be called the kernal, is non-singular. The basis inverse can be expressed in terms of the kernal as

$$(3.7)\qquad B^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ -A_2A_1^{-1} & I_2 \end{bmatrix} .$$

Instead of storing B^{-1} (in either explicit or product form), we can store the generally much smaller A_1^{-1}. Let h denote the dimension of A_1. At the beginning of the tour y = 0 and therefore h = 0.

The primal solution associated with the basis B is given by

$$(3.8)\qquad y_I^o = A_1^{-1}e_h \quad , \quad y_{I'}^o = 0$$

$$(3.9)\qquad s_{II}^o = e_{m-h}-A_2y_I^o \quad , \quad s_{II'}^o = 0 .$$

If $y^o \in Y(\lambda^o)$, then every non-basic variable has a non-negative reduced cost. For the non-basic slack variables this is

(3.10) $$x^o_{II'} = -(c - \lambda^o e_m)_I A_1^{-1} \geq 0$$

and for the non-basic structural variables

(3.11) $$r^o_{I'} = (c - \lambda^o e_m)_{I'} - x^o_{II'} A_3 \geq 0 .$$

Now consider increasing λ to $\lambda^o + \delta$. Then y^o does not change, but from (3.10) we get

(3.12) $$x_{II'}(\delta) = x^o_{II'} + \delta e_h A_1^{-1} \geq 0$$

since $(e_m)_I = e_h$. Define

(3.13) $$\sigma_{II'} = e_h A_1^{-1}$$

so that $\sigma_{II'}$ is just the vector of the column totals of A_1^{-1}. Then

(3.14) $$x_{II'}(\delta) = x^o_{II'} + \delta \sigma_{II'} \geq 0$$

which can be used to simplify

(3.15) $$r_{I'}(\delta) = [c - (\lambda^o + \delta)e_m]_{I'} - x_{II'}(\delta) A_3 \geq 0$$

to

(3.16) $$r_{I'}(\delta) = r^o_{I'} - \delta[e_{m-h} - \sigma_{II'} A_3] \geq 0.$$

It follows that y^o will remain optimal as long as $\delta \leq \min \{\delta_1, \delta_2\}$ where

(3.17) $$\delta_1 = \min_{(i,j)\in II'} \left\{ \frac{x^o_{ij}}{-\sigma_{ij}} \;\middle|\; \sigma_{ij} < 0 \right\}$$

and

(3.18) $$\delta_2 = \min_{(i,j)\in I'} \left\{ \frac{r^o_{ij}}{1-\sigma_{II'}\hat{A}^{ij}} \;\middle|\; \sigma_{II'}\hat{A}^{ij} < 1 \right\}$$

where $\hat{A}^{ij}$ is the (i, j) column of A_3. These limits on δ insure that the critical condition (2.13) remains satisfied, $y^o \in Y(\lambda^o + \delta)$.

The references to A_3 in the calculation of $r^o_{I'}$ and δ_2 are handled implicitly by an application of Rule 2. Let $x^o_{ij} = 0$ and $\sigma_{ij} = 0$ for all $(i, j) \in II$. Then

(3.19) $$x^o_{II'} \hat{A}^{ij} = x^o_{ij} + \sum_{(k,i)\in II'} x^o_{ki} \qquad \text{for } (i, j) \in I'$$

and

(3.20) $$\sigma_{II'} \hat{A}^{ij} = \sigma_{ij} + \sum_{(k,i)\in II'} \sigma_{ki} \qquad \text{for } (i, j) \in I'.$$

One step in the construction of $v(\lambda)$ is to take $\delta^* = \min\{\delta_1, \delta_2\}$ and move to

(3.21) $$\lambda^1 = \lambda^o + \delta^*.$$

We know that

(3.22) $$v(\lambda^1) = v(\lambda^o) - \delta^* e_m y^o$$

and that

(3.23) $$y^o \in Y(\lambda^1) .$$

Furthermore, the optimal tableau for (2.10) with $\lambda = \lambda^1$ contains at least one non-basic variable with a reduced cost of zero. This is either a slack variable or a structural variable depending on whether δ^* equals δ_1 or δ_2. This signals the possible existence of alternate optimal solutions, i.e., other members of $Y(\lambda^1)$ besides y^o. Choosing an entering basic variable with zero reduced cost and pivoting will move us to a new extreme point y^1 of P, unless the new variable enters the basis at zero.

Barring degeneracy, then, we arrive at

(3.24) $\quad y^1 \in Y(\lambda^1).$

When there is degeneracy, the pivot is simply a change of basis and we remain at y°. This is, in fact, very common since the problem is highly degenerate. Typically, several basis changes are followed by a breakthrough to a new extreme point. The slope of $v(\lambda)$ cannot change until such a breakthrough is made.

An illustration is given in Figure 3. In this case we have some $y^\circ \in Y(\lambda^\circ)$. Slope $q_o = -e_m y^\circ$. There is a basis change at λ^1 and $y^\circ \in Y(\lambda^1)$. There is another basis change at λ^2, this time accompanied by a move to a new extreme point y^1. Both y° and y^1 belong to $Y(\lambda^2)$. Slope $q_1 = -e_m y^1$. If y° and y^1 are both binary vectors, then the concavity of $v(\lambda)$ implies that y^1 has at least as many ones as y°.

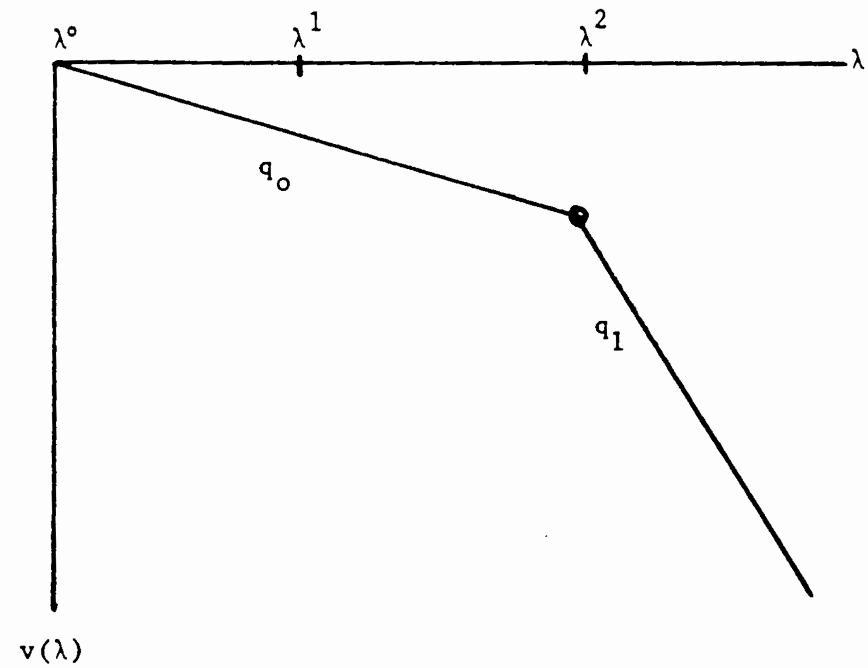

Figure 3. A step in the construction of $v(\lambda)$.

3.3 Determining the exiting basic variable.

The only matter remaining to be discussed is the question of whether or not a breakthrough to a new extreme point has occurred. This involves the determination of the exiting basic variable for each pivot. In the simplex method the entire pivotal column is computed. It will suffice, however, to compute only those h components that correspond to the rows in II'. There are two cases. First, however, note that since A_1 has a row for every $(i, j) \in II'$ it follows that A_1^{-1} has a column, $(A_1^{-1})^{ij}$, for every $(i, j) \in II'$.

Case 1. The entering basic variable is s_{uv} for some $(u, v) \in II'$. Refer to equation (3.4). Let

$$\alpha = A_1^{-1}(I_1)^{uv} = (A_1^{-1})^{uv} \tag{3.25}$$

where $(I_1)^{uv}$ is the column of the identity matrix I_1 corresponding to s_{uv}. The vector α has a component α_{ij} for each $(i, j) \in I$. Define $\alpha_{ij} = 0$ for all $(i, j) \in I'$.

Case 2. The entering basic variable is y_{uv} for some $(u, v) \in I'$. Referring again to (3.4), let

$$\alpha = A_1^{-1} (A_3)^{uv} \tag{3.26}$$

where $(A_3)^{uv}$ is the column of A_3 corresponding to y_{uv}. Applying Rule 2 we see that

$$\alpha = \begin{cases} \sum_{(k,u)\in II'} (A_1^{-1})^{ku} + (A_1^{-1})^{uv} & \text{if } (u, v) \in II' \\ \sum_{(k,u)\in II'} (A_1^{-1})^{ku} & \text{otherwise.} \end{cases} \tag{3.27}$$

Again, α has a component α_{ij} for each $(i, j) \in I$. This time define $\alpha_{ij} = 0$ for $(i, j) \in I'$ with $(i, j) \neq (u, v)$ and $\alpha_{uv} = 1$.

Now suppose that we increase the entering variable above zero by the amount τ. For either of the above cases we have

(3.28) $$y(\tau) = y^\circ - \tau\,\alpha$$

It is easy to show that we remain feasible as long as $\tau \leq \min\{\tau_1, \tau_2\}$. These are given by

(3.29) $$\tau_1 = \min_{(i,j)\in I} \left\{ \frac{y^\circ_{ij}}{\alpha_{ij}} \;\middle|\; \alpha_{ij} > 0 \right\}$$

(3.30) $$\tau_2 = \min_{(i,j)\in II} \left\{ \frac{-s^\circ_{ij}}{\alpha_{ij} + \sum_{(j,k)\in I} \alpha_{jk}} \;\middle|\; \alpha_{ij} + \sum_{(j,k)\in I} \alpha_{jk} <$$

These limits insure the non-negativity of the structural and slack variables, respectively. Let $\tau^* = \min\{\tau_1, \tau_2\}$. If $\tau^* = 0$, then this pivot is simply a basis change and we remain at y°. If $\tau^* > 0$, then we have a breakthrough and move to $y^1 = y(\tau^*)$ as given by (3.28). If $\tau^* > 0$ when $\delta^* = 0$, then we move to a new extreme point without changing the value of λ.

Once τ^* is known the exiting basic variable is also known. (There may be a choice among several in the case of ties.) Using A_1^{-1} and Rule 1 the necessary portion of the pivotal row is generated and then A_1^{-1} is updated by a pivot operation. If the entering variable is a slack and the exiting variable is a structural, then h (the dimension of A_1^{-1}) decreases by one. If the entering variable is a structural and the exiting variable is a slack, then h increases by one. In both of the remaining cases h remains the same.

Note that τ_2 involves considerably more calculation than τ_1. In many cases $\tau_2 = 0$ and this can be detected quickly by checking τ_3, which is defined as

$$(3.31) \qquad \tau_3 = \min_{(i,j)\in I} \left\{ \frac{1 - y^\circ_{ij}}{-\alpha_{ij}} \;\middle|\; \alpha_{ij} < 0 \right\}$$

If $\tau_3 = (1 - y^\circ_{ij}) / (-\alpha_{ij})$, then $y_{ij}(\tau) > 1$ when $\tau > \tau_3$. But, by (2.3), $y_{ij} > 1$ means that $s_{ij} < 0$ and $s_{ki} < 0$ for all k. Thus any structural variable exceeding one immediately forces n different slack variables negative. When $\tau_3 = 0$ any one of these slacks can be chosen as the exiting basic variable.

3.4 The algorithm.

The algorithm for making the grand tour of P can now be stated concisely as follows.

Step 0. Set $\lambda = c^*$, $v(\lambda) = 0$, $y = 0$, $p = n$.

Step 1. Determine δ^* and the entering basic variable.

Step 2. Set $v(\lambda + \delta^*) = v(\lambda) - \delta^* e_m y$. Then set $\lambda = \lambda + \delta^*$.

Step 3. Compute α.

Step 4. Determine τ^* and the exiting basic variable.

Step 5. Update A_1^{-1}.

Step 6. Set $y = y - \tau^*\alpha$. If $\tau^* > 0$ and y is binary, then y is a median of the network. Set $p = n - e_m y$.

Step 7. If $p = 1$, stop. Otherwise go to Step 1.

IV. Computational Results

Preliminary computational results have been very encouraging. For a small test problem with n = 10 nodes, the tour of P passed through exactly 10 extreme points including the origin. Each of these was an integer extreme point and hence a median of the network. There was a median for each p = 10, 9,..., 1. The $v(\lambda)$ function is presented in Table 1. There is one row of this table for each extreme point on the tour. The last four columns contain

$Z_p(\lambda)$ - the minimum weighted distance for p centers,

h - the size of the kernal upon arrival at this extreme point,

iter - the number of pivot operations required to reach this extreme point from the previous one,

time - the time, in CPU seconds on a CDC6400, required to reach this extreme point from the previous one.

The total number of pivot operations and the total time are given at the bottom of their respective columns. The entire tour took less than one second. The successive rows of Table 1 represent adjacent extreme points of P, hence (iter - 1) gives the number of basis changes preceeding each breakthrough. Note that $\delta^* = 0$ for p = 10, 9, 6, and 4.

Results for a 33-node network are given in Table 2. For this problem $c^* = 136$ and all of the node weights are equal. The distance matrix was taken from [Karg and Thompson, 1964]. Starting at the origin, integer extreme points corresponding to p-medians for p = 33 down to p = 10 were encountered. The solution for p = 10 is displayed in Figure 4. There is an arc drawn for each y_{ij} that is equal to one. (The direction of the assignment is indicated by an arrow when necessary.) The next extreme point visited was fractional with $e_m y = 23\frac{1}{2}$. This solution, shown in Figure 5, is therefore a "median" for $p = 9\frac{1}{2}$. In Figure 5, all of the arcs inside the circle in the northwest corner represent y_{ij}'s that are at $\frac{1}{2}$. The next two extreme points (Figures 6 and 7) were also fractional, with $p = 8\frac{1}{2}$ and $7\frac{1}{2}$ respectively. In Figure 8, the difficulty in the northwest corner has been resolved and we have a 7-median. Figures 9-13 portray the remainder of the tour down to the 4-median. The tour was halted at this point because of the storage space required for the kernal.

The tour did not produce a 9-median or an 8-median. Inspection of Figures 4-8, however, led to easy guesses as to the identities of these missing solutions. These guesses were subsequently verified by a different algorithm. The location of centers for the 9-median is $\{1, 6, 9, 12, 16, 20, 25, 29, 33\}$ and for the 8-median is $\{1, 6, 9, 12, 16, 20, 24, 29\}$. In general, the series of solutions that are available should make it easy to find the missing ones by means of a branch-and-bound search.

The computational burden increases as the number of centers decreases. This is apparently because the necessary changes in the configuration of the solution become more and more drastic. Let n_j represent the number of nodes, including node j, that are assigned to a center at node j. At the beginning of the tour every node is a center and hence $n_j = 1$ for all $j = 1, \ldots, n$. As the tour proceeds there are fewer and fewer centers, hence the nodes are grouped into fewer and larger clusters. This means most of the n_j's become zero and the rest become large since

$$\sum_{j=1}^{n} n_j = n \qquad (4.1)$$

must always hold. At the end, when $p = 1$, we have $n_{j^*} = n$ for some j^* and $n_j = 0$ for all $j \neq j^*$. It is clear however that closing a center at node j involves at least n_j reassignments. It is not surprising, then, that the number of basis changes preceeding a breakthrough increases as the number of surviving centers decreases. Unfortunately, these pivot operations must be performed on an inverse that is getting larger.

The present computer code for this algorithm is not very efficient, particularly in its handling of the inverse. Further experimentation is under way and should result in substantial improvements.

The next stage of this research will be to attempt to isolate network median problems in more complex location problems than the one treated here. The goal will be to develop efficient algorithms for a variety of problems that have heretofore been handled only heuristically.

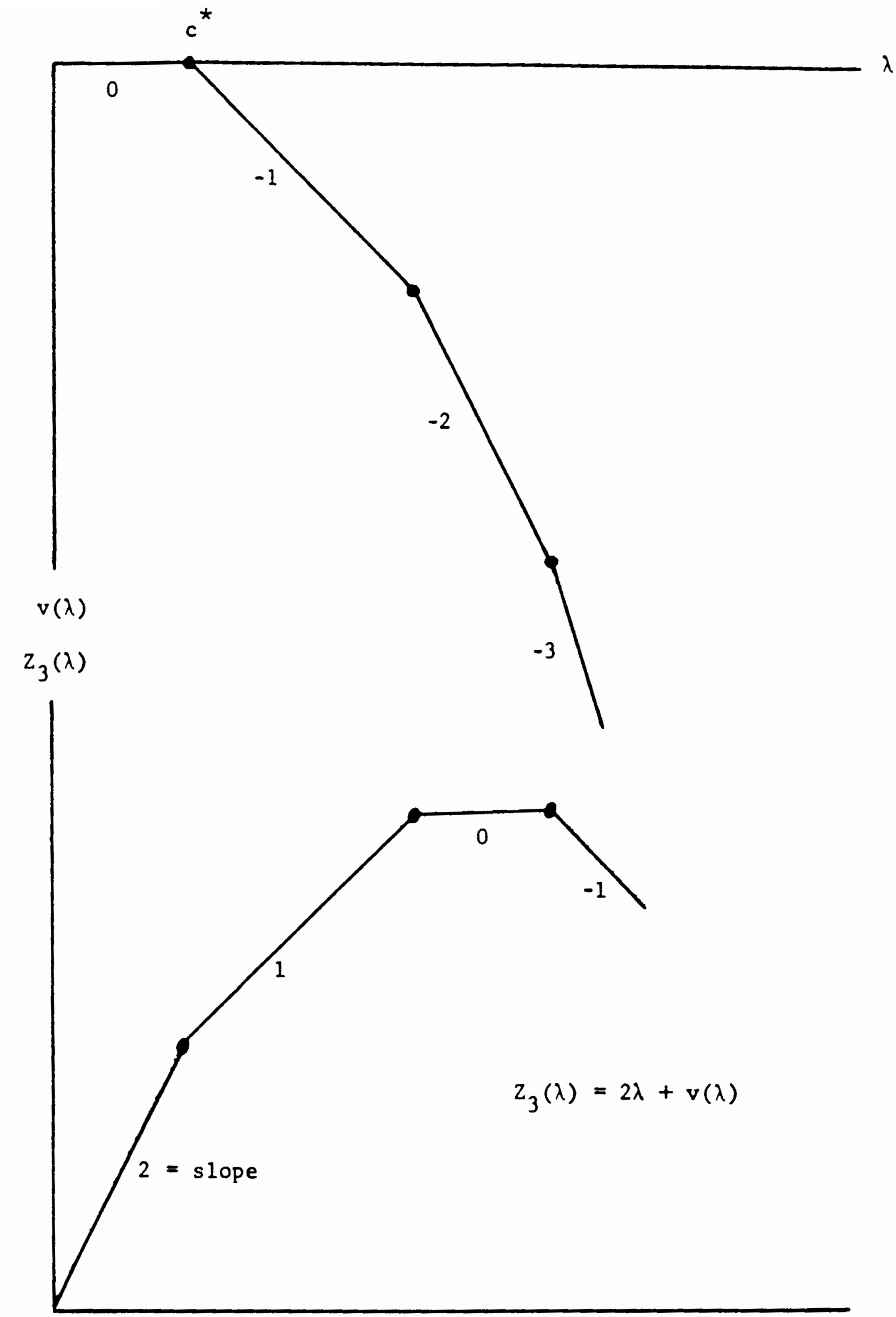

Figure 2. Portions of typical $v(\lambda)$ and $Z_3(\lambda)$ when $n = 5$.

Figure 4. 10-Median.

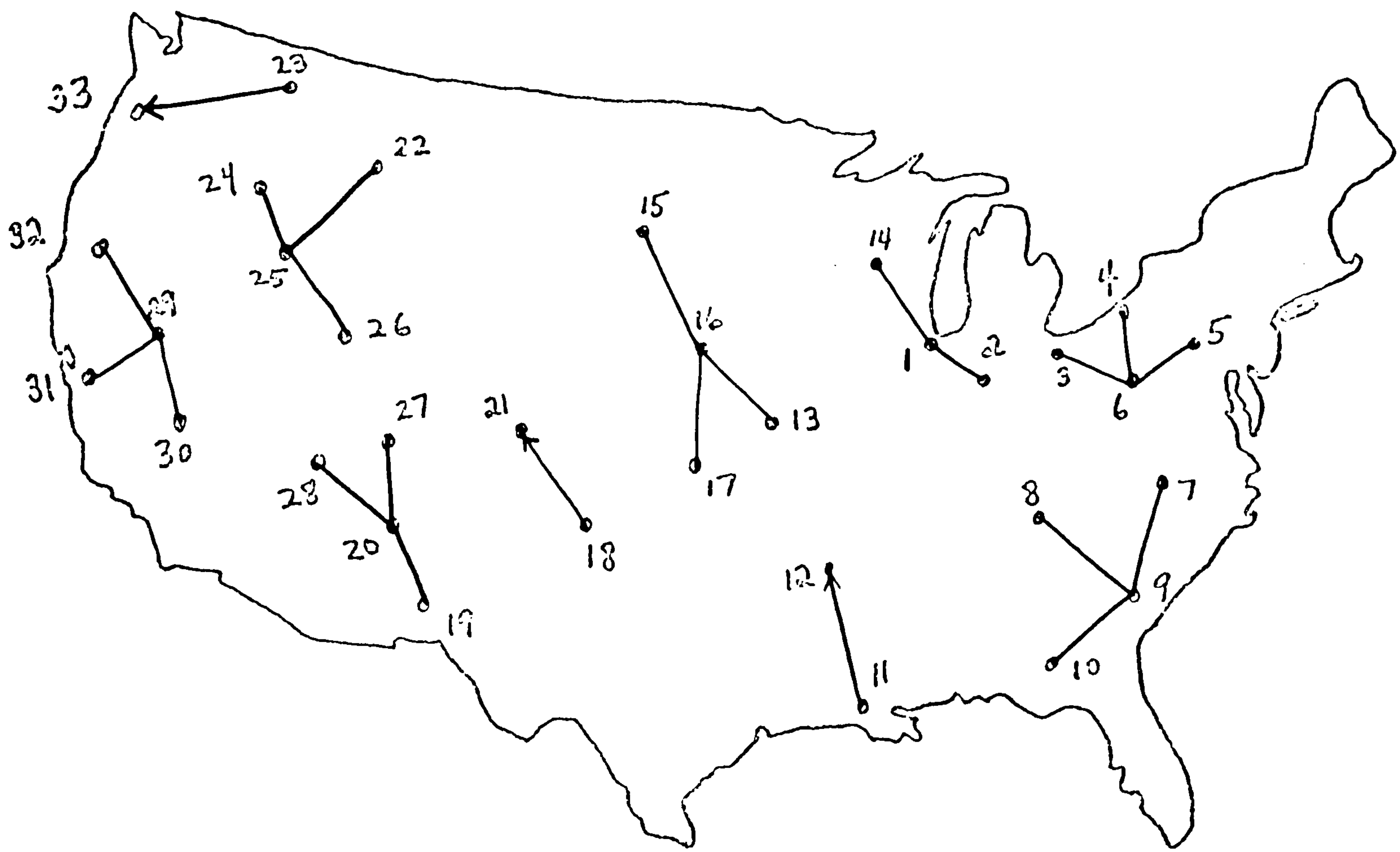

Figure 5. $9\frac{1}{2}$-Median.

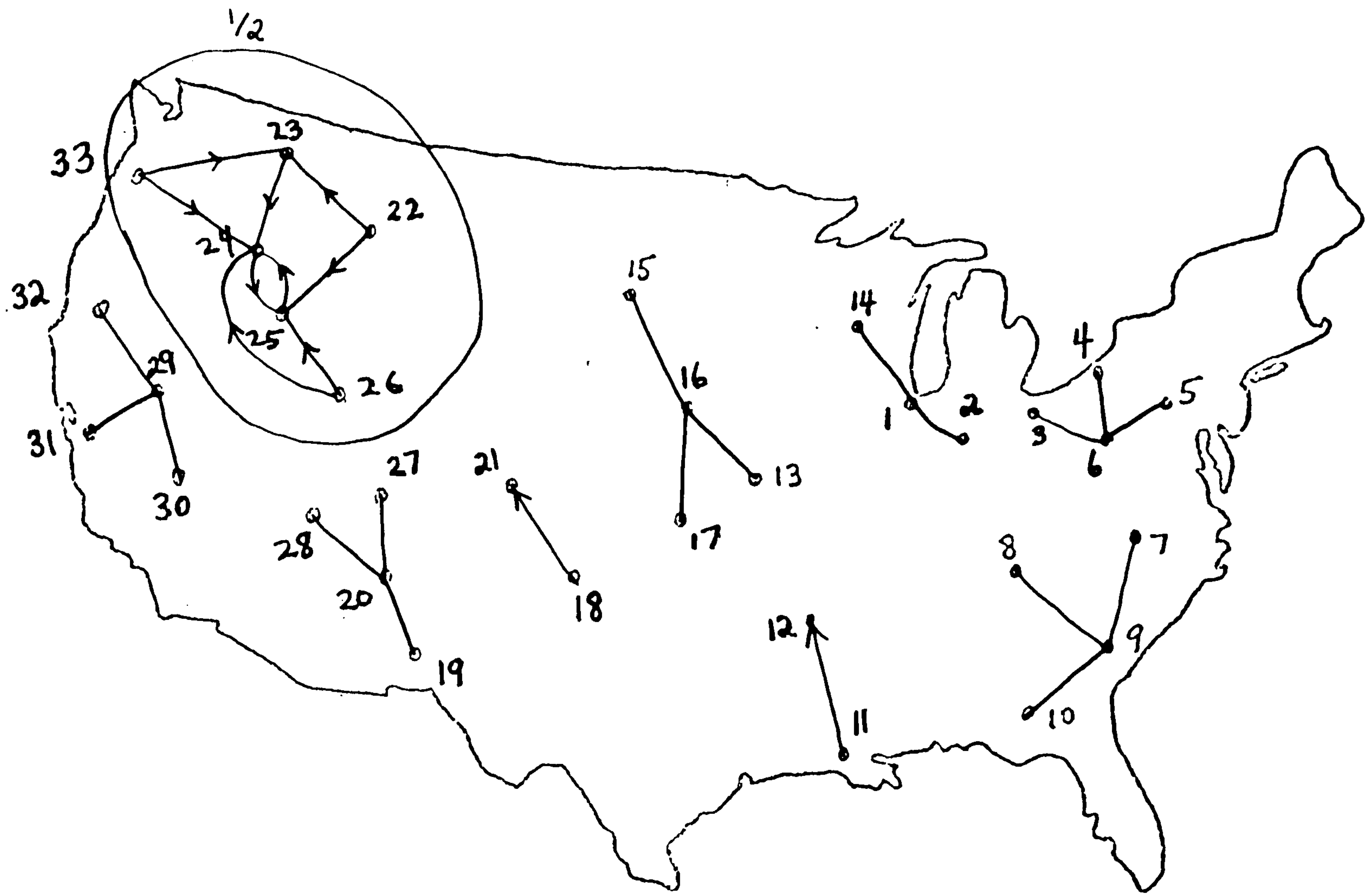

Figure 6. 8½-Median.

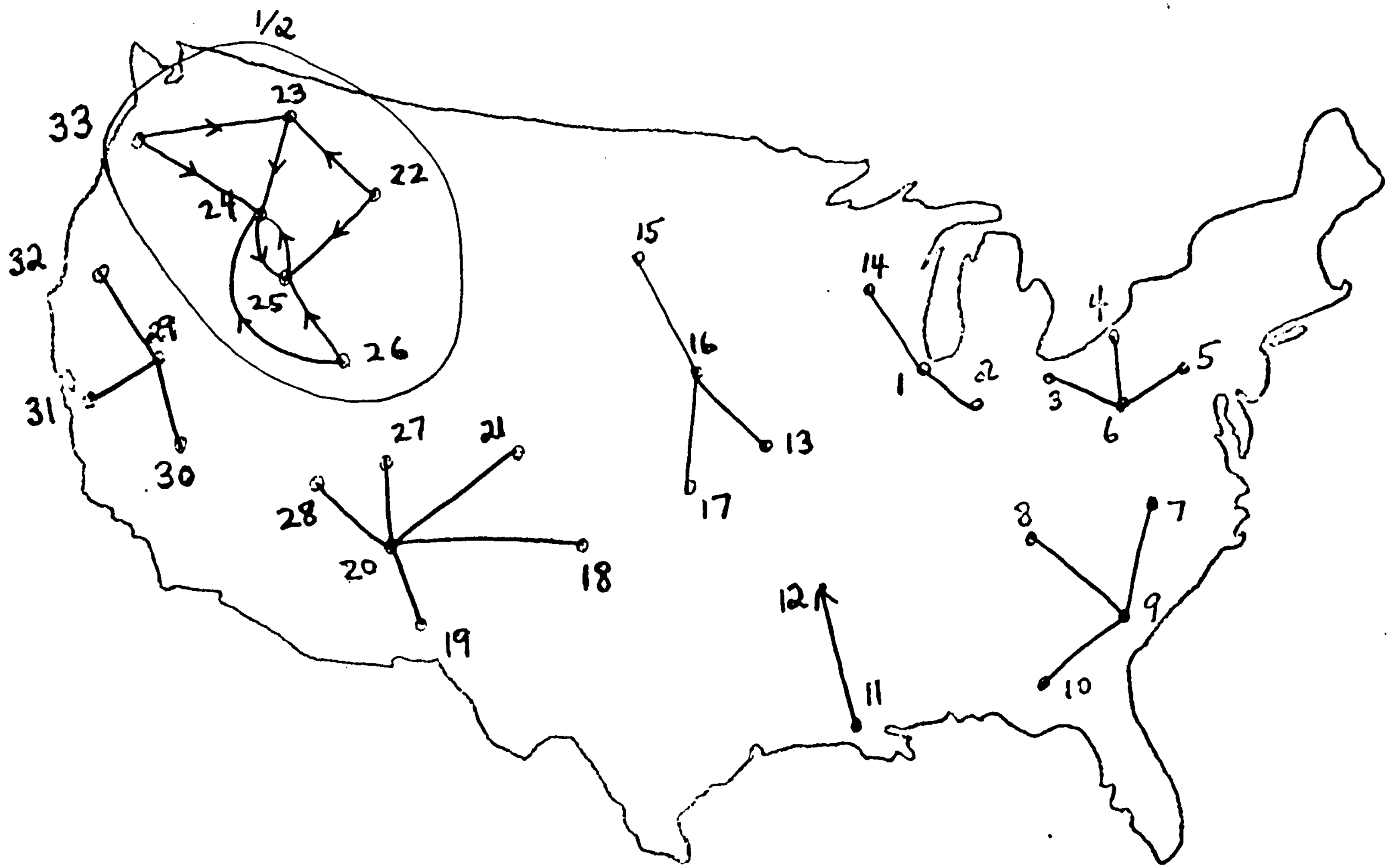

Figure 7. 7½-Median.

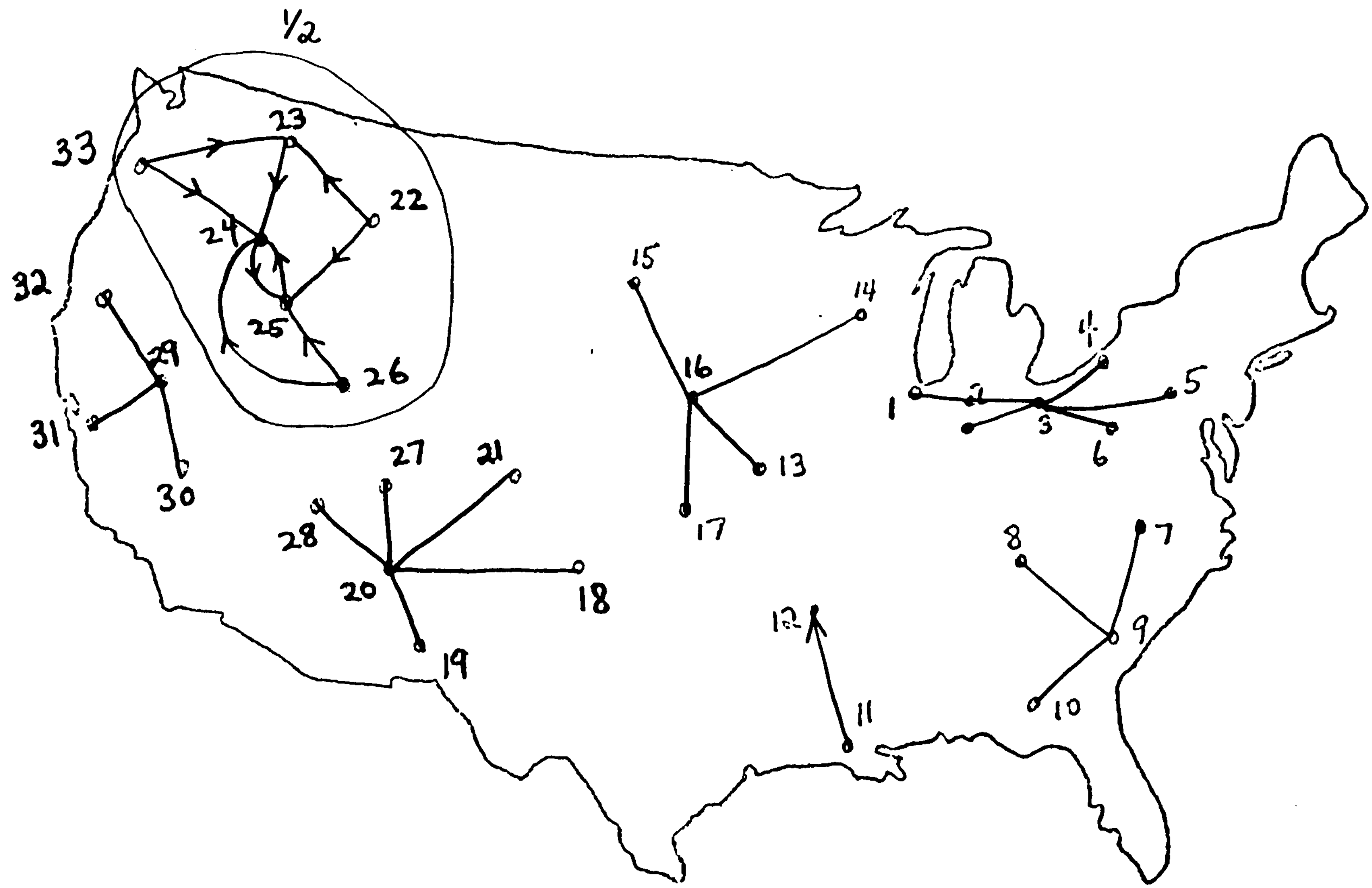

Figure 8. 7-Median.

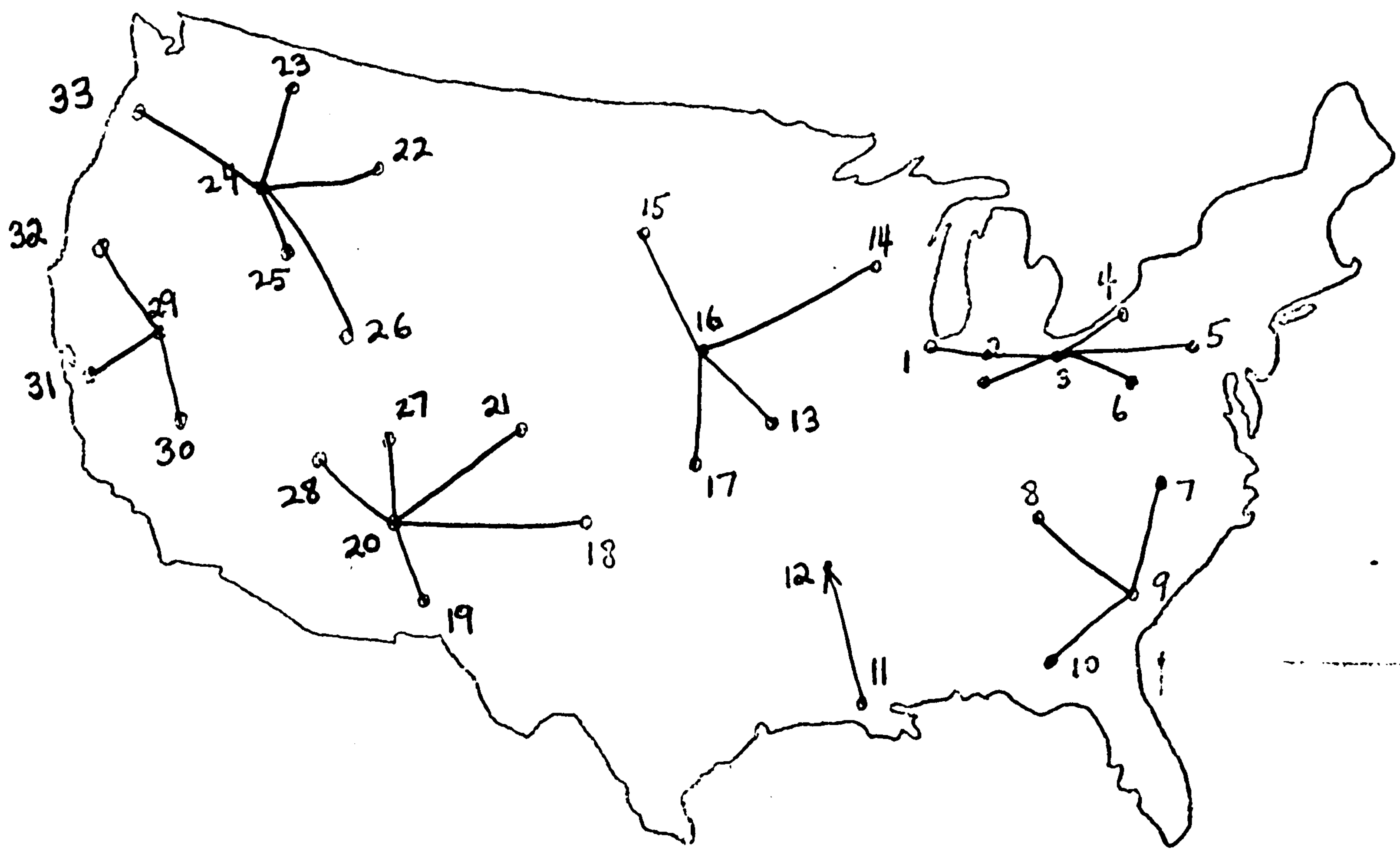

Figure 9. 6-Median.

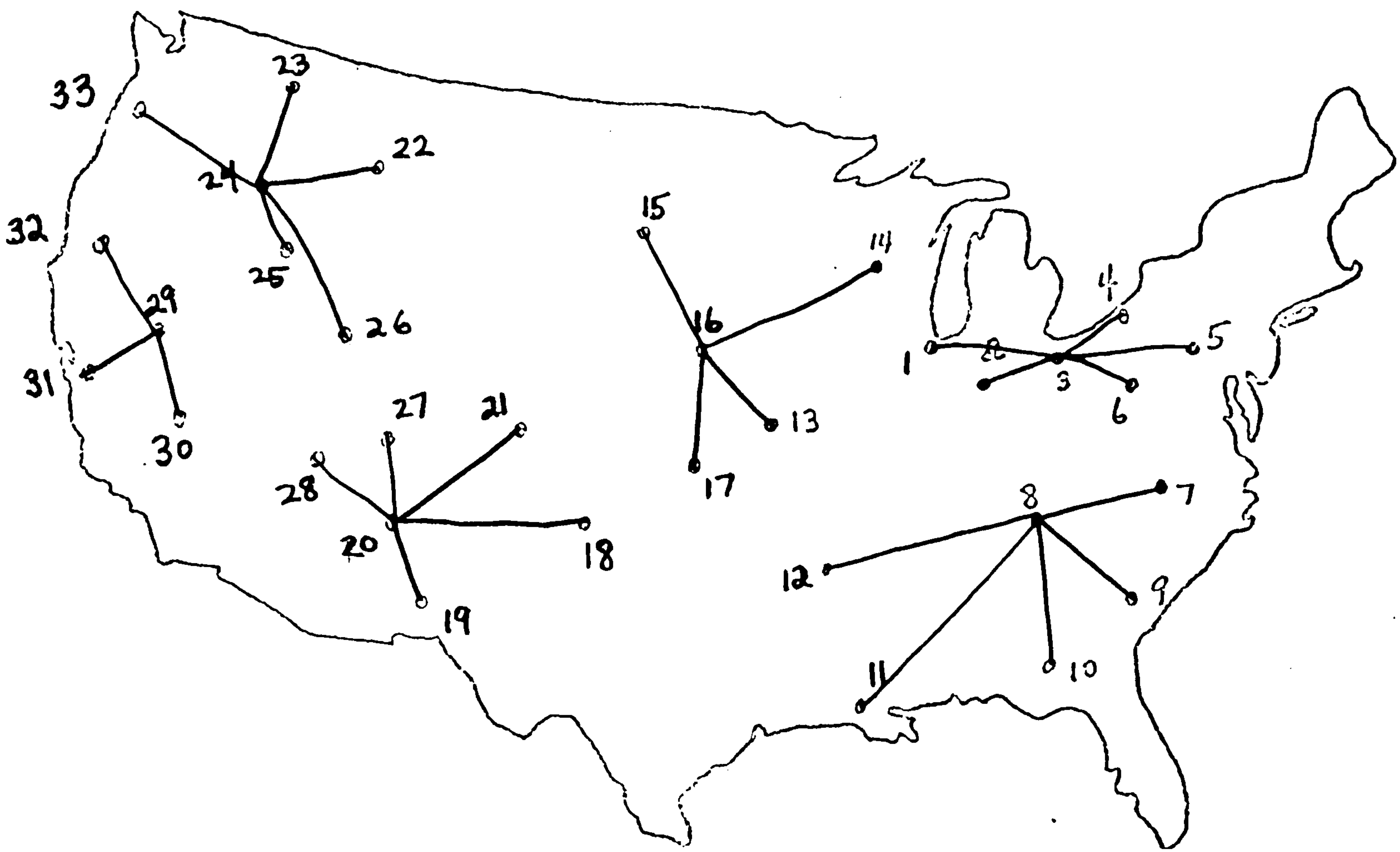

Figure 10. 5½-Median.

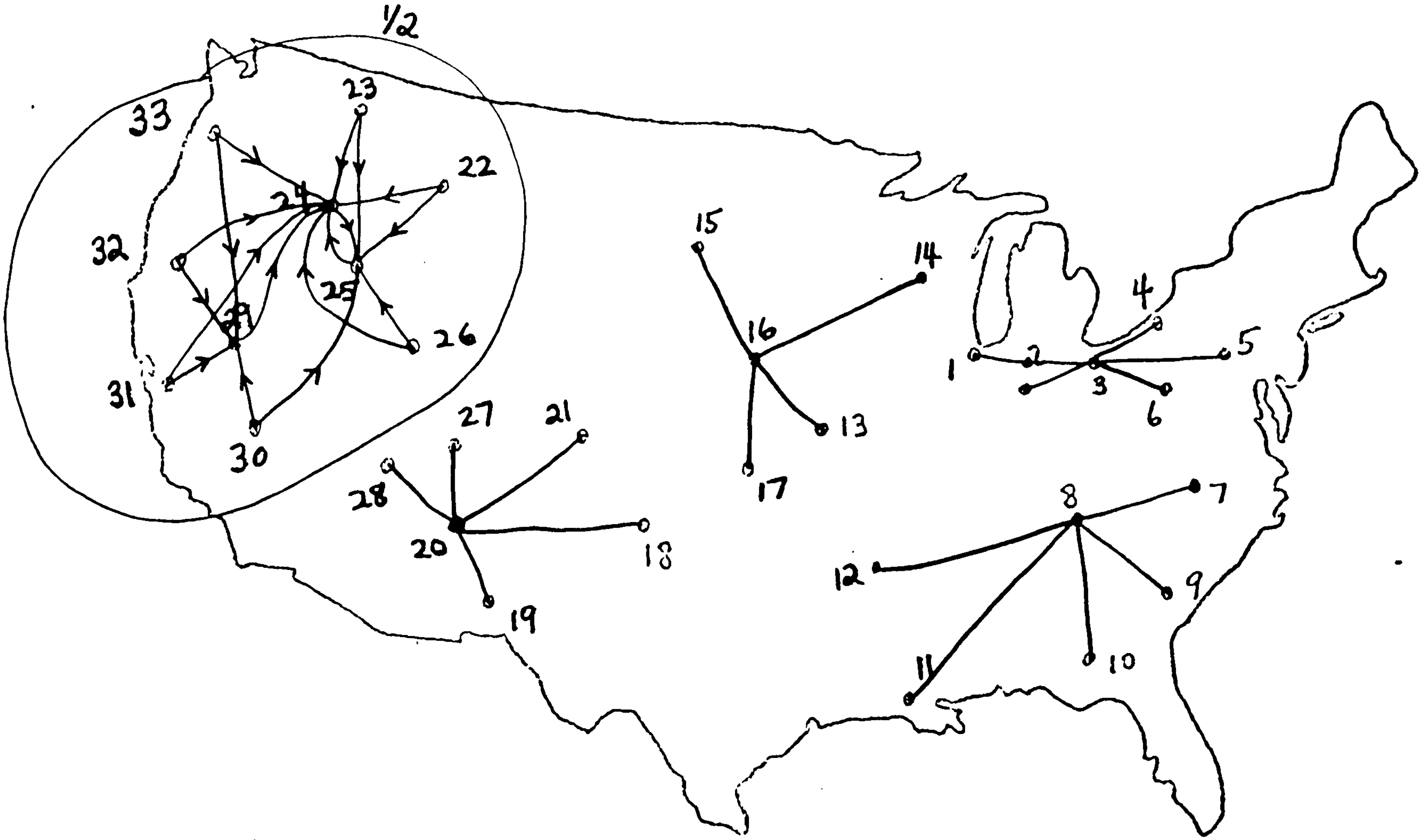

Figure 11. 5-Median.

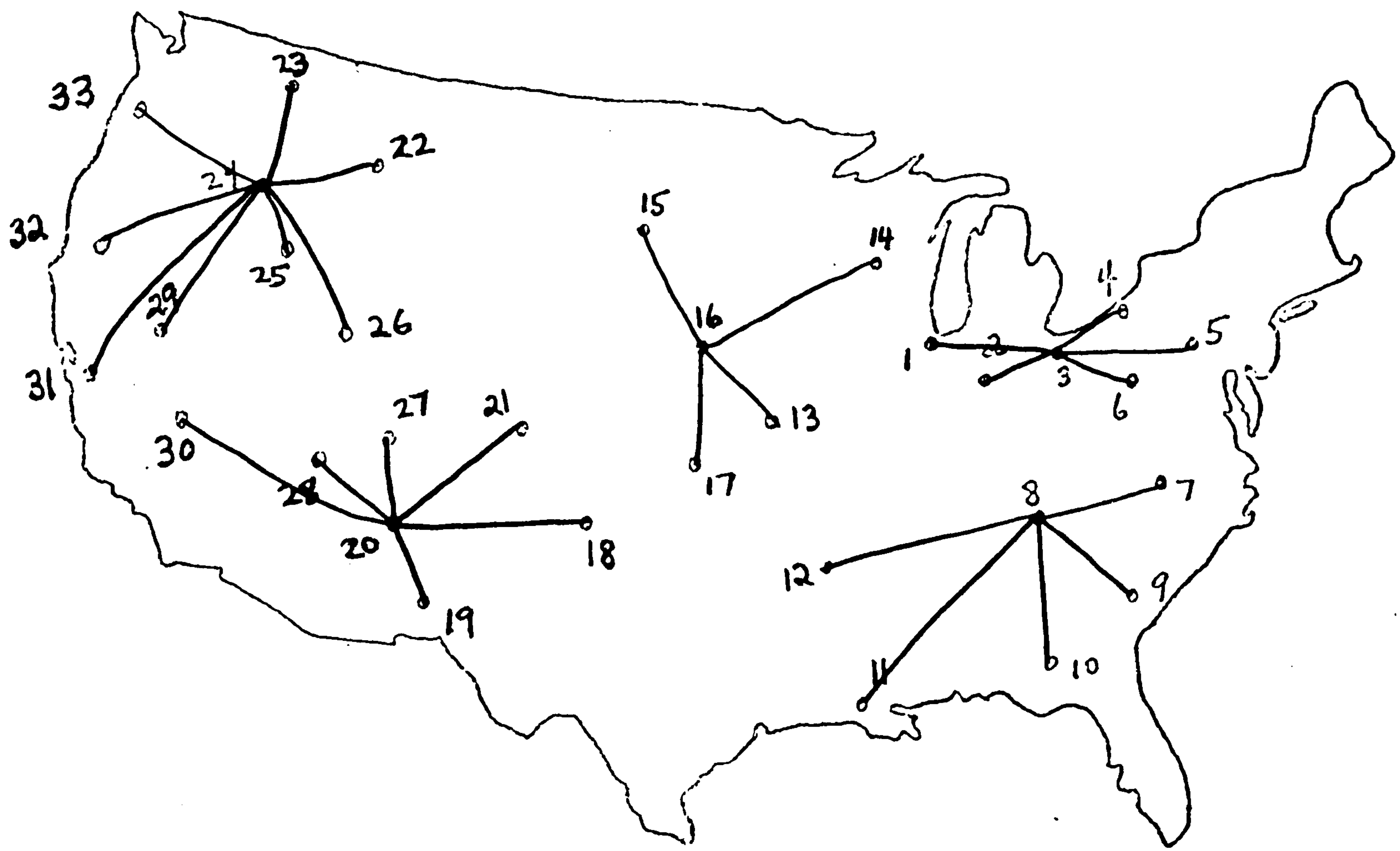

Figure 12. 4½-Median.

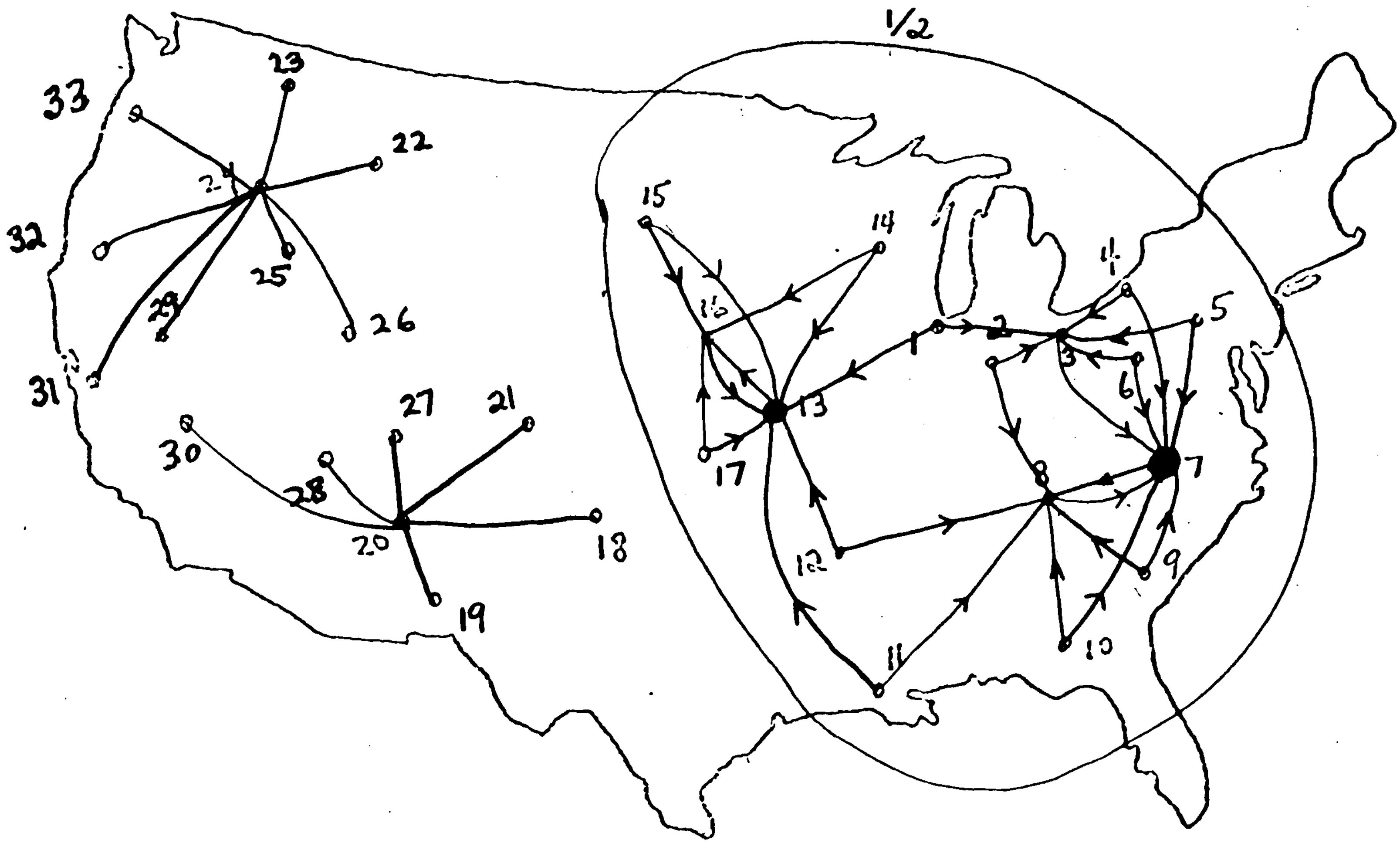

Figure 13. 4-Median.

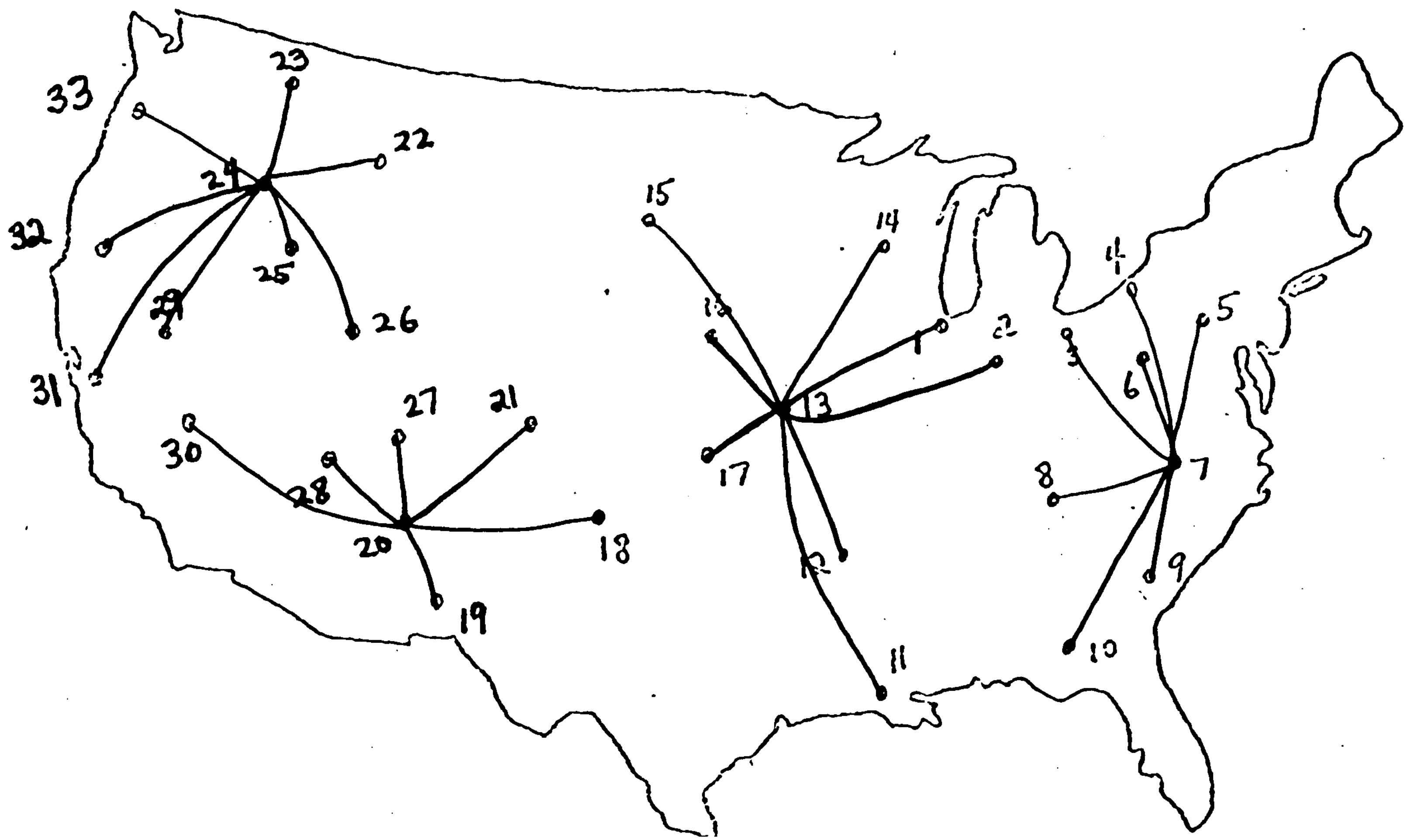

Table 1. Results for a 10-node network

λ	$v(\lambda)$	p	$Z_p(\lambda)$	h	iter	time
10	0	10	0	0	0	.000
10	0	9	10	1	1	.012
10	0	8	20	2	1	.024
22	-24	7	42	6	4	.045
28	-42	6	70	7	1	.033
28	-42	5	98	8	1	.025
41	-107	4	139	16	9	.117
41	-107	3	180	17	1	.035
63	-261	2	243	25	8	.189
106	-605	1	349	36	12	.421
					38	.901

Table 2. Results for a 33-node network

λ	v(λ)	p	$Z_p(\lambda)$	h	iter	time
136	0	33	0	0	0	.000
136	0	32	136	1	1	.027
184	-48	31	320	2	1	.038
195	-70	30	515	3	1	.039
195	-70	29	710	4	1	.013
200	-90	28	910	5	1	.038
211	-145	27	1121	6	1	.040
219	-193	26	1340	10	6	.142
224	-228	25	1564	11	1	.039
225	-236	24	1789	12	1	.041
237	-344	23	2026	13	1	.042
243	-404	22	2269	15	4	.132
251	-492	21	2520	16	1	.042
266	-672	20	2786	18	2	.075
282	-880	19	3068	22	4	.106
295	-1062	18	3363	28	6	.206
295	-1062	17	3658	28	2	.112
313	-1352	16	3971	30	2	.122
343	-1860	15	4314	34	5	.274
346	-1914	14	4660	35	1	.063
368	-2332	13	5028	40	5	.210
379	-2552	12	5407	41	2	.097
380	-2573	11	5787	42	1	.065
480	-4773	10	6267	61	23	1.752
577	-7004	9½	6555½	78	18	2.005
581	-7098	8½	7136½	79	1	.182
647	-8715	7½	7783½	87	10	1.679
671	-9327	7	8119	88	1	.196
713	-10419	6	8832	89	3	.559
1533	-32559	5½	9598½	154	81	19.274
1599	-34374	5	10398	158	10	3.669
1896	-42690	4½	11346	178	39	17.172
2034	-46623	4	12363	191	25	13.522
					261	61.973

References

Diehr, George (1972). "An Algorithm for the p-Median Problem", W.P. No. 191, Western Management Science Institute, UCLA, June.

Hakimi, S. L. (1964). "Optimal Locations of Switching Centers and the Absolute Centers and Medians of a graph", Operations Research, 12, pp. 450-459.

Hakimi, S. L. (1965). "Optimum Distribution of Switching Centers in a Communications Network and Some Related Graph-Theoretic Problems", Operations Research, 13, p. 462.

Hu, T. C. (1971). "Some Problems in Discrete Optimization", Mathematical Programming, 1, No. 1 (October), pp. 102-112.

Karg, R. L. and Thompson, G. L. (1964). "A Heuristic Approach to Solving Travelling Salesman Problems", Management Science, 10, No. 2 (January), pp. 225-248.

Re Velle, C. S. and Swain, R. W. (1970). "Central Facilities Location", Geographical Analysis, 2, No. 1 (January), pp. 30-42.

Teitz, M. and Bart, P. (1968). "Heuristic Methods for Estimated Generalized Vertex Median of a Weighted Graph", Operations Research, 16, pp. 955-961.