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STOCHASTIC PRODUCTION AND COST/PRODUCTION DUALITY

by

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ABSTRACT

This paper studies the implications of using standard models of production and cost to analyze technology when that technology is noisy, i.e. the production process is stochastic. A model of a competitive, expected profit-maximizing firm facing a stochastic production process is posed and analyzed. Two implications for the analysis of such firms based on the usual observables of output levels, input levels and prices, and costs are presented. Direct analyses of technology via estimation of a production function will typically be plagued by misspecification unless the models reflect a special type of separability. Analyses of technology based on cost models are also affected: here the standard model of cost using observed output will only be consistent with a constant returns-to-scale technology. Cost functions based on expected output will provide all relevant technological implications. Examples are provided.

1. Introduction

Most production processes are stochastic in nature. Many firms function by setting production goals and developing output control procedures to help the firm come close to their goals. In general while enterpreneurs may plan to produce some specified level of output, they often do not exactly fulfill their plans: shortfalls or overages occur. In many firms output control is an everpresent part of the firm's activity; automotive manufacturers, food processors and transportation companies are obvious examples.*

Randomness in the production process is not a new topic. Zellner, Kmenta and Dreze [23] resurrected the direct estimation of the Cobb-Douglas model by considering a multiplicative term and firms that maximize expected profit. Feldstein [9] considers a more general Cobb-Douglas case and derives results for the estimation procedure and implications for factor shares of returns-from-production. While there have been a number of papers concerned with stochastic input prices, Rothenberg and Smith [15] appear to be the first to trace out some of the general equilibrium resource allocation effects of assuming the noise to be in input variables rather than the prices. More recently a number of papers (see, e.g., [13] and [16]) have been concerned with estimating stochastic production functions to provide insight about issues of efficiency in production.

What is new in this paper is the direction of the inquiry. While for the most part the literature has been concerned with the econometrics of analyzing cost and production, our interest will concern the models of cost

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*In fact the impetus for this paper comes from ongoing work concerned with estimating multi-product cost functions for railroad firms, where some of the outputs include speed of delivery and schedule reliability.

and production themselves. Specifically we study some of the implications of using standard models for analysis of technology when there is noise (i.e. random disturbances) in the production process. It is shown that ignoring noise corresponds to an implicit, and severe, restriction of the admissible class of models of production processes. This sheds light on how such stochastic effects must be accounted for in analyses of production and cost.

The randomness being analyzed is not the type that is usually addressed (except as noted above) by econometric analysis. We are not concerned for example, with problems of measurement error or differences amongst firms. We consider noise that enters the production process that the enterpreneur acknowledges, plans for, and acts upon: failures of machines, variability of labor quality, imperfections in quality control schemes (as well as quality control itself), theft, accidents, etc. We examine the issue for model functional structure that recognizing (or not recognizing) noise in the process implies.

In what follows we examine the implications of assuming particular functional structures to represent production or cost functions. Two types of analysis will be addressed:

1. Direct analyses of production wherein some form of the production function is chosen (e.g. Cobb-Douglas, CES [1], Translog [6] or Generalized Leontief [8]) and estimated.

2. Analyses of production via a cost function, relying upon the duality ([11], [17]) between production and cost to provide the link.

In both cases we assume a firm that maximizes expected profit and chooses inputs accordingly. Observed output will fluctuate according to a distribution that, while assumed to be known to the entrepreneur, is unknown to the economic analyst. Thus the analyst has the following observed information at his disposal:

1. Observations on output, y, and cost C
2. Prices of input factors, q.
3. Levels of input factors purchased, x.

We shall refer to the standard approach as one wherein either a production function F(x) or a cost function C(y,q) is posed (i.e. a functional form posed) and estimated (see, e.g. [22 , Ch. 4]). We will see that by so doing we ignore the fact that the entrepreneur may have faced a stochastic production problem and that in many cases our model will be seriously misspecified and produce very misleading results.

Section two provides a simple model of an expected profit-maximizing firm that will be used in the rest of the paper. Section three provides two analyses. First we examine the conditions under which standard analyses of production functions (ignoring noise) will be correct if the actual production function is stochastic; we then provide an example wherein this would not be true. Then we turn to analyses of technology via cost functions. Here we find that cost functions must be estimated using expected output instead of observed output (as is usually done). Extending the production example shows that failure to do this may result in major estimation errors.

2. A Model of Stochastic Production and Cost Minimization

In this section we present a model of a firm producing a stochastic output and choosing the deterministic inputs so as to maximize expected profits. The first order conditions are then used to define an implicit cost minimization problem that the enterpreneur faces that relates factor level usage to expected output and factor prices. The relationships between the stochastic production function, the expected production function and the cost function will be explored in the rest of the paper.

Consider a firm producing a single output y using an input vector* x, $x \in R^n_+$, which it purchases at given prices $q \in R^n_{++}$. Production follows a production function $f: R^n_+ \times D \to R_+$, represented as $f(x,\omega)$, with the following properties:

1. $f(0,\omega) = 0 \quad \forall \omega \in D$
2. $D \subseteq R$
3. f is continuous with continuous first and second derivatives in x and ω.
4. $\nabla_x f(x,\omega) > 0 \quad \forall \omega \in D$

Furthermore, ω is distributed G (i.e. $\omega \sim G$), with G assumed known to the firm.

Let p be the price of output; thus profits will be taken to be $\pi = py - q'x = pf(x,\omega) - q'x$. We assume that the firm maximizes expected profits, i.e. it chooses x that solves the following problem:

$$\text{(PM)} \qquad \max_x \int_D (pf(x,\omega) - q'x)\,dG$$

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*$R^n_+ = \{x \in R^n : x_i \geq 0,\ i = 1,\ldots,n\}$, $R^n_{++} = \{x \in R^n : x_i > 0,\ i = 1,\ldots,n\}$

The first order conditions are straightforward:

$$p \int_D f_i dG = q_i \qquad i = 1,\ldots,n \tag{1}$$

where $f_i \equiv \partial f(x,\omega)/\partial x_i$. Since, by assumption, $f_i \in R_{++}$ and $q_i \in R_{++}$ then (1) implies that

$$\frac{E(f_i)}{E(f_j)} = \frac{q_i}{q_j} \qquad \forall_{i,j} \tag{2}$$

where $E(f_i) \equiv \int_D f_i dG$, $\forall i$. The n-1 ratios in (2) express the first order condition that the marginal rate-of-technical-substitution of the expected production function should equal the ratio of input prices. This can be viewed as an efficiency-of-expected production condition that arises from the following cost minimization problem (CMP):

(CMP) $\min \quad q'x$

$\text{S.T.} \quad E(f(x,\omega)) = u$

with q as above and u parametric ($u \equiv E(y)$). The result of (CMP) is the cost function $C(E(y),q)$; the first order conditions for (CMP) are the constraint and the (n-1) ratios (2) above.*

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*In all the above discussion we have implicitly assumed appropriate sufficient conditions on $E(f)$ are in force.

3. Functional Structure Implications

Noise in the production process implies that only in special cases can we ignore its presence when performing an economic analysis. It will be shown that our econometric models of production and cost must be extended to allow for testing for the presence of noise that the entrepreneur has accounted for in his choice of input and output levels.

In the sections that follow we will use basic notions of separability due to Leontief [12] and Sono [18], to provide a notion of stochastic separability. Most models of inherent noise have assumed, it turns out, stochastic separability for the processes modeled (see, e.g. [13], [16], [23]). The second section provides a simple example of a production function that is not stochastically separable. Here we assume a Cobb-Douglas production process with one of the output elasticities being random (see, e.g. [9]). If this is what the entrepreneur actually faces and if the analyst then attempts to directly estimate a Cobb-Douglas production function using the observable data listed in section one above, the analyst may end up with the completely erroneous conclusion that the entrepreneur is operating off of his expansion path. On the other hand, in general, estimating a cost function of the form C(y,q) on the observed data will only be consistent with the actual production process if it is homogeneous of degree one in inputs, clearly something to be tested rather than assumed. The fourth section extends the earlier example to examine analyses via cost functions.

3.1. Stochastic Separability and the Expansion Path

Consider first an extension of the Leontief-Sono separability condition to the problem of separability of the random variable from the non-random inputs in the production process. For convenience of discussion, we will refer to such separability as stochastic separability.

Definition. $f(x,\omega)$ is stochastically separable (s.s.) if

$$\frac{\partial (f_i(x,\omega)/f_j(x,\omega))}{\partial \omega} = 0 \qquad \forall_{i,j}$$

Theorem 1. if $f(x,\omega)$ is s.s. then there exist functions $k: R^n \to R$ and $\tilde{f}: R^2 \to R$ (with $\tilde{f}_1 > 0$) such that

$$f(x,\omega) = \tilde{f}(k(x),\omega)$$

This is a direct extension of separability in the deterministic case. See [2] for the deterministic case.

From Theorem 1 we see that

$$f_i(x,\omega) = k_i(x)\tilde{f}_1(k(x),\omega)$$

$$\Rightarrow \qquad E(f_i(x,\omega)) = k_i(x)E(\tilde{f}_1(k(x),\omega))$$

Therefore, the first order conditions (2) for problem (PM) (and for (CMP)) are:

$$\frac{E(f_i)}{E(f_j)} = \frac{k_i(x)E(\tilde{f}_1(k(x),\omega))}{k_j(x)E(\tilde{f}_1(k(x),\omega))} = \frac{q_i}{q_j}$$

i.e. (2) becomes

$$\frac{k_i(x)}{k_j(x)} = \frac{q_i}{q_j} \qquad \forall_{i,j}$$

Thus, if $f(x,\omega)$ is s.s. then the distribution of ω (i.e. G) is irrelevant: we may proceed as if the process being studied was deterministic. In other words, if $f(x,\omega)$ is s.s. then we can proceed without modeling the noise process itself (i.e. G and how ω enters f).

Clearly, the reverse issue is more important: what is the structure of $f(x,\omega)$ such that the form of G can be ignored? The answer is that $f(x,\omega)$ must be s.s. To see this we require only that there be functions $T_{ij}:R^n_+ \to R_{++}$ such that

$$T_{ij}(x) = \int_D f_i dG / \int_D f_j dG \qquad \forall_{i,j}$$

for arbitrary G. An obvious candidate is k(x) with $T_{ij}(x) = k_i(x)/k_j(x)$.

The condition above is equivalent to

$$\int_D (f_i(x,\omega) - T_{ij}(x) f_j(x,\omega))\,dG = 0$$

Since this must hold for arbitrary G then we must have:

$$\frac{f_i(x,\omega)}{f_j(x,\omega)} = T_{ij}(x)$$

which implies s.s. Therefore s.s. is a necessary and sufficient condition for using a deterministic production function to model first order conditions for a stochastic production process if G can be arbitrary:

Theorem 2. $f(x,\omega)$ is s.s. if and only if $\exists T_{ij}:R^n_+ \to R_{++}$, $\forall_{i,j}$ such that

$$T_{ij}(x) = \int_D f_i dG / \int_D f_j dG \qquad \text{for all } G.$$

As an example, $f^A(x,\omega) = (x_1\omega)^\alpha x_2^\beta$ is s.s. while $f^B(x,\omega) = (x_1 + \omega)^\alpha x_2^\beta$ is not. Certainly, a priori, there is no theoretical reason to prefer one specification to the other. Thus, the real import of the above theorem is that stochastic separability is a property that should be tested (by using a model sufficiently general to accept or reject it) rather than assumed. The cost of not testing is to misspecify the model and may result in very misleading results. An example in which the expansion path of the firm is misanalyzed is shown in the next section.

3.2. A Non-S.S. Example

To illustrate the implication of s.s., we will examine a simple example of a stochastic production process which is not stochastically separable. Consider the production process represented by:

$$f(x,\omega) = x_1^{\omega} x_2^{\beta}$$

with $0 \leq \omega \leq a$, $a > 0$, $\omega \sim U[0,a]$. $f(x,\omega)$ is Cobb-Douglas in (x_1,x_2) with ω distributed uniformly on $D = [0,a]$ (i.e., $dG = d\omega/a$) and β non-stochastic. The expected production function $E(f(x,\omega))$ is simply:

$$E(f(x,\omega)) = \int_0^a x_1^{\omega} x_2^{\beta} \frac{d\omega}{a} = \frac{(x_1^a - 1)x_2^{\beta}}{\ell n x_1^a} \qquad x_1 \neq 1$$

Finally, it is trivial to show that $\lim_{x_1 \to 1} E(f(x,\omega)) = x_2^{\beta}$ from both the left and right and thus $E(f(x,\omega))$ is continuous and given by

$$E(f(x,\omega)) = \begin{cases} \dfrac{(x_1^a - 1)x_2^{\beta}}{\ell n x_1^a} & x_1 \neq 1 \\ x_2^{\beta} & x_1 = 1 \end{cases}$$

Now if $f(x,\omega)$ were estimated directly (by, say, taking logs and estimating the random coefficients regression model (see [20], [21])), then the resulting estimated function would be $f(x,E(\omega))$, i.e., estimating $\ell ny = \omega \ell n x_1 + \beta \ell n x_2 + \varepsilon$ would yield expected values of ω and β, thereby implicitly providing $f(x,E(\omega))$. Thus, we will compare $E(f(x,\omega))$ and $f(x,E(\omega))$ for our example. To see this we express the first order conditions as follows:

$$\frac{E(f_1)}{E(f_2)} = \frac{ax_2}{\beta x_1} \frac{x_1^a \ell n x_1^a - x_1^a + 1}{(x_1^a - 1)\ \ell n x_1^a} = \frac{f_1(x,E(\omega))}{f_2(x,E(\omega))} Z(x_1,a) = \frac{q_1}{q_2}$$

where $f_1(x,E(\omega)) = af(x,E(\omega))/2x_1$, $f_2(x,E(\omega)) = \beta f(x,E(\omega))/x_2$ and

$$Z(x_1,a) = \begin{cases} 2\,\dfrac{x_1^a \ell n x_1^a - x_1^a + 1}{(x_1^a - 1)\ell n\, x_1^a} & x_1 \neq 1 \\ 1 & x_1 = 1 \end{cases}$$

It can be shown that

$$\lim_{x_1 \to 1} Z(x_1,a) = Z(1,a) \qquad \forall_a$$

$$\lim_{x_1 \to +\infty} Z(x_1,a) = 2 \qquad \forall_a$$

$$\lim_{x_1 \downarrow 0} Z(x_1,a) = 0 \qquad \forall_a$$

$$Z(x_1,a) \begin{cases} > 1 & x_1 > 1 \\ < 1 & x_1 < 1 \end{cases} \qquad \forall_a$$

$$\frac{\partial Z(x_1,a)}{\partial x_1} > 0 \qquad x_1 \neq 1 \qquad \forall_a$$

Consider now the expansion path of the firm, i.e. let

$$\mathcal{P} = \{x \in R_+^2:\ \ q_2 E(f_1) = q_1 E(f_2)\}$$

for a given $q = (q_1,q_2)'$ and consider the following two rays from the origin:

$$A = \{x \in R_+^2:\ \ q_1 \beta x_1 = q_2 \frac{a}{2} x_2\}$$

$$B = \{x \in R_+^2:\ \ q_1 \beta x_1 = q_2\, a\, x_2\}$$

A is the "expansion path" that would be associated with $f(x,E(\omega))$, i.e. associated with the following cost minimization problem

$$\min \quad q'x$$
$$\text{S.T.} \quad f(x,E(\omega)) = u$$

with u parametric. On the other hand B is the asymptote for $\mathcal{P}$. From

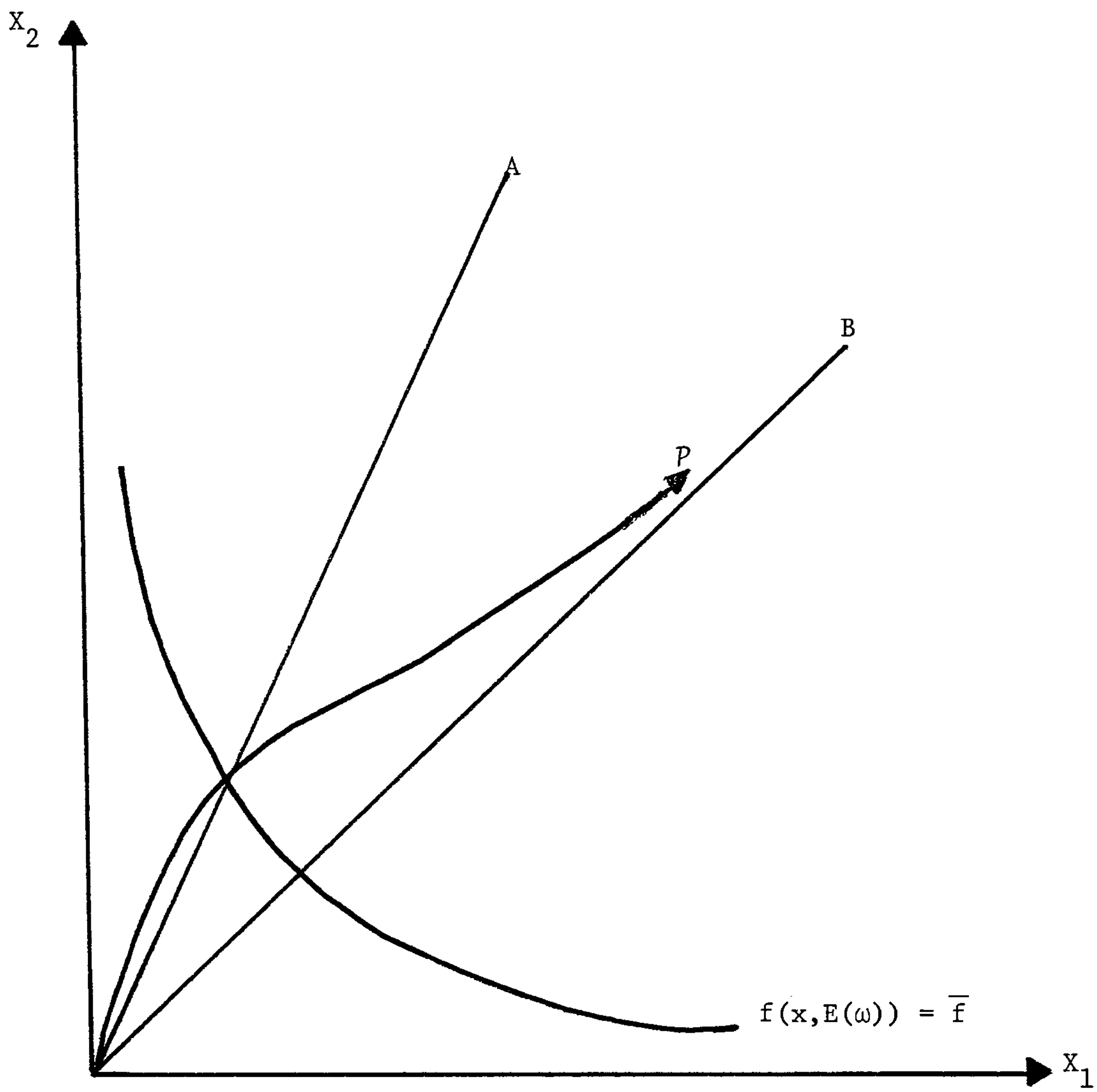

Figure 1

Expansion Path Relationships

the analysis of $Z(x_1,a)$ we see that P represents a continuous function that starts at the origin, travels initially above A, cuts A at $x_1 = 1$, and asymptotically approaches B. This is illustrated in Figure 1 with $\bar{f}= (2\beta q_1/aq_2)^{\beta}$ providing the isoquant of $f(x,E(\omega))$ where P crosses A.

Therefore, by not explicitly modeling the presence of noise in the production process (and instead having implicitly relegated it to a multiplicative error form) we might be misled into concluding that some sort of input usage bias existed for the firm under study. Since the firm will be operating on P (not A), observations of input combinations will occur on (or about) P, not on (or about) A. This will become more acute the greater the range of operation of the firm. Since the observations will not lie along A, an improper conclusion of bias in the use of inputs might occur, when in fact the firm is acting efficiently

3.3. Cost/Production Duality and Stochastic Production Functional Structure

What are the implications of estimating a cost function rather than the production function? We shall see that the "obvious" cost function to estimate will only be consistent with an extremely narrow range of production functions.

To lay the groundwork for what will be shown, we observe that the solution to (CMP) produces a production function which is the convex hull of E(f). To see this we consider a (expected) production possibility set $Y \subset R^{n+1}$ and an inputs requirement set $V(E(y))$:

$$V(E(y)) = \{x \in R^n_+ : (E(y),-x) \in Y\} ,$$

with expected isoquants $Q(E(y))$:

$$Q(E(y)) = \{x \in R^n_+ : x \in V(E(y)), x \notin V(E(y) + \alpha) \ \forall \alpha > 0 \},$$

and finally we consider the cost function-implied technology V*(E(y)):

$$V^*(E(y)) = \{x \in R^n_+ : q'x \geq C(E(y),q) \ \forall q > 0\}$$

where the cost function is C(E(y),q), the result of (CMP). The above sets are direct extensions of their non-stochastic counterparts in the literature (see [11], [17], and [22]). C(E(y),q), V(E(y)) and $V^*(E(y))$ have the usual regularity, monotonicity, etc. properties (see [22], Chapter 1). These will not be repeated here, except to note that the above functions and sets are well-defined and come from a specified technology (which we denote $\{f(x,\omega),G\}$) and a given vector of input prices q.

Now, returning to the first section of the paper, if an analyst observes output (y), costs (C), input prices (q) and factor levels (x) then the "obvious" cost function to estimate is C(y,q), perhaps augmented by factor demand equations x = x(y,q). It has been argued that C(y,q) should have as general a form as possible but it should be noted that the supposedly obvious cost function C(y,q) is a function of the random output y, not the expected output E(y). This is because we observe realizations of y and <u>not</u> E(y). If, for example, C(y,q) is estimated via a regression model then the resulting estimated model is E(C(y,q)). Note that E(C(y,q)) = C(E(y),q) if and only if*

$$C(y,q) = y\,\ell(q) + d(q)$$

where $\ell: R^n_{++} \rightarrow R_+$ and $d: R^n_{++} \rightarrow R_+$. Since all inputs are variable and C(E(y),q) must be a cost function, it follows that C(0,q) = 0 and therefore d(q) = 0. Thus $C(E(y),q) = E(y)\ell(q)$. Finally, since cost functions should be posi-

*This is true since we must allow for arbitrary distributions on y which reflect the fact that we did not start with a specific technological description $\{f(x,\omega),G\}$.

tively linearly homogeneous (PLH) in prices, $\ell(q)$ is a homogeneous function of degree one. Hence, using a well-known result of Shephard [17], $E(f(x,\omega))$ must be homogeneous of degree one in the inputs (x). Stated more exactly we have the following theorem:

Theorem 3: Estimation of a cost function $C(y,q)$ of noisy output y and given input prices q is consistent with an underlying stochastic production process $\{f(x,\omega),G\}$ with G arbitrary only if $f(x,\omega)$ is (PLH) in x.

$f(x,\omega)$ must be (PLH) since if $E(f(x,\omega))$ is required to be (PLH) then by Eulers theorem

$$\int_D (f(x,\omega) - \sum f_i(x,\omega)x_i)dG = 0$$

for arbitrary G. Thus $f(x,\omega)$ must be (PLH) also.

3.4. Implications for Estimating Cost Functions: An Example

In the previous section we found that:

(1) Cost is a function of expected output, and not observed output;

(2) cost functions estimated using observed output will provide valid technological information only if the firm faces constant returns to scale.

The use of cost functions to provide technological information (e.g. returns-to-scale) has become a standard analytical tool. Cost functions have been estimated for such diverse areas as electric utilities [5], railroads [4], hospitals [14], the Canadian Economy [7], and garbage collection [19] to name but a few.

In general, most such studies* have used observed output. From the analysis above we know that if output is stochastic then using observed output

*Two exceptions are [3] and [10].

is incorrect; expected output should be used instead. Clearly, however, it is important to attempt to see how bad such a misspecification might be. It is certainly possible that even though the use of observed output is theoretically incorrect, it is not particularly damaging in practice.

Such a proposition is difficult to address theoretically, since the associated cost functions are very difficult to derive. Instead, we will examine a simple example based on the example from section 3.2 above. Let

$$f(x,\omega) = x_1^{\omega} x_2^{\beta}$$

again with $\omega \sim U[0,a]$, $a > 0$. As was shown earlier, the expected production function is :

$$E(f(x,\omega)) = \begin{cases} \dfrac{(x_1^a - 1)\, x_2^{\beta}}{\ell n\, x_1^a} & x_1 \neq 1, \\ x_2^{\beta} & x_1 = 1. \end{cases}$$

In particular, let $a = 1$ and let $\beta = .5$. While the cost function dual to $E(f)$ is not easily derivable one can use a computer to simulate a firm with such a technology; this is precisely what was done. Prices q_1 and q_2 were independently drawn from a uniform distribution on [10, 20]. Optimal input levels were computed so as to minimize cost subject to meeting a required expected output level (eight levels were used: $E(f) = 2,3,4,5,6,7,8,9$). Six price pairs were drawn, providing 48 observastions. The density, $U[0,1]$ was sampled 48 times (independent draws) to provide realizations of ω.. These were used, with the (x_1^*, x_2^*) values to produced "actual" output. Finally, after the cost was compuated, a normal (0,1) deviate was added to the cost. Thus, a simulated data base providing observed cost, observed actual output, expected output, input prices, and input levels was constructed providing 48 observations on a cost-minimizing, price-taking firm facing the stochastic production function specified above.

A transcendental-logarithmic cost function was fit to the data. The cost function was augmented with a factor share equation (for variable 1) as discussed by Christensen and Greene [5]. The cost model is as follows:

$$(3) \qquad \ell n\, C = \alpha_0 + \alpha_z \ell n_z + \alpha_1 \ell n q_1 + \alpha_2 \ell n q_2 + \alpha_{zz}(\ell n z)^2/2 + \alpha_{11}(\ell n q_1)^2/2 + \alpha_{22}(\ell n q_2)^2/2 + \alpha_{z1}\ell n z \ell n q_1 + \alpha_{z2}\ell n z \ell n q_2 + \alpha_{12}\ell n q_1 \ell n q_2$$

$$(4) \qquad S_1 = \alpha_1 + \alpha_{11}\ell n q_1 + \alpha_{12}\ell n q_2 + \alpha_{z1}\ell n z$$

where

C = observed cost/sample mean,

z = output/sample mean,

q_i = observed input i price/sample mean,

S_1 = price of input one x quantity of input one/observed cost.

This system was estimated via full-information maximum likelihood using the WYMER package, written by Cliff Wymer, available on Northwesterns CDC6600. Since we are estimating a cost function, homogeneity in factor prices was enforced by adding the constraints:

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_{11} + \alpha_{12} = 0$$

$$\alpha_{12} + \alpha_{22} = 0$$

$$\alpha_{z1} + \alpha_{z2} = 0.$$

Two functions, were estimated: one with output as E(f) and one with output as $f(x,\omega)$. Table one provides the estimation results (asymptotic standard errors are provided in parentheses)

TABLE I

Parameter	Expected Output Model	Actual Output Model
α_0	-.440216	-.429391
	(.141938)	(.171476)
α_z	1.254480	.656391
	(.294479)	(.224451)
α_1	.579532	.579670
	(.090284)	(.096495)
α_2	.420468	.420330
	(.090284)	(.096495)
α_{zz}	.142559	.018345
	(.499665)	(.321297)
α_{11}	-.190921	-.114360
	(.174750)	(.197385)
α_{22}	-.190921	-.114360
	(.174750)	(.197385)
α_{z1}	-.226397	-.150679
	(.164960)	(.128745)
α_{z2}	.226397	.150679
	(.164960)	(.128745)
α_{12}	.190921	.114360
	(.174750)	(.197385)

One of the major uses of a cost model is to provide implications about returns-to-scale. In a single output model, returns are simply (for the translog model) α_z^{-1} if we are evaluating the returns at the point-of-means, i.e. for all variables taken at their sample means. In the case of the expected output model our scale-economies prediction would be $(1.25)^{-1}$ or .8. In the case of the actual output model we get a prediction of over 1.5. These are vastly different predictions. In one case we find decreasing returns-to-scale, while in the other case we find increasing returns-to-scale. If this were infact a study aimed at providing policy implications, then the use of actual output, instead of expected output, would reverse the implication.

Now, it might be argued that the standard errors for α_z are relatively large and that infact neither model rejects constant-returns-to-scale. This argument would be wrong. The appropriate way to test for constant-returns-to-scale is to so restrict the model and then perform a log-likelihood test (see Theil [21]). If we do this we find that the critical type I error value for the expected output model is approximately .11 while the critical value for the actual output model is less than 0.005. In other words, in order to reject constant-returns-to-scale in the expected output model, one would have to risk a type I error of at least .11, while in order not to reject constant-returns-to-scale in the actual output model one would have to require a type I error level of less than .005.

Thus, the two cost functions provide very different implications. From the theory developed above we know that the cost function using expected output is dual to the technology relevant to firm decisions (the expected production function) and thus it provides useful information. From this example we can see that not only is a cost function based on actual output not dual to the appropriate technology (when output is stochastic), it can provide grossly misleading results in terms of statistical analysis. In short, using expected output (in cases where output is stochastic) is not only theoretically correct, it is critical from the viewpoint of practice.

4. Summary and Conclusions

In the preceeding sections both the first order optimizing conditions and the cost function were derived for a competitive,expected profit-maximizing firm faced with a stochastic production process. The subsequent analysis showed that models of technology for such firms based on the usual observables (costs, input prices and levels, output levels) can be severely biased unless the stochastic nature of the technology is properly accounted for.

In the case of direct analyses of technology via estimation of a production function we found that only under special conditions (i.e. stochastic separability of the production function) would the standard approach to estimating a production function be appropriate. An example of a non-stochastically separable production function showed how misrepresentation of the firm's expansion path was possible if the stochastic nature of the production function was not explicitly modeled.

An alternative approach to analysis of technology has often been available via estimation of the cost function of a firm. Here again we find that not properly accounting for the stochastic aspects of a technology may result in a misspecified model: the cost function estimated on the observables of input prices and output levels will only be consistent with a constant returns-to-scale technology. This is because commitments on input factors are made in anticipation of an expected output level, thereby linking cost with planned output and committed purchases of inputs.

Extending the non-stochastically separable example used earlier, we saw that estimation of the cost function using standard procedure (i.e. using observed costs, input prices and observed outputs) provided estimates that reversed the technological implications that were actually present: even though the firm did not enjoy returns-to-scale (at the point-of-means), the standard procedure predicted returns-to-scale. Thus, applied studies using such procedures may produce grossly inaccurate technological implications, thereby providing incorrect policy prescriptions.

In general then, cost function analysis can be used to characterize the relevant aspects of firm technology, if the cost function is correctly posed as being a function of input prices and expected output. Then the cost function is dual to the expected production functions, which is all that the expected profit maximizing firm needs to know anyway.

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