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CONNECTING AND RESOLVING SEN'S AND ARROW'S THEOREMS

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ABSTRACT. As shown, the source of Sen's and Arrow's impossibility theorems is that Sen's Liberal condition and Arrow's IIA counter the critical assumption that voters' have transitive preferences. As this allows transitive and certain cyclic preferences to become indistinguishable, the Pareto condition forces cycles. Once the common cause of these perplexing conclusions is understood, resolutions are immediate.

After several decades Arrow's (1952) Impossibility Theorem and Sen's (1970) "The impossibility of a paretian liberal" correctly retain their central, seminal positions in the large literature they have spawned. On the surface, these results are different. Sen, for instance, comments that "unlike in the theorem of Arrow, we have not required transitivity of social preference. We have required ... merely the existence of a best alternative in each choice situation." Sen further notes that he has "not imposed Arrow's much debated condition of 'the independence of irrelevant alternatives.' " Nevertheless, as demonstrated here, Sen's and Arrow's Theorems share the same (surprisingly elementary) explanation. Once we understand this common cause, the conclusions become obvious and resolutions are immediate. It is of interest that the same argument explains related problems from other areas about other issues; e.g., consumer surplus, etc.

1. SEN'S THEOREM

In his theory, which is of interest only for $k \geq 3$ alternatives, Sen uses the axioms

(U) (*Unrestricted Domain.* Every logically possible set of individual orders is included in the domain of the collective choice rule.)

(P) (*Pareto.* If every individual prefers any alternative a to another alternative b, then society must prefer a to b.)

(L*) (*Minimal Liberalism.* There are at least two individuals where each is decisive over at least one pair of alternatives; i.e., there is a pair $\{a, b\}$ such that if

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a decisive individual over this pair prefers $a \succ b$ (respectively $b \succ a$), then society prefers $a \succ b$ (respectively $b \succ a$).

Sen proves that all procedures satisfying these axioms admit cycles; these cycles frustrate the identification of a maximal element. The preferences of the $n \geq 2$ voters are, of course, transitive. After all, Sen's conclusion has no interest should voters have cyclic preferences because (P) would mandate a cyclic outcome for the unanimity cyclic profile $a \succ b, b \succ c, c \succ a$. This cycle is not a problem; it only manifests the "garbage in, garbage out" adage that if structure is not imposed on inputs, we cannot expect structure on outputs. Transitivity of preferences, then, is a critical assumption for choice theory.

The surprise is that, contrary to expectations and intentions, *all procedures satisfying L* counter the transitivity assumption; L* procedures are specifically required to service voters with the more primitive cyclic and other nontransitive preferences.* Indeed, if a procedures can serve only voters with transitive preferences, such as the standard plurality vote, it is eliminated by L*. So, from the perspective of lost transitivity, rather than being worrisome or paradoxical, Sen's conclusion is to be expected; the real mystery would be if his conclusion were otherwise.

To explain, recall that a role of axioms is to exclude procedures. Thus, to find all methods satisfying specified axioms, we could go down the list and systematically exclude procedures that fail to satisfy each particular condition. For instance, (P) dismisses all positional methods (i.e., procedures where points are assigned to candidates according to how a voter ranks them) which award a single point to each of a voter's j top-ranked candidates and zero to all others. (So, $j = 1$ is the plurality vote, $j = k - 1$ is the antiplurality vote.) The plurality vote is excluded, for example, because its $a \succ b \sim c$ ranking for the unanimity profile $a \succ b \succ c$ fails to respond to the unanimous $b \succ c$ preference.

To appreciate the kinds of procedures dismissed by L* consider Sen's proof for the $k = 3$ alternatives $\{a, b, c\}$ where voters one and two determine, respectively, the $\{a, b\}$ and the $\{a, c\}$ outcomes. Only the decisive voter determines the outcome of a designated pair, so restrictions cannot be imposed on the other voters' binary rankings for the pairs labelled "none" in Table 1. Consequently, a L* procedure can neither recognize nor use these binary rankings; it cannot even check whether they support or deny transitivity.

(1)

Voter	$\{a, b\}$	$\{b, c\}$	$\{a, c\}$
1			none
2	none		
Others	none		none

Instead of embracing the critical assumption of transitivity, the true domain of a L* admissible procedure (for $k = 3$ and where the decisive voters have power over the indicated pairs) is the set of all profiles where the "none" slots can be filled in any desired manner. Denote this domain by $SEN^n(3)$. Notice that $SEN^n(3)$ only requires a voter to rank each pair; the pairwise rankings need not satisfy *any* sequencing requirement such as transitivity. It is equally obvious that a L* procedure is defined on $SEN^n(3)$. (This automatically eliminates all positional voting methods because they require at least acyclic preferences.) Thus, the L* procedures are intended for societies so primitive that the voters are assumed capable only of ranking pairs.

While Sen obviously did not anticipate nor intend to use $SEN^n(3)$, it is a mandatory domain for L* procedures. The transitivity assumption and U, then, constitute a *profile restriction* to $T^n(3) = \{$all n-person profiles where each person has transitive preferences of the three candidates and where there are no pairwise ties.$\}$ To connect this profile restriction with standard choice issues, recall that even though (by assumption) voters have transitive preferences, the pairwise election outcomes can be cyclic. One resolution to remove these cycles is to find sufficiently severe profile restrictions, such as Black's single-peakedness (Black 1958, Saari 1994, 1995b), to ensure acyclic outcomes. Similarly, in our process of systematically removing procedures, the next step is to retain only those L* procedures which adequately serve the sophisticated voters modelled by $T^n(3)$ where Sen's definition of "adequate" requires the outcomes to be at least acyclic.

In order to identify these L* procedures, a first step is to find the procedures capable of distinguishing between transitive and nontransitive preferences; that is, first find the L* procedures which at least recognize that $T^n(3)$ and $SEN^n(3)$ are different domains. To see what is involved by using Sen's example (but with a significantly different interpretation), the following table displays the portions of his profile recognized by a L* procedure. If a L* procedure can distinguish between $T^n(3)$ and $SEN^n(3)\backslash T^n(3)$, then either (1) it is impossible to fill in the blanks to create transitive rankings (so this partial profile cannot occur with the $T^n(3)$ restriction), or (2) the only way to fill the blanks results in transitive preferences (so some procedure may recognize transitive preferences).

$$
(2) \qquad \begin{array}{cccc}
1 & a \succ b & b \succ c & \\
2 & & b \succ c & c \succ a \\
\text{Others} & & b \succ c &
\end{array}
$$

As it is trivial to fill the blanks without respecting transitivity (so 2 fails), it remains (1) to determine whether they can be filled with transitive rankings. They can; for voter-one, use $a \succ c$, for voter-two use $b \succ a$, and for all others use $a \succ b$, $a \succ c$. Thus the $T^n(3)$ profile restriction fails because a L* procedure cannot distinguish between transitive and nontransitive preferences; the transitivity dismissed by L* remains lost. By modifying the above argument it is easy to show the following:

Proposition 1. *Let* $\mathbf{p} \in SEN^n(3)$. *There exists a profile* $\mathbf{p}_t \in T^n(3)$ *so that a* L^* *procedure cannot distinguish between* $\mathbf{p}$ *and* $\mathbf{p}_t$. *Conversely, for* $\mathbf{p}_t \in T^n(3)$, *not only are there profiles* $\mathbf{p}_c \in SEN^n(3)\backslash T^n(3)$ *that a* L^* *procedure cannot differentiate from* $\mathbf{p}_t$, *but* $\mathbf{p}_c$ *can be chosen so that all voters have cyclic preferences.*[1]

According to Prop. 1, the transitivity restriction has *not*, in any manner, changed or restricted the perceivable domain for a L* procedure; all L* procedures behave

[1] Technically, two profiles are equivalent if they agree on all pairwise rankings except those noted as "none" in Table 1; the equivalence classes of this relationship partition $SEN^n(3)$. The proposition states that each equivalence class has at least one profile from $T^n(3)$ as well as at least one profile from $SEN^n(3)\backslash T^n(3)$ where each voter is cyclic. Thus, $T^n(3)$ does not change the number of $SEN^n(3)$ equivalence classes.

as though the domain remains $SEN^n(3)$. This is because the portion of a $SEN^n(3)$ profile recognized by a L* procedure also occurs for some profile in $T^n(3)$. Indeed, a L* procedure cannot even determine whether each voter is transitive or cyclic! Thus all of the L* outcomes arising with the primitive voter domain $SEN^n(3)$ also occur with transitive voters!

It now is easy to construct a transitive profile with cyclic outcomes. Start with a profile $\mathbf{p}_c \in SEN^n(3)$ where a cycle is the natural outcome (because of (P) or some other condition such as maximin principle used in (Gaertner, Pattanaik, and Suzumura, 1992)), and then construct one of the guaranteed (by Prop. 1) transitive profiles $\mathbf{p}_t \in T^n(3)$ that is L* indistinguishable from $\mathbf{p}_c$. A natural candidate for $\mathbf{p}_c$ is the unanimity cyclic profile $a \succ b$, $b \succ c$, $c \succ a$ because (P) forces the same cyclic outcome. A choice for the indistinguishable $\mathbf{p}_t$ is the transitive profile constructed for Table 2. This means that the only fair (i.e., (P) respecting) outcome for a profile with the partial listing of Table 2 is a cycle. From this perspective, rather than being a surprise, cyclic outcomes must be expected. Transitivity remains a L* hostage that (because of Prop. 1) cannot be easily released.

The same argument holds for $k \geq 4$ where voters one and two determine respectively, the $\{a, b\}$ and the $\{c, d\}$ rankings. As a L* procedure cannot recognize the binary preferences nondecisive voters have over these pairs, the L* effective domain imposes no restrictions on how a nondecisive voter ranks them. Again, counter to expectations and intentions, L* destroys the assumption of transitive preferences. Instead, a procedure satisfying L* is defined over a larger domain[2] $SEN^n(k)$ where most preferences fail transitivity and (by P) many L* outcomes are cyclic. Again, the obvious, immediate extension of Prop. 1 asserts that for every $\mathbf{p} \in SEN^n(k)$, there is a L* indistinguishable transitive profile $\mathbf{p}_t \in T^n(k)$. Consequently the profile restriction $T^n(k)$ does not, in any way, alter or restrict the domain for a L* procedure from that of $SEN^n(k)$.

Again, cyclic outcomes are generated by choosing $\mathbf{p}_c \in SEN^n(k)$ where the only (P) fair outcome is a cycle and then finding an indistinguishable transitive profile. One $\mathbf{p}_c$ choice assigns all voters the cyclic rankings $a \succ b, b \succ c, c \succ d, d \succ a$ so (because of P) this cycle is in the outcome. To find an indistinguishable $\mathbf{p}_t \in T^n(k)$, list the $\mathbf{p}_c$ portions that a L* procedure can examine.

(3)

Voter	$\{a, b\}$	$\{b, c\}$	$\{c, d\}$	$\{a, d\}$
1	$a \succ b$	$b \succ c$		$d \succ a$
2		$b \succ c$	$c \succ d$	$d \succ a$
Others		$b \succ c$		$d \succ a$

Generating a L* equivalent, transitive profile $\mathbf{p}_t$ is immediate; e.g., let voter-one have the preferences $d \succ a \succ b \succ c$ and all other voters have $b \succ c \succ d \succ a$.

Salles (1994) escalates the complexity of the difficulty by creating a troubling situation where the same transitive profile generates *two* cycles. Using our notation,

[2] Define $SEN^n(k)$ to be where triplets of pairs recognized by L* satisfy transitivity, but there is no restriction on the rankings of pairs decided by other voters. (In Table 3, this restricts voter-one's missing rankings to be $a \succ c$, $d \succ b$.) Equivalently, start with transitive preferences and then vary the rankings of the indicated pairs in all possible ways. This second representation makes the extension of Prop. 1 immediate.

his Mr. Prude and Mr. Lascivious have, respectively, the preferences $a \succ b \succ c \succ d$ and $d \succ b \succ c \succ a$. Salles' description of these alternatives permits him to appeal to Hammond's (1982) condition about privately unconditional preferences so that P is decisive over $\{a, b\}$, $\{c, d\}$ while L is decisive over $\{a, c\}$, $\{b, d\}$. This combination of decisiveness and specified preferences defines

$$\begin{array}{llllll} & P & a \succ b & & c \succ d & & b \succ c \\ (4) & L & & c \succ a & & d \succ b & b \succ c \end{array}$$

To understand Salles' cycles note that Table 4 also admits the indistinguishable nontransitive unanimity profile $a \succ b$, $c \succ a$, $d \succ b$, $c \succ d, b \succ c$ creating the two (P) cycles $a \succ b \succ c \succ a$, $b \succ c \succ d \succ b$. Again, the problem arises because the critical transitivity assumption is lost whenever "natural conditions" change the actual domain to admit nontransitive voters. More complex examples now can be designed.

With the failure of L* procedures to recognize $T^n(k)$, rather than being a mystery or surprise, Sen's Theorem is to be expected. From this perspective, his theorem only asserts that no procedure intended for primitive preferences meets the needs of sophisticated (transitive) voters; transitivity is too weak of a profile restriction. While we could try to impose effective profile restrictions[3], stronger profile restrictions do not alter the fact that we still are dealing with crude procedures defined for the nontransitive preferences of $SEN^n(k)$. Consequently, even with severe profile restrictions, stilted conclusions and procedures must be expected. A better approach is to replace L* and (P) with more reasonable conditions where the actual domain better approximates transitive voters; this is discussed in Section 4.

2. Arrow's Theorem

For $k \geq 3$ alternatives, Arrow uses the axioms

(IIA) (*Independence of irrelevant alternatives.* The relative group ranking of any two candidates only depends upon the voters' relative ranking of this pair.)

(ND) (*No dictator.* The group outcome cannot always be the same as the ranking of a particular voter.)

to conclude that if the voters' preferences are transitive and if the outcomes must be transitive, then the only procedure satisfying U, P, IIA is a dictator; namely, U, P, IIA, and ND are in conflict. As true with Sen's Theorem, Arrow's conclusion would have no interest should voters have cyclic preferences (because (P) would require cyclic outcomes). Nevertheless, against expectations and intentions, IIA welcomes all voters with cyclic or other nontransitive kinds of preferences; an unexpected IIA consequence is that it retains only those procedures that service voters with these primitive preferences! Because IIA (as true for L*) dismisses the critical assumption of transitive preferences, rather than being surprising or disturbing, Arrow's conclusion is to be anticipated.

If the k candidates are $\{c_1, c_2, \dots, c_k\}$ and $B(c_i, c_j) = \{c_i \succ c_j,\ c_j \succ c_i\}$, then $B(k) = \prod_{i<j} B(c_i, c_j)$ lists all $2^{\binom{k}{2}}$ ways to (strictly) rank each pair of alternatives.

[3] An effective profile restriction eliminates those *equivalence classes* with a $SEN^n(k)$ profile where P requires the L* procedures to have cyclic conclusions; the restriction need not be stated in terms of transitive preferences!

The product of $B(c_i, c_j)$ over all n voters, $B^n(c_i, c_j)$, lists all ways the voters can rank this pair while $B^n(k) = \prod_{i<j} B^n(c_i, c_j) = B(k) \times \cdots \times B(k)$ is the space of profiles where each voter is required only to rank each pair. As the pairwise rankings defining preferences in $B^n(k)$ need not be connected in any manner, they need not be transitive, quasitransitive, acyclic, or anything else. It is immediate that an IIA procedure must be defined on $B^n(k)$. Actually, IIA imposes an even more severe restriction. For a given procedure F, define $F_{c_i,c_j}(\mathbf{p})$ to be the $\{c_i, c_j\}$ relative rankings in $F(\mathbf{p})$. (So, if $F(\mathbf{p}) = c_3 \succ c_2 \succ c_4 \succ c_1$, then $F_{c_1,c_2}(\mathbf{p}) = c_2 \succ c_1$ and $F_{c_3,c_4}(\mathbf{p}) = c_3 \succ c_4$.) A procedure F satisfies IIA if and only if for each $\{c_i, c_j\}$, the domain for F_{c_i,c_j} is $B^n(c_i, c_j)$. As a consequence, IIA only requires the voters to be able to rank each pair of alternatives; transitivity of personal preferences is a separate assumption.

It follows from this statement that IIA eliminates all procedures that serve only transitive, or even only acyclic voters. Consequently, IIA excludes all positional methods including the Borda Count (BC). (For k candidates, the BC assigns $k - j$ points to a voter's jth ranked candidate; $j = 1, \dots, k$.) In particular, all IIA procedures can be used by voters with the crude preferences represented by $B^n(k)$. Rather than being anyone's choice, this domain, which includes cyclic and other nontransitive outcome, is a hidden IIA consequence. But because IIA dismisses transitivity, Arrow's conclusion must be anticipated.

As with Sen's Theorem, (U) imposes $T^n(k)$ as a *profile restriction* to further reduce the class of IIA procedures. The goal is to determine whether the $T^n(k)$ constraint allows any of the mappings intended for the crude $B^n(k)$ preferences to recognize when it is dealing with rational voters (at least so that only transitive outcomes occur). A dictator is one such procedure; when restricted to transitive preferences, the dictator's binary rankings can be assembled in only one manner.

In Sen's framework, the decisive agents are identified; this allows the same equivalence classes of profiles to apply to all L* methods (Prop. 1). In Arrow's formulation, however, the equivalence classes for a procedure are its "level sets" (of $B^n(k)$ profiles) defined for each outcome.[4] Nevertheless, the argument remains essentially the same; because $B^n(k)$, $n \geq 2$, $k \geq 3$, is so huge, some level set of profiles contains a $\mathbf{p}_t \in T^n(k)$ and a $\mathbf{p}_c \in B^n(k) \setminus T^n(k)$ where the nontransitive profile $\mathbf{p}_c$ determines the procedure's "fair" but nontransitive outcome.

This assertion can be illustrated with the important pairwise vote (which is defined over $B^n(3)$). To do so, I must find a transitive $\mathbf{p}_t$ and a nontransitive $\mathbf{p}_c$ which are indistinguishable with the pairwise vote and where the "fair" nontransitive outcome is determined by $\mathbf{p}_c$. To do so (Saari 1994, 1995b), decompose the Condorcet profile into its binary parts as given in the following table.

(5)

$a \succ b \succ c$	$(a \succ b)_1$	$(b \succ c)_2$	$(a \succ c)_3$
$b \succ c \succ a$	$(b \succ a)_3$	$(b \succ c)_1$	$(c \succ a)_2$
$c \succ a \succ b$	$(a \succ b)_2$	$(c \succ b)_3$	$(c \succ a)_1$

By satisfying anonymity, the pairwise vote cannot determine who cast what ballot, so it cannot distinguish the Condorcet profile from the profile $\mathbf{p}_c$ where the pairwise

[4] Also, an assertion such as Prop. 1 does not hold in Arrow's setting. For instance, with the pairwise vote, there is no transitive profile that is indistinguishable from the unanimous cyclic profile $\mathbf{p}_c \in B^n(3)$ where $a \succ b$, $b \succ c$, $c \succ a$.

rankings of voters 1, 2, 3 are specified by the subscripts. Thus, the pairwise vote (or any procedure satisfying IIA and anonymity) is incapable of distinguishing between the Condorcet profile and the profile of cyclic, confused voters where voters one and two have the cyclic preferences $a \succ b$, $b \succ c$, $c \succ a$ and voter three has the cyclic preferences $a \succ c$, $c \succ b$, $b \succ a$. By being indistinguishable, both profiles have the same pairwise outcomes; this common conclusion must be the $\mathbf{p}_c$ "fair" outcome of a cycle. ("Fairness" for the pairwise vote is measured by the number of voters with each ranking.) Thus the only fair pairwise outcome for a profile with the binary components given by Table 5 is a cycle. Stated in another manner, the real source of (the widely studied topic of) pairwise voting cycles is that the procedure ignores the transitivity of preferences. (Notice how this comment impugns the integrity of methods based on pairwise rankings; this includes the widely accepted standard of the "Condorcet winner." For different, stronger arguments, see (Saari, 1995b).)

While the Table 5 decomposition applies for those IIA procedures satisfying anonymity, it is not applicable for a procedure where individuals have varying levels of influence on the outcome. To handle all possible situations, think of an IIA procedure as defining a sense of "fairness" as manifested by which of the voters' binary rankings determine each relative ranking of each pair. As I now show, whatever this sense of fairness, each ND, IIA procedure defines equivalence classes of profiles (i.e., profiles among which the procedure cannot differentiate) with transitive and nontransitive profiles but where some of the nontransitive profiles dictate a nontransitive outcome based on the procedure's "fairness" criteria.

This goal is accomplished by using the proof of (an extended version of) Arrow's Theorem in (Saari, 1994, 1995b). This proof reduces to examining the properties of a partial profile where two voters are decisive over different pairs, say $\{a, b\}$ and $\{b, c\}$. A voter, however, may be decisive over an assigned pair only in *certain specified situations* (i.e., all other voters may need to have specified rankings of the pair). The argument used to generate a cycle requires voter-one to change $\{a, b\}$ rankings while keeping fixed a specified $\{b, c\}$ ranking and an unspecified $\{a, c\}$ ranking. Similarly, voter-two has to vary $\{b, c\}$ rankings while keeping fixed a specified $\{a, b\}$ and an unspecified $\{a, c\}$ ranking.[5] If a ND, IIA procedure is capable of recognizing when it is dealing with rational voters, then either it is impossible to fill in the "fixed" regions with binary rankings that are transitive (so this situation cannot occur with the $T^n(k)$ profile restriction), or these regions can be filled only in a transitive manner.

(6)

Voter	$\{a, b\}$	$\{b, c\}$	$\{a, c\}$
1	Varies	Specified	Fixed
2	Specified	Varies	Fixed
Others	Specified	Specified	Fixed

Without transitivity, the "fixed" blanks can be filled in any desired way, so it remains to determine whether they can be filled while preserving transitivity. They can; vary the first voter's rankings between $a \succ b \succ c$, $b \succ a \succ c$ or between $c \succ a \succ b$, $c \succ b \succ a$ where the choice depends upon the specified $\{b, c\}$ ranking.

[5] To see why this suffices, after the fixed $\{a, c\}$ ranking is determined, say it is $c \succ a$, then voters one and two can choose preferences so that the group's rankings are $a \succ b$ and $b \succ c$.

(In either case the $\{a, c\}$ and $\{b, c\}$ rankings remain fixed.) Similarly, the second voter varies between $a \succ b \succ c$, $a \succ c \succ b$ or $b \succ c \succ a$, $c \succ b \succ a$ depending on the required $\{a, b\}$ ranking. For each of the remaining voters, once the $\{a, b\}$ and $\{b, c\}$ ranking is specified, an $\{a, c\}$ choice can be made to ensure transitivity. Because IIA prohibits imposing any requirement upon the $\{a, c\}$ ranking, other than it remains fixed, the procedure cannot distinguish between certain transitive and nontransitive profiles. Thus cycles can be viewed as manifesting a "fairness" property of the IIA procedure which is dictated by a nontransitive profile; the partial portion of the nontransitive profile is indistinguishable from the partial profile of transitive preferences. In particular, the $T^n(k)$ restriction does not allow ND, IIA respecting procedures to recognize transitive preferences. As this effectively returns us to the realm of nontransitive voters, Arrow's conclusion is to be expected.

3. IIA TYPE CONDITIONS

Similar assertions, supported by the same simple arguments, must be anticipated in any situation (beyond L^* and IIA) where portions of voters' preferences are ignored. Namely, by ignoring substitutes, equivalence classes of preferences are defined where most preferences are not transitive. Such conditions only require us to assume transitive preferences over each of the indicated subsets of alternatives. Using the "mix-and-match" approach of Table 5 with sufficiently heterogeneous preferences and enough voters (or by using the argument applied to Arrow's assertion), it may be impossible to distinguish transitive and intransitive preferences. Therefore, unless the spirit of IIA is discarded by imposing a condition to connect different sets of substitutes, we must expect the transitivity assumption to be useless.

To illustrate with $k = 4$, consider an IIA condition where the relative ranking of any triplet only depends upon the voters' relative rankings of this triplet. The effective domain for an admissible procedure requires each voter to rank each triplet in a transitive manner, but, because IIA decouples the sets, these rankings need not be related in any manner. This admits, for instance, a voter with the cyclic preferences $a \succ b \succ c$, $b \succ c \succ d$, $c \succ d \succ a$, $d \succ a \succ b$. Again, $T^n(k)$ does not suffice to allow a procedure to recognize when it is dealing with rational voters. This can be seen by using the "mix-and-match" approach of Table 5 to show that a procedure satisfying the triplet IIA condition and anonymity cannot distinguish the transitive Condorcet profile $a \succ b \succ c \succ d$, $b \succ c \succ d \succ a$, $c \succ d \succ a \succ b$, $d \succ a \succ b \succ c$ from the nontransitive four-voter profile

1, 2	$a \succ b \succ c$	$b \succ c \succ d$	$c \succ d \succ a$	$d \succ a \succ b$
3	$b \succ c \succ a$	$c \succ d \succ b$	$d \succ a \succ c$	$a \succ b \succ d$
4	$c \succ a \succ b$	$d \succ b \succ c$	$a \succ c \succ d$	$b \succ d \succ a$

where voters one and two have the cyclic preferences and voters two, three and four define a Condorcet cycle over each triplet (so, with a positional method, their preferences cancel). This inability to distinguish a rational from a confused voter profile suggests, for instance, that positional election outcomes over the triplets need not correspond to the positional election outcome for the same profile over all four candidates. (While the proof uses radically different arguments, this is the case; see (Saari 1995d) and the cited references.) Because the spirit of IIA is so prevalent, there are many consequences of this observation that IIA vitiates the assumption of transitive preferences. For instance, when a profile defines the conflicting $a \succ b \succ c \succ d$ and $c \succ b \succ a$ rankings with certain procedures, we must wonder whether a (not c) is the voters' preferred candidate. Notice how this assertion questions the rationale for runoff elections.

More generally, whenever there are situations where a procedure ignores some of the agents' available substitutes without providing connectivity, transitivity should be assumed to be lost — the actual (rather than the intended) domain for the specified procedure includes an unexpected variety of perverse preferences. Here, the domain only requires transitive preferences for the sets of alternatives identified by each agent. Notice how this situation extends outside of choice theory to include topics from, say, probability where probabilities replace preferences (Saari, 1991, 1995d), statistics, some game theory issues (Saari, 1991), topics in economics such as consumer surplus, or, more generally, the behavior of the aggregate excess demand functions for the different subsets of commodities (where each agent holds fixed his or her holding of the goods not in that subset). (This argument introduces a different interpretation for the last two sections of (Saari, 1995c) and the cited references.) So, the hidden but critical issue is to determine whether imposing the profile restriction of transitive preferences suffices for a procedure to distinguish between transitive and nontransitive profiles. Because many procedures in actual use cannot make these distinctions, difficulties must be anticipated. (New implications for positional voting obtained by examining consequences of this statement will be reported elsewhere.)

To reclaim the lost transitivity, the ranking of different sets cannot be done in isolation of available substitutes. Therefore, resolutions must involve discovering appropriate connectivity conditions. One approach to design procedures is to mimic the large literature explaining when binary rankings satisfy transitivity. As an example, if IIA is replaced by a condition allowing the group's relative ranking of $\{a, b\}$ to involve not only each voter's $\{a, b\}$ ranking, but also how each voter ranks all pairs involving a or b, then, as indicated below, the BC is an admissible procedure. Observe how these connectivity conditions violate IIA.

To end this section with some technical comments, observe that the Pareto condition is a mere convenience for Sen's or Arrow's conclusion; it can be replaced with any condition ensuring that the outcomes for enough pairs can change. For instance, for Sen's assertion with $k = 3$ the decisive voters already supply the necessary change. To see this, for any specified $\{b, c\}$ outcome, choose binary preferences for the decisive voters to create a cycle; Prop. 1 ensures that this partial profile is supported by indistinguishable transitive and cyclic profiles. The extension of Arrow's Theorem in (Saari, 1994, 1995b) replaces (P) with an "involvement" condition that (for $k = 3$) only requires the rankings of at least two pairs to change.

Similarly, the only need (for the proof) for transitive outcomes is that any $\{a, b\}$ binary ranking coupled with a particular $\{b, c\}$ ranking uniquely dictates the transitive $\{a, c\}$ ranking and that this condition holds when the pairs are interchanged in any manner. (See footnote 5.) As Arrow's assertion extends to any setting with this functional relationship, his conclusion holds when indifference is allowed, for set valued mappings, for certain game theoretic analysis, etc. (See Saari, 1991.) Relaxing transitivity to include, say, quasitransitive outcomes introduces freedom because the $a \sim b$ ranking combined with any $\{b, c\}$ ranking fails to completely specify the $\{a, c\}$ ranking; this additional flexibility is what allows Gibbard's (1969) oligarchy conclusion. The reason Gibbard's conclusion remains unacceptable for modern society is another manifestation of the fact that IIA restricts attention to procedures intended for primitive voters incapable of doing anything more than ranking each pair of alternatives. If we must build a vehicle using only oxen and carts, do not expect a Porche.

Technically, all of these issues can be identified with elementary algebra problems involving m unknowns in n equations, $m \geq n$. The equations are the relative rankings of each pair of candidates. (This is immediate in Sen's setting; in Arrow's formulation these are the F_{c_i,c_j} equations.) The issue is to determine whether these equations admit certain solutions; e.g., does there exist a $\mathbf{p}$ so that, for example, $F_{a,b}(\mathbf{p}) = a \succ b$, $F_{b,c}(\mathbf{p}) = b \succ c$ and $F_{a,c}(\mathbf{p}) = c \succ a$? In algebra, the existence follows immediately from "linear independence" conditions. Mimicking the approach developed in (Saari, 1995b), it turns out that the IIA and L* conditions provide the analogous functional independence.

4. Resolutions

If we wish to resolve these problems by using stronger profile restrictions, observe that IIA and L* define equivalence classes of indistinguishable profiles. Thus an effective restriction must eliminate all classes with nontransitive profiles where a "fairness" criteria (e.g., (P)) demands a nontransitive outcome. Indeed, profile restrictions, such as Black's single-peakedness or the replicated preferences common in economic models, can be viewed as finding conditions where the decomposed profiles cannot be reconstructed to create a dominate number of nontransitive preferences (Saari, 1995b). Unfortunately, as true with L* and $k = 3$, these restrictions can be sufficiently Draconian to kill interest in the resulting system. Of more importance, by trying to address the needs of a sophisticated society with procedures designed (thanks to IIA or L*) for primitive voters, the resulting methods tend to be highly stilted with no practical interest. This becomes apparent with the kinds of procedures admitted through profiles restrictions; e.g., see (Kalai, Muller, 1977), (Kalai, Ritz, 1980), (Gibbard, Hylland, Weymark, 1987), and, for the weakest possible profile restriction (where only one voter is restricted from just one ranking) (Saari, 1991, 1995b).

An important lesson learned from these impossibility theorems is that although IIA and related concepts such as L* are attractive, their hidden cost of lost transitivity render them unrealistic and unusable. Instead, whenever transitive preferences can be expected, the merits of a particular set of alternatives cannot be viewed in total separation of other available substitutes. As we now know, this requires conditions allowing different subsets of alternatives to be compared and connected. From this observation, all sorts of conditions are possible. Indeed, the following are only suggestions included to show how this approach removes the roadblock of impossibility assertions.

To explain in terms of Sen's Theorem, recall that his argument invokes the notion of individual liberties where an individual should have the right to determine what to wear, or what to read, or what personal vices to enjoy. But should other voters retain a right to consider even one these alternatives (perhaps by comparing it with other alternatives), transitivity can be lost without some sort of connectivity. Maybe the real question is to determine when such rights are absolute, and when can society interfere. History proves that the answer changes with time and location. For instance, maybe I should have a right to read or sell whatever I want, but what if it involves hate literature in a troubled society or child pornography in a society plagued by child abuse? Smoking in public settings was accepted as an individual decision in the early 1970s, but it no longer is true in much of the Western world. While I can teach my university courses in blue jeans and a faded sweater rather than a sports coat, this would have been seriously frowned upon before the late 1960s. Even something as seemingly innocuous as the color of a shirt may matter; as "colors" can designate gang membership, some public schools restrict the admissible colors and types of clothing. An issue, then, is to determine when an individual right becomes a community decision.

A first, simple, pragmatic measure is that "individual rights" over alternatives may be rightful community concerns when a sufficient number of others possess strong enough opinions about this choice that they will sacrifice other desirable alternatives in order to register their objections. This theme already is suggested by the transitive preferences of Tables 2, 3 where, rather than exhibiting indifference about the decisive voters' choices, the nondecisive voters exhibit strong opinions. So, perhaps the "intensity of preferences" (Saari, 1995a, b) should play an important role in a society sufficiently sophisticated to justify assuming transitive preferences.

A way to measure the $a \succ b$ intensity is to count the number of available alternatives an individual uses to separate a and b in a *transitive* preference ranking. For instance, voters with $d \succ c \succ a \succ b$ or $a \succ b \succ c \succ d$ have a weak (level-zero) $a \succ b$ preference (as no available alternative is used to separate them), while $a \succ d \succ b \succ c$ exhibits a slightly stronger (level-one) $\{a, b\}$ ranking as a and b are separated by one alternative, while $a \succ c \succ d \succ b$ has a level-two $a \succ b$ intensity. By including this intensity level (which nearly is a minimal way to indicate that preferences are transitive), the effective domain of a procedure better matches that of transitive preferences, so more rational outcomes can be expected.

To illustrate with $k = 3$, replace IIA with

(IIA*) (IIA applies to all voters except voter-one. For voter-one let IIA apply except for $\{a, b\}$ rankings where the intensity level is specified.)

The IIA* effective domain allows voters 2 to n to have no sequencing restrictions on their binary rankings, but voter-one must be at least partially rational. The reward is that, in addition to the procedures designed only for $B^n(3)$, new nondictatorial procedures are admitted which can exploit this partial transitivity. One class, for instance, requires voter-one to be a dictator over $\{a, c\}$, $\{b, c\}$ rankings, and a dictator over $\{a, b\}$ rankings only when this ranking is level-one. When voter-one's $\{a, b\}$ ranking is level-zero, the other voters can determine the $\{a, b\}$ outcome by, say, a majority vote.

The reasons the nondictatorial procedures remain undesirable is that IIA* admits only methods designed for a fairly primitive society. Here, the partially rational voter is a partial dictator for the primitive voters who are permitted to participate only in certain settings on one pair. But this is as it should be because IIA* tacitly assumes that these voters are incapable of doing anything more. By further relaxing IIA to recognize the rationality of more voters, more procedures are admitted. In fact, with

(IIIA) (*Intensity IIA*. Society's relative ranking of a pair depends only on the voters' relative rankings of this pair *and their intensity levels of this ranking.*)

we finally obtain realistic procedures.

Theorem. *For $k \geq 3$ alternatives and $n \geq 2$ voters with transitive preferences, there exist procedures with transitive rankings that satisfy anonymity, (P), and IIIA.*

One such procedure is the BC (Saari, 1995a); it is the only positional method admitted by the theorem. From here a rich selection of other admissible procedures can be created; e.g., a BC runoff, Black's method (where a Condorcet winner is selected when one exists, otherwise the BC winner is chosen), and so forth. If anonymity is relaxed to (ND), then we can admit methods such as a weighted BC system where some voter's vote counts as though they were cast by j voters. The important point is that by restoring the transitivity of individual preferences realistic possibility assertions emerge!

The only part of the proof of the theorem that needs explanation is to explain why the BC satisfies IIIA. To do so, use the alternative BC formulation (Borda 1781, Saari 1995b) where each voter votes on each pair of alternatives; the sum of points a voter gives a candidate over all pairwise election agrees with the BC points he assigns her. So, a transitive voter with the relative ranking $a \succ b$ assigns one more point to a than b based on the $\{a,b\}$ election. In all other pairwise $\{a,c\}$ and $\{b,c\}$ elections, a receives a point more than b if and only if this voter ranks c between a and b. Thus, IIIA is satisfied, but IIA is not because the $\{a,b\}$ outcome requires information about the voters' $\{a,c\}$ and $\{b,c\}$ beliefs.

Notice that IIIA can be replaced with alternative conditions that connect different sets of alternatives. Another one, for instance, is to allow the group ranking for each pair $\{a,b\}$ to depend only on how the voters rank this pair and any other pair containing either a or b. Again, it is clear that one resulting procedures is the BC; by using the results of (Saari, 1995b, d) it follows that the BC is the only positional method to satisfy this condition.

Just as with Arrow's Theorem, a way to resolve Sen's concern is to relax his conditions in order to admit more procedures. While one approach is to modify L* (e.g., see (Gaertner, Pattanaik, and Suzumura, 1992)), I show how this also can be accomplished by modifying (P). To see the idea, consider whether I should wear a white (a) or blue (b) shirt. Now, it may be relevant for society to determine whether this choice is truly an individual right. The cyclic outcome caused by the extreme $\{a,c\}$ transitive preferences for Table 2, for instance, suggests that most voters disagree with the notion that voter-two should be decisive over $\{a,c\}$. Similarly, the transitive preferences for Table 3 admit the interpretation that society does not want voter-one to make the $\{a,b\}$ choice. In other words, society may have the right to determine whether I can be decisive over $\{a,b\}$, but after I am given decisive powers, society cannot express a ranking over these alternatives.

This distinction between society granting the right to be decisive over $\{a,b\}$ and the actual choice admits a generalization. If this is truly an individual right of no concern to anyone else, then there is no reason to impose (P) on any pair that includes a and/or b. Namely, instead of excluding a and b from other voters' set of alternatives (which is a natural approach), they can be included by relaxing (P) in the following manner.

Modified Pareto (P*). If individual j is decisive over a pair $\{a, b\}$, then (P) holds except for those pairs involving either a or b.

This condition leads to the following possibility theorem.

Theorem. *Assume there are $n \geq 2$ agents with transitive preferences. Assume there are at least two agents who are decisive over mutually distinct sets of alternatives. There exist procedure with acyclic outcomes which satisfy (P*).*

In other words, it is possible to view Sen's problem as being caused by (P): when L* is present, (P) imposes a strong connection over the cyclic preferences that causes the problems with the indistinguishable transitive preferences. It is somewhat ironic that by relaxing (P) a condition connecting pairs to (P*) procedures are admitted which partially reclaim the lost transitivity.

The proof of the theorem is immediate. Let $\mathcal{A}$ be the set of all alternatives and let $\mathcal{D}$ be the set of alternatives over which some agent is decisive. Use the BC to rank the alternatives $\mathcal{A} \backslash \mathcal{D}$, and let the decisive voters rank the various $\mathcal{D}$ subsets. The rankings among the various subsets of $\mathcal{D}$ and with $\mathcal{A} \backslash \mathcal{D}$ cannot be determined by P*, so the procedure can impose this ranking.

To illustrate with the example of Table 2 (which does not satisfy the conditions of the theorem because two decisive agents have a as an alternative in their assigned set), after the two sets are ranked, the $\{b, c\}$ ranking can be chosen to be consistent with transitivity. In Table 3, we could just assume that the $\{a, b\}$ ranking always is ranked above the $\{c, d\}$ ranking.

References

Arrow, K. J. (1951). *Individual Values and Social Choice*, Wiley (2nd ed. 1963).

Black, D (1958). *The theory of committees and elections*. Cambridge University Press.

Borda, J.C. (1781). *Memoire sur les elections au Scrutin*. Histoire de l'Academie Royale des Sciences.

Gaertner, W., P. K. Pattanaik, and K. Suzumura (1992). *Individual rights revisited*, Economica **59**, 161-177.

Gibbard, A. (1969). *Intransitive social indifference and the Arrow dilemma*, Mimeographed notes.

Gibbard, A., Hylland, A., Weymark, JA (1987). *Arrow's theorem with a fixed feasible alternative*, Soc Choice Welfare **4**, 105-115.

Hammond, P. J. (1982). *Liberalism, independent rights and the Pareto principle* , Logic, Methodology and the Philosophy of Social Science (L. J. Cohen et al. eds.), Amsterdam, pp. 607-620.

Kalai, E., Muller, E. (1980). *Characterization of domains admitting nondictatorial social welfare functions and nonmanipulable voting procedures*, JET **16**, 457-469.

Kalai, E., Ritz, Z. (1979). *Characterizations of the private alternatives domains admitting Arrow social welfare functions*, JET **22**, 457-469.

Saari, D. G. (1991). *Calculus and extensions of Arrow's Theorem*, Jour. Math Econ **20**, 271-306.

Saari, D. G. (1992). *The aggregate excess demand function and other aggregation procedures*, Economic Theory **2**, 359-388.

Saari, D. G. (1994). *Geometry of Voting*, Springer-Verlag.

Saari, D. G. (1995a). *Inner consistency or not inner consistency; a reformulation is the answer*, Social Choice, Welfare, and Ethics (Barnett, W., Moulin, H., Salles, M., and Schofield, N., ed.), Cambridge University Press.

Saari, D. G. (1995b). *Basic Geometry of Voting*. Springer-Verlag.

Saari, D. G. (1995c). *Mathematical complexity of simple economics*. Notices of American Mathematical Society **42**, 222-230.

Saari, D. G. (1995d). *A chaotic exploration of aggregation paradoxes*. SIAM Review **37**, 37-52.

Salles, M.(1994). *Rights, permission, obligation: Comments on Prasanta K. Pattanaik, "On modelling individual rights: Some conceptual issues"*, Draft paper; CREME.

Sen, A. (1970). *The impossibility of a paretian liberal*. Jour. of Political Economy **78**, 152-157.