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OPTIMAL MOTION TOWARDS A STOCHASTIC DESTINATION

by

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"Optimal Motion Towards a Stochastic Destination"

Abstract

We consider a vehicle (ship, tank, helicopter, airship, automobile, etc.) moving on a Euclidean plane towards a stochastic final destination, to be chosen from a discrete set after a decision period. The decision period itself may be deterministic or stochastic. We assume the vehicle can move at variable speed associated with a monotone nondecreasing variable cost, and it can also stop and wait anywhere. There is a fixed cost per time unit "carried" by the vehicle as well. The vehicle's optimal trajectory in the X-Y-Speed space during the decision period is investigated. The applications of the problem include shipping and desert warfare problems, and by using non-Euclidean norms (such as rectilinear or graph distances), it can be extended to the operation of emergency service vehicles.

1. Introduction

Tramp ships, and even line freight ships, may sometimes change their destination port en route. Sometimes, the next port of call is known to be tentative in nature even when it is officially entered into the log book (the fact that it is tentative is not likely to be logged, though). We may say then, that the ship is actually headed towards a stochastic destination. A similar situation where the final destination may even be distributed continuously (with or without mass points), can possibly be encountered in naval and desert warfare. Another instance is emergency services: take a helicopter on its way to base after a rescue mission, but with enough fuel for another mission. The pilot can be diverted to a new emergency before reaching base, so his base is only a tentative destination! If we use the appropriate distance norm, we have a similar problem for regular (wheeled) emergency vehicles.

If the decision period is deterministic, the problem can be reduced to a regular plant location problem or the Weber problem, and the version we choose to discuss in detail is the Euclidean plant location problem, also known as the Fermat problem. See Kuhn [4] for some historical background, and Francis and Cabot [2] where an extensive reference list is supplied. If the decision period is stochastic, the problem of identifying the optimal trajectory in terms of location--i.e., in the X-Y-Speed space--is a dynamic location problem which can be solved by dynamic programming (or DP), or other calculus of variations or numerical methods.

In section 2 we formulate and solve the deterministic decision time case. Section 3 is devoted to the stochastic decision period version. We also present some results of a computer program written in 1977 for a special case where the decision period is distributed exponentially. The lack of memory associated with this distribution made possible the application of a relatively simple dynamic programming model for this case [6]. In Section 4 we discuss the potential gain associated with applying our model. Finally, we conclude with a brief discussion of the possible extensions of the model, including an ermergency vehicle example.

2. Motion Towards a Stochastic Destination with a Deterministic Decision Period.

The Problem: On a Euclidean plane let N be a set of n points (x_i, y_i); $i = 1, \ldots n$ (also referred to by index alone); let n probabilities p_i; $i = 1, \ldots, n$ be given such that $\Sigma p_i = 1$; let a starting point, point (x_0, y_0) (or point 0) also be given; let $f(S)$; $S \geqslant 0$ be a function such that

(1) $$f(0) = 0,$$

(2) $$f(S) > 0; \ \forall \ S > 0,$$

(3) $$f(S + \varepsilon) \geqslant f(S); \ \forall \ S, \varepsilon > 0,$$

and such that except for $S = 0$, $f(S)$ is (not necessarily strictly) convex and represents the cost of motion at speed S; let $F > 0$ be given (the fixed time value); and finally let $T > 0$ be given (the decision period). It is required to minimize the following function by choosing $t \equiv (x_t, y_t)$ and S (i.e., choose a point t, to be at, T time units from now, and a speed S with which to proceed afterwards, so that the expected total expense will be minimized):

(4) $$Z(t,S) = FT + f(d(0,t)/T)d(0,t) + (f(S) + F/S) \sum_{i=1}^{n} p_i d(t,i),$$

where $d(i,j)$ is the Euclidean distance between points i and j. We denote the optimal S by S^*, and similarly we have t^* and $Z^* = Z(t^*,S^*)$. Note that FT is a constant, so we can actually neglect it; the second term is the cost of getting to t during T time units, i.e., at a speed of $d(0,t)/T$. Now, clearly the problem of finding S^* can be solved separately, and indeed we start by solving it.

The Speed Choice Problem

If we look at the list of stipulations for $f(S)$, (1) just means that we can stop and wait at zero marginal cost (which is why regular airplanes are not listed among the vehicles of interest here), (2) is evident, and (3) is redundant, given (1), since if f is not monotone for $S > 0$, then it has a global minimum for that region at some S, say S_{min}, where the function assumes the value $f_{min} \leq f(S);\ \forall\ S > 0$. Now suppose we wish to move at a speed of $\lambda S_{min};\ \lambda \in (0,1]$, during T time units, thus covering a distance of $\lambda T S_{min}$; then who is to prevent us from waiting $(1 - \lambda)T$ time units, and then go at S_{min} during the remaining λT time units, at a variable cost of f_{min} per distance unit? As for the convexity requirement, which we actually need from S_{min} and up only, this is not a restriction at all! Not only do all the vehicles we mentioned behave this way in practice generally, but even if they did not (e.g., some speeds cause vibrations, etc.), we could use the convex support function of f as our "real" f, by a policy, similar to the one discussed above, of moving part time at a low speed and part time at a higher one at a cost which is a linear convex combination of the respective f's. Figure 1 "sums" our treatment of an ill-behaved function (in dotted lines where never used). We will also assume that f is continuously differentiable, but see [9] or [7] as to why this is not restrictive in practice. Hence, our only real assumption is that we can stop and wait at zero cost, i.e., (1). We assign this result to memeory in "cell" Lemma 1.

Lemma 1: Let $\bar{f}(S)$; $S > 0$ be any positive cost function associated with moving at speed S continuously and let (1) hold, then by allowing mixed speed strategies, we can obtain a function $f(S)$; $S > 0$ such that f is positive, monotone nondecreasing and convex, and reflects the real variable costs.

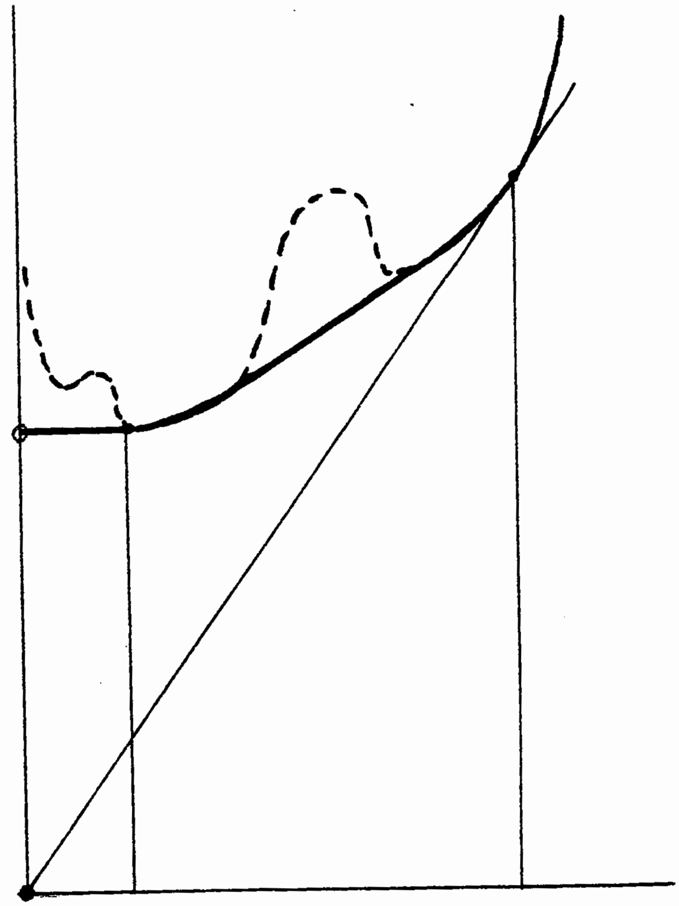

Figure 1

Now, since each time unit costs F, and we can go S distance units during it, each distance unit's "fair share" is F/S. To this add f(S), to obtain the cost of a distance unit at a speed of S when we know where we are going, and require our fixed costs to be covered. (On the other hand, not knowing what we want to do means that we may have to lose the F money, or part of it.) Denote the total cost as above by TC(S), or

$$(5) \qquad TC(S) = f(S) + F/S.$$

But, F/S is strictly convex in S, and f(S) is convex too, so TC(S) is strictly convex. Further, $\lim_{\varepsilon \to 0^+} TC(\varepsilon) = \infty$, so TC(S) has a unique minimum, S^*. Since we practically assume differentiability, then

$$(6) \qquad S^* = \arg\{f'(S) = F/S^2\},$$

and we can obtain it numerically. (S^* is also depicted in Figure 1, where a ray from the origin supports f.)

Choosing t Optimally

Our problem is to find the point t, or the "decision point," where we elect to be at the end of the decision period. Then, we will know with certainty what we have to do, so we will proceed at S^* to whichever point i chosen, at a cost of $TC(S^*)d(t,i)$. Denoting $TC(S^*) = TC^*$, we may rewrite (4) as follows:

$$(7) \qquad Z(t) = FT + f(d(0,t)/T)d(0,t) + TC^* \sum_{i=1}^{n} p_i d(t,i).$$

Theorem 1: Z(t) is strictly convex in t.

Proof: Clearly FT is a constant, so it is convex. Let $h(w) = f(w/T)w$, hence our second term, $f(d(0,t)/T)d(0,t)$ is $h(d(0,t))$. By differentiation we can show that $h(w)$ is strictly convex, monotone increasing and nonnegative. $d(0,t)$ is convex (being a norm), and it follows that $h(d(0,t))$ is strictly convex as well (see Theorem 5.1 in [5], for instance). As for the third term it is clearly convex (since $\{p_i\}_{i=1,\ldots n}$ are nonnegative probabilities), and our result follows for the sum. □

It follows that a unique minimum, Z^* exists for Z, within the convex hull of the n + 1 points 0,1,...,n. In order to find this minimum we look for a point such that the gradient ∇Z is zero. However, it may happen that t^* merges with a point of N, or with the starting point, 0. In such a case we follow [2], and define the gradient to be zero there (it would include 0/0 undefined elements). What we propose to do is to check all the given points $N \cup \{0\}$, and see if the optimum is one of them. If not, we can forget this issue. (Except that it is not advisable to stop at a fixed point during a search algorithm, so upon the rare occurrence that a search procedure takes us to a nonminimal fixed point, we simply move slightly aside, and continue.) We now examine the two components of ∇Z, by x_t and y_t.

(8) $$\frac{\delta Z}{\delta x_t} = \frac{x_t - x_0}{d(0,t)}(f(S) + Sf'(S)) + TC^* \sum_{i=1}^{n} p_i \frac{x_t - x_i}{d(t,i)},$$

(9) $$\frac{\delta Z}{\delta y_t} = \frac{y_t - y_0}{d(0,t)}(f(S) + Sf'(S)) + TC^* \sum_{i=1}^{n} p_i \frac{y_t - y_i}{d(t,i)},$$

where

(10) $$S = d(0,t)/T.$$

The "length" of the gradient L_t is

(11) $$L_t = [(\delta Z/dx_t)^2 + (\delta Z/\delta y_t)^2]^{1/2}.$$

It remains to discuss the check we propose for the fixed points. To that end we define

(12) $$p_0 \triangleq \begin{cases} (f(S) + Sf'(S))/TC^*; & S \neq 0 \\ f_{min}/TC^* & ; \quad S = 0 \ (\Longleftrightarrow t = 0), \end{cases}$$

and with p_0 we define L_i (replacing L at i = 0,1,...,n) as follows:

(13) $$L_i \triangleq TC^*([(\sum_{\substack{j=0 \\ j\neq i}}^{n} p_j \frac{x_i - x_j}{d(i,j)})^2 + (\sum_{\substack{j=0 \\ j\neq i}}^{n} p_j \frac{y_i - y_j}{d(i,j)})^2]^{1/2} - p_i)$$

L_i is positive if the n points j = 0,...,i-1, i + 1,...,n "pull" stronger than i, and it indicates that $t^* \neq i$. If $L_i \leqslant 0$, $t^* = i$. (Formally, if $L_i < 0$ we say that $\nabla Z(i) = \underset{\sim}{0}$.) As for p_0, the weight we assign to the starting point, note the following: (i) if $S \leqslant S_{min}$, implying $f'(S) = 0$, then $p_0 = f_{min}/TC^*$, i.e., the same as for S = 0; (ii) since $\sum_{i=1}^{n} p_i = 1$, then clearly $\sum_{i=0}^{n} p_i \geqslant 1 + f_{min}/TC^* > 1$; (iii) even if $\inf(f(S)) = f(0^+)$, p_0 is well defined when we set $f_{min} = f(0^+)$ (however, in reality, this is not likely to happen).

We can gain some more insight into the problem if we consider two limiting cases: (i) $T \to \infty$; and (ii) $T \to 0$.

(i) <u>The $T \to \infty$ Case</u>: Here we assume $S_{min} > 0$ (and not 0^+). Under this assumption, at a cost of f_{min} per distance unit, we can arrive anywhere we want during the decision time. Hence we have $p_0 = f_{min}/TC^*$, and our problem is solved as in the static plant location. As usual, denote the solution point by t^*, and clearly for T large enough we are not going to move during the whole decision period, but rather only during T^* time units of it, where

$$(14) \qquad T^* = d(0,t^*)/S_{min}.$$

Hence the same solution is obtained for any $T > T^*$.

It may be advisable to try solving under the assumption that $T > T^*$, and then check the assumption. This way, even if $f'(s)$ jumps at S_{min}, we will not have any problems with it. If $T \geqslant T^*$, we are through, and else we know that $S > S_{min}$.

(ii) <u>The $T \to 0^+$ Case</u>: Recall that by (5), with S^* as per (6) we have $TC^* = f(S^*) + F/S^*$, and from (6) we easily obtain

$$(15) \qquad F/S^* = S^*f'(S^*).$$

Now substitute (15) to TC^*, and we have

$$(16) \qquad TC^* = f(S^*) + S^*f'(S^*).$$

Also recall that W(S) as defined below

$$(17) \qquad W(S) \overset{\Delta}{=} f(S) + Sf'(S),$$

was the relative weight of the starting point 0 in (8) and (9) and the numerator in (12). We observe that W(S) is a monotone increasing function (since f, f' and $f'' > 0$), and that $W(S^*) = TC^*$. But at t^* (8) and (9) are zero, hence

(18) $$\frac{x_0 - x_t^*}{d(0,t^*)} W(S) = TC^* \sum_{i=1}^{n} p_i \frac{x_t^* - x_i}{d(t^*,i)},$$

(19) $$\frac{y_0 - y_t}{d(o,t)} W(S) = TC^* \sum_{i=1}^{n} p_i \frac{y_t^* - y_i}{d(t^*,i)}.$$

Squaring (18) and (19), adding them and taking the square root again, we obtain

(20) $$W(S) = TC^*[(\sum_{i=1}^{n} p_i \frac{x_t^* - x_i}{d(t^*,i)})^2 + (\sum_{i=1}^{n} p_i \frac{y_t^* - y_i}{d(t^*,i)})^2]^{1/2}.$$

Clearly $W(S) \leq TC^*$ (the magnitude of a vector sum is less than the sum of the magnitudes), with equality only in the special case where all the points, including the starting point are colinear, and both 0 and t are to the same side of all the rest (in which case we can behave as if we know where we are going, since we have to reach the first point at least, and we know the decision will be made by the time we get there). But W(S) is monotone, hence if $W(S) < TC^*$ then $S < S^*$, and

(21) $$d(0,t^*) \leq TS^*.$$

Following (20) we define G(t) for any $t \in \{E^2 - N\}$ (i.e., any point on the plane and 0, but not $i \in N$)

(22) $$G(t) = TC^*[(\sum_{i=1}^{n} p_i \frac{x_t - x_i}{d(t,i)})^2 + (\sum_{i=1}^{n} p_i \frac{y_t - y_i}{d(t,i)})^2]^{1/2}.$$

For $t = t^*$, and S chosen optimally (20) plus (22) yield

(23) $$G(t^*) = W(S).$$

Now (for the first time) we use the data $T \to 0$, and by (21) we have

(24) $$\lim_{T \to 0^+} d(0,t^*) = 0.$$

I.e., we only have to determine in which direction and at what speed to proceed, but we will not get very far. The direction we choose is that of $-\nabla Z(t^*)$, as we always have to; but now we can take 0 instead of t^*, using (24), so we do not have to search for this value. As for the speed, we choose S^∇ (the "gradient" speed), such that

(25) $$S^\nabla \triangleq \arg\{W(S) = G(0)\},$$

since by (23) this is the value for $t^* = 0$.

Since the speed is one of the parameters we are interested in, we present a theorem which will also hold for the stochastic decision period case.

Theorem 2: The gradient speed S^∇ as defined at any point, is an upper bound for the optimal speed at that point, and S^* is an upper bound for S^∇.

Proof: By Theorem 1, $Z(t)$ is strictly convex, hence along the segment $\overline{0,t^*}$ it is also strictly convex, and since $Z(t^*) \leqslant Z(t)$; $\forall$ t, it is monotone decreasing along the segment. Let $g(t)$ be the absolute value of the directed derivative along $\overline{0,t^*}$. Clearly $g(t)$ is monotone decreasing for λ when $t = \lambda \cdot 0 + (1 - \lambda)t^*$ (i.e., $x_t = \lambda x_0 + (1 - \lambda)x_{t^*}$, $y_t = \lambda y_0 + (1 - \lambda)y_{t^*}$, 0 being an index and not a number here). For $\lambda = 0$ the slope $g(0)$ is bounded from above by $G(0)$, since $G(0)$ reflects the steepest descent (in the direction of $-\nabla Z$). At t^*, the direction of $\overline{0,t^*}$ is the steepest descent direction itself, by (20). Summing these assertions we have

$$(26) \qquad G(0) \geqslant g(0) > g(t^*) = G(t);\ 0 \neq t^*$$

It follows that the gradient speed S^∇ is an upper bound on the speed for <u>any</u> movement from 0, and we can designate any point as 0. I.e.,

$$(27) \qquad S \leqslant S^\nabla \leqslant S^*. \qquad \square$$

So S^∇, which is rather easy to compute, is an upper bound on our speed anywhere, and it would be very easy to extend the proof to the stochastic decision period case, using the basic attributes of the expectation. (We will omit that formal extension, however.)

<u>The Stopping Line and the Waiting Region</u>

For $T \geqslant T^*$, we obtained $S = S_{min}$, and by (17) it follows that $W(S) = f_{min}$. Using (23) we have

$$(27) \qquad G(t^*) = f_{min}.$$

Now, starting at different points, but such that $G(0) > f_{min}$ and $T > T^*$ as defined for them we should stop at different decision points respectively (unless we start from colinear points, on $\overline{o,t^*}$), each satisfying (27)). Actually there is a locus of points satisfying (27), which we call D as follows

$$(28) \qquad D = \{t \in E^2 | G(t) = f_{min}\}.$$

We call D the stopping line (although it may happen to be a point). Now denote the area within D, inclusive, as C, or

(29) $$C = \{t \in E^2 | G(t) \leq f_{min}\}.$$

C is also called the waiting area, since being there during the decision period would imply waiting. Clearly $C \subseteq D$, with C = D for the special case where one of the points $N \cup 0$ is the only solution for a large T. In case $C \neq D$, however, we have a nonempty set E as follows

(30) $$E = C - D \text{ (or } C/D).$$

Specifically, there is a point in C, and in E if $E \neq \emptyset$, for which G = 0 (if C = D, we have to define G as 0 there, since it includes 0/0 terms); we denote this point by t_{min}, i.e.,

(31) $$G(t_{min}) = 0.$$

Clearly, in order to identify t_{min}, we do not need any information about the starting point or any of the costs we carry, but just the information on N and $\{p_i\}$. t_{min} is actually the point minimizing the expected distance to i, and locating it is an instance of the (real) single static plant location. Similar to the procedure described above (for the problem including the starting point), we can check all the nodes in N first by applying (13), but with the summation running from 1 to n instead of 0 to n, to obtain values which we will denote by L_i^- (instead of L_i as in (13), to differentiate between them); now, if L_i^- is negative, the solution is there, since

(32) $$L_i^- < 0 \Rightarrow t_{min} = i.$$

Usually, $L_i^- = 0$ could be included except if all the points are colinear and they are also divisible into two sets, each to one side, each "weighting" 1/2, in which case two points and any convex combination of them could serve as t_{min}, so $t_{min} = i$ does not necessarily imply $L_i^- < 0$. Still, it is true that

(33) $$t_{min} = i \Rightarrow L_i^- \leqslant 0.$$

(Note that TC^* is a positive constant and we can delete it.)

It turns out that for the positive L_i^- valued points (and if $n > 2$, or even $n = 2$, $p_1 \neq 1/2$, we have some such positive values), we can tell very easily if they belong to the waiting area C or to $\bar{C}$, and in the former case also if they belong to the stopping line D, or to E. This is due to the fact that L_i^- is G(i) in the descent direction from i to t_{min}, while $L_i^- + 2TC^*p_i$ is G(i) in the opposite direction. Denote these by $G(i)^-$ and $G(i)^+$, respectively, and we have the following possibilities:

(34) $$L_i^- = G(i)^- > f_{min} \Rightarrow i \in \bar{C};$$

(35) $$G(i)^- \leqslant f_{min},\ G(i)^+ \geqslant f_{min} \Rightarrow i \in D(\subseteq C);$$

(36) $$G(i)^+ < f_{min} \Rightarrow i \in E.$$

Also it can be shown, and is intuitively clear, that

(37) $$p_i > (1 + f_{min}/TC^*)/2 \Rightarrow t_{min} = i.$$

We are now ready to discuss the stochastic decision period case. In that connection, note that some of our results so far, such as Theorem 1, the stopping line, etc., are not dependent upon T, hence they can serve us for the stochastic decision period case as well.

3. The Stochastic Decision Period Case

Our problem is exactly as before, except that T is a random variable (RV) now. Conceivably the p_i values could be influenced by information such as "the decision has not yet been made," but we do not consider this case in detail (i.e., we assume statistical independence between T and the choice). Our RV may be discrete (contact with management is at predetermined times), continuous or mixed. We discuss the discrete case in detail, and show how to accommodate the continuous case by the discrete one. We assume that the distribution of T is given. (Bayesians will find no fault with that assumption, hopefully. Others will have to take it at face value.) Let

(38) $$P(T = h_j) = q_j;\ j \in J = \{1,2,\ldots\},$$

where $q_j \geq 0$; $\forall$ j and they sum to one, of course; J may be finite or not, the index 0 is maintained for the start as before, and we may assume $h_0 = q_0 = 0$ for it. Our problem is to find the best set of decision points t_j (or t_j^* when optimality is assumed), such that as long as the decision is not yet made by $T = h_j$ we proceed from t_j^* to t_{j+1}^* (starting at $t_0 = t_0^*$). Let v_j be the conditional probability of decision at h_j, given it has not been made yet, i.e.,

(39) $$v_j = P(T - h_j | T > h_{j-1}) = q_j/(1 - \sum_{k=1}^{j-1} q_k);$$

and, following (7) we define $Z(t_{j-1}, t_j)$:

(40) $$Z_j(t_{j-1}, t_j) = F(h_j - h_{j-1}) + f(d(t_{j-1}, t_j)/(h_j - h_{j-1}))d(t_{j-1} \cdot t_j)$$
$$+ v_j TC^* \sum_{i=1}^{n} p_i d(t_j, i) + (1 - v_j) Z_{j+1}(t_j, t^*_{j+1}).$$

This formulation lends itself to dynamic programming very naturally, and assuming optimality we define $Z^*_j(t_{j-1})$:

(41) $$Z^*_j(t_{j-1}) = \min_{t_j}\{Z_j(t_{j-1}, t_j)\} = Z_j(t_{j-1}, t^*_j).$$

Before proceeding further with the general solution, two limiting cases will help us to confine our search to a manageable area. These are analogs of cases we discussed above, and here is the payoff for the effort there.

<u>The $P(T < T^*) \to 0$ Case:</u> This is the analog of the $T \geq T^*$ case, so we proceed at S_{min} to the correct spot along the stopping line. We refer to the solution as the "slow" trajectory.

<u>The $P(T < \varepsilon) \to 1;\ \forall\ \varepsilon > 0$ Case:</u> This case to which we refer as the "gradient" case, is analog to the $T \to 0^+$ case, since it stipulates that with probability approaching 1 this is indeed anticipated. Therefore we move at a speed of S^∇ in the $-\nabla Z(t)$ direction. Now, under the stipulation, the probability that we will go far before the decision is negligible, but this does not deter us from defining the steepeset descent, or (minus) gradient trajectory all the way

until the stopping line. The "gradient" speed we use is a function of t, which may be obtained by

$$(42) \qquad S^{\nabla}(t) = \arg\{f(S^{\nabla}(t)) + S^{\nabla}(t)f'(S^{\nabla}(t)) - G(t) = 0\},$$

which is a direct extension of (25).

Since by Theorem 2 (which extends almost directly to the stochastic decision period case) $S^{\nabla}(t)$ is an upper bound on S. We also refer to this as the fast trajectory. It is interesting (although intuitively clear) to note that S^{∇} is decreasing along the fast trajectory.

Lemma 2: When moving along the gradient trajectory, which we denote by X(t), in the $-\nabla Z(t)$ direction, $S^{\nabla}(t)$ is monotone nonincreasing.

Proof: Let z(t) be the expected distance to the final destination from t given a decision (recall, a decision is due immediately), i.e.,

$$(43) \qquad z(X(t)) = TC^{*} \sum_{i=1}^{n} p_i d(t,i)$$

Then G(t) as per (22), is z's directional derivative along X(t), i.e.,

$$(44) \qquad G(t) = |z'(X(t))|$$

We want to show that G(t) is monotone nonincreasing (which wil imply our lemma by (42) and the montonicity of W(S)). We know that G(t) decreases from G(0) to f_{min} without changing signs along X(t), so it will suffice to show that $z''(X(t)) \geq 0$. But X(t) is a trajectory in the steepest descent direction, hence if we differentiate it by t, twice, we get

(45) $$\dot{X}(t) = -\nabla z(X(t)),$$

(46) $$\ddot{X}(t) = -\nabla^2 z(X(t))\, \dot{X}(t) = \nabla^2 z(X(t)) \nabla z(X(t)).$$

We continue and differentiate z(X(t)), twice again, to obtain

(47) $$z'(X(t)) = \nabla z(X(t))^T\, \dot{X}(t),$$

(48) $$z''(Xt)) = \dot{X}(t)^T\, \nabla^2 z(X(t))\, \dot{X}(t) + \nabla z(X(t))^T\, \ddot{X}(t).$$

Finally, by substituting (46) in (48) we have

(49) $$z''(X(t)) = (\dot{X}(t) + \nabla z(X(t)))^T\, \nabla^2 z(X(t))(\dot{X}(t) + \nabla z(X(t))),$$

which is a bilinear form $(Y^T \nabla^2 z Y)$. Now, z is clearly a convex function, hence $\nabla^2 z$ is positive semidefinite (at least), and our result follows. □

Figure 2 depicts the results of a program run for an exponentially distributed decision period, for seven expectations θ, and for seven randomly chosen points of randomly chosen weights (probabilities) [6]. For large θ's, the trajectories were virtually the same as the slow trajectory. For small θ's, a similar behavior was observed relative to the fast trajectory. Interestingly, though all the trajectories were within the convex hull of the area between these two trajectories, one of them actually <u>intersected</u> the fast trajectory. The speed choice obeyed Theorem 2. In general one might say that the faster is the decision due, the faster we should move, and the longer our total trajectory may be--since we do not expect to stick to it for a long

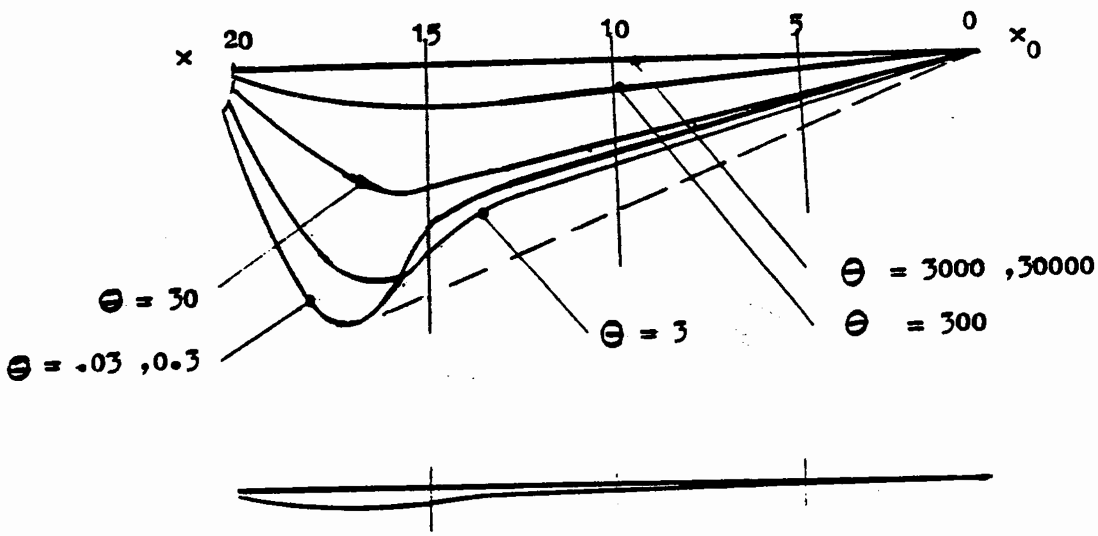

Figure 2

while; the slower is the decision due, the more we tend to go slowly and along a "mildly curving" trajectory (if not exactly straight). In any case, as far as the physical location of the trajectory is concerned, we should not find ourselves out of the area defined by the convex hull of the extreme case trajectories, minus E (which we should never enter even if it is in that convex hull). If we do, we can always find a better trajectory for immediate or delayed advantage, within this area. We denote this result in Theorem 3.

Theorem 3: One should never leave the convex hull of the slow and fast trajectories, minus E.

Proof: By negation, as described above. ▯

Incidentally, the results in Figure 2 even show that the convex hull of a "medium" speed trajectory and the slow one contains all the "slower" speed trajectories. However, this may not be extendable to more general distributions, where the relative "speeds" may not be uniquely implied. Note, however, that if we do not use exaggerated vertical scale, all trajactories seem rather straight! Practically there is no doubt that locationwise we should pick some straight trajectory, even the slow one, and just optimize the speed choice. This would yield most of the potential gain, with the additional benefit of a less complex navigational problem.

If we return now to (40) or (41), we can see that the stopping line and the search area are what we need practically to obtain a working dynamic programming model. We may start by assuming the slow trajectorey, which would make our decision variable univariate, and we can fold back satisfactorily by assuming that the, say, k^{th} step will bring us to the stopping line. It may happen, that we overshoot the starting point, but it should not be difficult to adjust. However, we do not suggest using this method here, since it does not seem to justify the programming effort, and we can simply use a multivariable library search method instead. A problem might be if local minima existed besides the global minimum; the next theorem removes this obstacle.

Theorem 4: The problem of locating the decision points t_j^* so as to minimize $Z_1^*(t_0)$, is convex.

Proof: By iterative application of Theorem 1. ▯

The Continuous RV Decision Period Case

There is no conceptual problem in discretizing a continuous RV, and we can even do it in such a manner that the q_i values are equal, if we are so inclined. If we want to avoid the need to use too many discrete points for computational comfort, it might help if we would make the steps of the discrete variable cumulative distribution function straddle on both sides of the continuous curve. Figure 3 illustrates a very simple possibility, and explains itself.

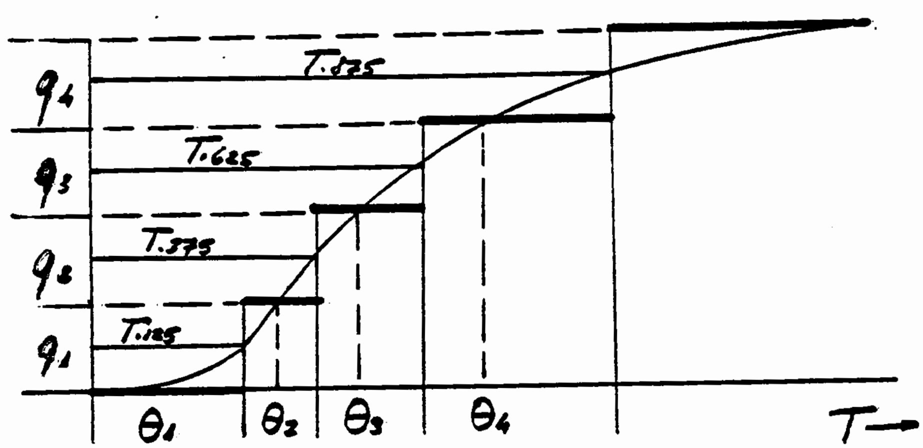

Figure 3

We may similarly gain if we do not take the resulting step function speed trajectory as is, but smooth it. This can be done very efficiently by connecting the centers of these steps by a natural cubic spline function which, in turn, can be done in linear time relative to the number of the steps (see [1] or [7]). A piecewise linear speed trajectory may be considered good enough, though. For the planar trajectory similar smoothing can be performed, but practically we can certainly do with a straight (and not even piecewise linear) trajectory, as discussed above.

4. On the Expected Gain by the Model

Our model is based on the fact that the decision period is going to be completely wasted, unless we utilize it. This places an obvious upper bound on our expected gain, V, namley

$$(50)\qquad V \leqslant FT.$$

In the stochastic case, similarly

$$(51)\qquad V \leqslant FE(T).$$

Clearly, the only way we can approach this upper bound is if G(t) approaches TC^* along the trajectory, throughout the decision period; for instance, if the points are close to each other and far from the start. In this case we behave as if the destination is known. However, in both the deterministic and the stochastic decision period case, if T is large, we cannot do anything at least part of the time. It is obvious that in the deterministic case the gain cannot exceed FT^*, which makes us rewrite (t), and similarly (51).

$$(52)\qquad V \leqslant F \min\{T,T^*\},$$

$$(53)\qquad V \leqslant F \min\{E(T),T^*\}.$$

But, suppose now that $T \geqslant T^*$, can we really expect to gain even FT^*? The answer of course is no. In this case G(t) is rather low, at least towards the stopping line where it reaches f_{min}. We may easily compute V for this case by the following formula

(54) $$V = TC^*\left(\sum_{i=1}^{n} p_i(d(0,i) - d(t^*,i))\right) - f_{min}\, d(0,t^*),$$

where the gross gain is the improvement in the expected "future" total costs to reach the final destination, but we have to subtract the "present" variable costs, in this case $f_{min}\, d(0,t^*)$. By substituting $f(d(0,t^*)/T)d(0,t^*)$ for these costs, we obtain the expected gain in the deterministic case.

(55) $$V = TC^*\left(\sum_{i=1}^{n} p_i(d(0,i) - d(t^*,i))\right) - f(d(0,t^*)/T)d(0,t^*).$$

In the stochastic case, similarly, we have the following result.

(56) $$V = FE(T) + TC^*\left(\sum_{i=0}^{n} p_i d(0,i)\right) - Z_1^*(0).$$

A similar result can be obtained at any stage, given that we reached it without decision, but we omit it. Note however that this expected gain is monotone nonincreasing. For instance, once we reach the stopping line it drops to zero, since there is nothing useful we can do any more. Note that if we start anywhere in C, all the formulas above, including the bounds (52) and (53) yield $V = 0$ (e.g., $T^* = 0$ in this case).

5. Conclusions and Extensions

Not surprisingly, our model of dynamic location is quite similar to its static plant location parent. Even the stochastic (discrete) case may be viewed as a multiplant location problem with interconnections between plants (which rule out decomposition), discussed in [2]. However, we have to use the speed costs function wisely, as implied by Lemma 1 to ensure the convexity of our version.

Although the model was developed mainly with shipping in mind, it has Euclidean distances, rather than spheric ones. An obvious extension would be to introduce such distances. If we do, then practically we can make do with large circle trajectories, analog to straight lines, since we see that this is attractive even in the Euclidean case.

Other distances we should extend to are rectilinear and graph distances. It stands to reason to assume that the strong analogy with the static plant location problem we observed will hold for these cases as well [3]. We discuss a rectilinear city application below.

We may want to improve the model by incorporating constraints into it. In shipping for instance, it has been known for quite a while that ships need (deep enough) water in order to propel their way. This type of constraint is conceptually easy to introduce to the model (although it actually makes it a hybrid-distance model, since some distances would be graph-like, and others regular). A tougher constraint may be if we have a fuel constraint and if the total consumption en route to t^* plus the worst case would violate it. In such a case we may want to weight the distant point unproportionally. Note, however, that generally the model tends to save fuel.

An Example: An emergency vehicle, equipped with radio and attentive operators, is on its way from concluded service (or whatever) to base, in a rectilinear city. Assume the base is located at t_{min} and also assume symmetry. (That is what we would like, to minimize the expected service time. Although minmax is considered superior by some [3], with symmetry these coincide.) Now if you ask the operator, "quo vadis?", knowledge of Latin alone will not suffice her/him to answer you, since she/he does not know where she/he is going! If a call comes when the vehicle is en route, it will be diverted towards it; else it will stop at the base. Figure 4 illustrates some possible trajectories the vehicle could take, depending on the cost of turning. Two points are clear by direct extension of our results:

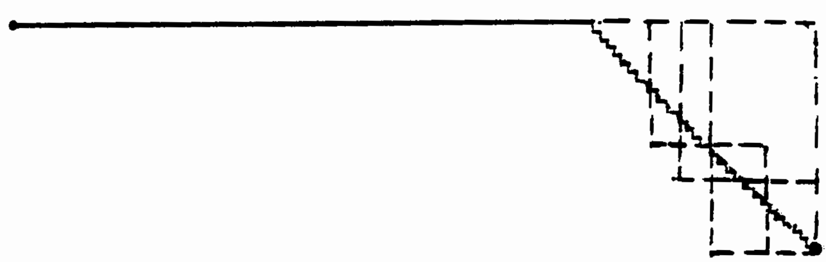

Figure 4

(i) due to the symmetry, a situation where the x (or y) distance is greater than the y (or x) distance calls for moving in the x (or y) direction until the opposite occurs; (ii) the farther the vehicle is from base, and the more likely and sooner a call is to come, the faster it should move. (Moral: stop for cigarettes near base, and not out yonder--if you cannot give them up completely.)

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