Discussion Paper #237

ASSIGNMENT MODELS IN VOTING THEORY(+)(*)

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I. Introduction

Many voting procedures have been proposed to combine individual preferences into a group preference. They range from such simple methods as majority voting, plurality voting, or Borda count to more elaborate mechanisms such as Condorcet winner, extended Borda count (P. Young's 'scoring functions' [26]), and Condorcet completion methods e.g. (Copeland).[1]

As the formal study of these procedures has progressed, their basic features have been uncovered, often leading to suggestions for the correction of their unattractive features. Astute voters quickly discover that plurality voting, for instance, easily lends itself to individual preference misrepresentation.[2] Also it is wasteful of information on individual preferences as these are never fully elicited. This last feature prompted Borda to suggest a simple scoring function to aggregate the complete individual preferences, the 'Borda count'. To remedy the manipulability of the latter, Condorcet proposed that simple majority voting be applied to every pair of alternatives to determine the Condorcet winner, i.e. the alternative (if any) winning a simple majority vote against every other. As Condorcet noticed, however, that method may fail if voting yields an intransitive group preference. As a (partial) solution he suggested that the pair with the smallest number of (majority) votes be deleted. A century later, the Rev. C.L. Dodgson elaborated on this suggestion by proposing to adopt as group preference that ranking which requires a minimal number of individual vote reversals on pairs of alternatives for simple majority rule not to cycle. The formal statement

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[1] A thorough simulation study comparing the outcomes under these rules can be found in [13]

[2] Farquharson [12], for example, quotes a letter from Pliny the Younger to Titus Aristo relating such an occurrence in a vote of the Roman Senate.

of this criterion led Bowman and Colantoni [9] to a general formalization of majority rule under transitivity constraints as a linear integer programming problem. Also, independently, Blin and Whinston [3][4] formulated the problem as a quadratic assignment problem. An extensive discussion of the properties of this solution was provided by Levenglick.[19]. Finally Merchant and Rao [20], [21] extended some of these results, and provided some solution algorithms. In view of the above noted quadratic assignment nature of the (constrained) transitive majority decision procedure, the following issues remain to be explored: (i) What special features of the quadratic assignment formulation characterize the case of restricted domains - e.g. single-peaked individual preferences? (ii) What is the relationship between other voting procedures and quadratic assignment type problems? (iii) Do other assignment type problems - say, linear also lead to other, possibly new, voting procedures?[1] We deal with the first question in Section 3 while the last two issues are taken up in Section 4. The basic notation, the problem statement, a brief review of known results leading to the findings of sections 3 and 4, and some extensions are discussed in Section 2.

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[1] Other relationships between assignment-type problems and group decision making have also been studied. In particular P. Gardenfors [ ] has considered the feasibility of using ordinal preferences rather than ratings in formulating certain linear assignment problems e.g. the man-machine assignment problem; while in [ ] he considered the use of some voting rules in assignment problems based on bilateral preferences e.g. the 'stable marriage problem', Also Kenneth A. Shepsle has proposed an an assignment model to represent Congressional Committee assignments. [25]. Although these contributions suggest another link between assignment problems and voting procedures, we limit our discussion to the three issues listed above.

II. Constrained Transitive Majority Decisions: Some preliminaries.

2.1 Notation and assumptions. We consider (1) a finite set of l group members G indexed by h $(h=1,\ldots,\ell)$; (2) a finite set A of m alternatives $(i=1,\ldots,m)$ e.g. candidates, bills, outcomes or, in general, any item to be decided upon collectively; (3) individual preferences over the outcomes are represented by complete, irrflexive, asymmetric and transitive binary relations[1] on A, denoted P_h; we use the $\{0,1\}$ relation matrix to represent P_h throughout. For instance if $a \succ_h b \succ_h c$ ("$\ldots \succ_h \ldots$ preferred to ... by voter h") we note $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ etc. $\mathcal{P}$ denotes the set of all conceivable strict rankings P_h. Here the preference relations are considered as data and are not subject to strategic misrevelations by individuals; with this assumption we can focus on the formal relations between voting procedures and assignment problems[2]. A voting procedure specifies (i) a complete group ranking $P \in \mathcal{P}$ from the individual rankings P_{hi}; or (ii) a non empty subset (possibly a singleton) $S \subset A$ of "socially preferred" alternatives. In the former case we speak of a "social welfare function" (as in Arrow [1]) i.e. a mapping $\varphi : \prod_{h=1}^{\ell} \mathcal{P}_h \to \mathcal{P}$. The latter case is referred to as a "social choice function", a mapping $\gamma : \prod_{h=1}^{\ell} \mathcal{P}_h \to \{2^A \setminus \emptyset\}$, the power set of A excluding the empty set.

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[1] To simplify the proofs we consider strict orders on A rather than preorderings. However the results do extend to preorderings with some simple modifications.

[2] It is well known, of course, that once a group has adopted some collective decision rule, astute voters may find themselves in a position to benefit from misrepresenting P_h. See, for instance [18], [23], [5] on this point.

Given ℓ and m, the constrained transitive majority decision rule (TMDR) is a mapping φ which constructs the group ranking P by (1) considering a sequence of simple majority decisions on all $\binom{m}{2}$ pairs of items; while (2) constraining the resulting binary relation P to be transitive. More formally step (1) involves forming the linear combination of the individual orderings. The resulting "voting matrix" is a point in Q^{m^2} (where Q denotes the rationals).

$$(1) \quad V = \frac{1}{\ell} \sum_{h=1}^{\ell} P_h$$

An entry v_{ij} of V is the relative frequency of individual votes in favor of i over j. We denote by $\mathcal{V}$ the set of all voting matrices, for a given ℓ and m. And step (2) requires that we pick a best transitive binary relation from $\mathcal{V}$.

2.2 TMDR as a quadratic assignment (QA) problem

If a single alternative is to be chosen, simple majority rule chooses the winning alternative from the pair with the largest majority. Since we wish to choose a full ordering P on A, TMDR proposes to pick as the winning ordering the one with the largest total number of majority votes. Formally we wish to maximize the functional

$$(2) \quad \max_{\rho \in \mathcal{L}} f(\rho) = \sum_{i,j} \rho_{ij} v_{\rho(i)\rho(j)}$$

where ρ is a permutation in the set $\mathcal{L}$ of all permutations on A. If ρ^* maximizes f then the socially preferred decision is $P^* \in \mathcal{P}$ [1].

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[1] $\mathcal{L}$ and $\mathcal{P}$ are in one-to-one correspondence with each other given a permutation ρ on A, there is a single relation matrix $P \in \mathcal{P}$ representing ρ and conversely.

To guarantee that P* be transitive ($P^* \in \mathcal{P}$) it suffices to use as interaction matrix $[q_{ij}]$, the transitive closure relation matrix:

$$(3) \quad q_{ij} = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 \\ \vdots & 0 & 1 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

In words the formulation of equation (2) seeks that ranking of A such that simultaneous permuting of the rows and columns of the voting matrix $[v_{ij}]$ according to that ranking maximizes the sum of the upper diagonal elements of the permuted matrix $[v_{\rho(i)\,\rho(j)}]$ - i.e. it maximizes overall agreement over all pairs. Since this is equivalent to minimizing the sum of the lower diagonal elements of $[v_{\rho(i)\,\rho(j)}]$ we can also say that we minimize desagreement over all pairs - when disagreement is measured by the sum of the frequencies of individual votes for j over i when socially, i prevails over j. This is, of course, nothing but Dodgson's principle as nentioned earlier. In this formulation we take the relative frequencies of votes (in v_{ij}) over pairs rather than the absolute number of votes as Dodgson proposed. But, of course, dividing through by a constant does not change the solution to the maximum (or minimum) problem. Consequently we use both measures interchangeably in the sequel. Formally the minimal disagreement problem reads

$$(4) \quad \underset{\rho \in \mathcal{L}}{\text{Min}} \; g(\rho) = \sum_{i,j} q_{ij} \, v_{\rho(j)\rho(i)}$$

and by the above reasoning we have

<u>Lemma 1</u>: Problems (2) and (4) are equivalent.

It can also be noted that varying the interaction matrix q_{ij} in (2) or (3) would lead to different scores f or g for any permutation. As we show later this suggests a way of treating other voting procedures within the general quadratic assignment (QA) framework. But, first we consider a geometric interpretation of the problem.

2.3 TMDR as a closest vertex solution

We first note

Lemma 2[(1)] Given ℓ and m, the set $\mathcal{V}$ forms a convex polytope in Q^{m^2} space defined by

$$
(5)\quad \begin{cases} \text{(i)} & v_{ij} + v_{ji} = 1 & \forall\ i \neq j \\ \text{(ii)} & 0 \leqq v_{ij} \leqq 1 & \forall\ i,j \\ \text{(iii)} & v_{ij} + v_{jk} \geqq v_{ik} & \forall\ i,j,k \end{cases}
$$

Proof:

Properties (i) and (ii) follow directly from the definition of a voting matrix while property (iii) can be easily verified by considering all six permutations of (i,j,k): P(i,j,k), P(j,i,k), P(j,k,i), P(k,j,i), P(k,i,j), P(i,k,j). Letting $|P(i,j,k)|$ denote the number of elements in the set of individual rankings (i,j,k) we find that

$$(6)\quad |P(i,j)| = |P(i,j,k)| + |P(k,i,j)| + |P(i,k,j)|$$

$$(7)\quad |P(j,k)| = |P(i,j,k)| + |P(j,k,i)| + |P(j,i,k)| \quad \text{and}$$

$$(8)\quad |P(i,k)| = |P(i,j,k)| + |P(j,i,k)| + |P(i,k,j)|$$

Hence $|P_{ij}| + |P_{jk}| \geqq |P_{ik}| \Longleftrightarrow$ (iii)

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[1]This lemma is stated as theorem in [ ] with the exception of property (iii) which is expressed as $v_{ij} + v_{jk} + v_{ki} \leqq 2$.

Convexity of $\mathcal{V}$ is easily verified for any two arbitrary matrices $[v^1]$ and $[v^2]$ in $\mathcal{V}$.

Q.E.D.

We also note that property (iii) follows from the transitivity requirement on the individual preference orders P_h. If we drop this requirement we obtain a set $\tilde{\mathcal{V}} \supset \mathcal{V}$ defined by

$$(9) \quad \begin{cases} \text{(i)} & v_{ij} + v_{ji} = 1 \\ \text{(ii)} & 0 \leqq v_{ij} \leqq 1 \end{cases}$$

$\tilde{\mathcal{V}}$ is a convex polytope in Q^{m^2} spanned by all linear convex combinations of the points $\tilde{P}_h$, i.e. the complete asymmetric, irreflexive (but not necessarily transitive) individual binary preference relations. When these relations are further required to be transitive (e.g. P_h) their convex hull spans the set $\mathcal{V}$. Finally, we note that if we add an integrality constraint to (5) and (9), namely (iv) $v_{ij} = 0$ or 1, we obtain the vertices of $\mathcal{V}$ and $\tilde{\mathcal{V}}$, respectively. The geometric interpretation of TMDR is as follows: pick the 'closest' transitive vertex, say P^* of $\mathcal{V}$ i.e. a point satisfying (4-i,ii,iii,iv) where closest is to be understood in the Euclidean distance sense. Formally

$$(10) \quad \underset{P \in \mathcal{P}}{\text{Min}} \quad d^2(P,V) = \sum_{i,j} |P - v_{ij}|^2$$

and

<u>Lemma 3</u> (Levenglick [ ], Theorem 4) Problem (2) and (10) are equivalent.

Similarly if we pick the city-block distance we obtain

$$(11) \quad \underset{P \in \mathcal{P}}{\text{Min}} \quad d(P,V) = \sum_{i,j} |P - v_{ij}|$$

<u>Lemma 4</u>: Problem (11) and (4) are equivalent

<u>Proof</u> If we set p_{ij} for some (i,j) pair then $(1-v_{ij} = v_{ji})$ by definition of V; i.e. v_{ji} is the proportion of individual votes disagreeing with the social ranking of i over j. Similarly if we set $P_{kt} = 0$, we rank t over k socially and the term v_{kt} once again measures disagreement over the (k,t) pair. Thus minimization of the city-block distance (equation (11)) is equivalent to minimizing overall disagreement (equation (4)).

Q.E.D.

In view of the equivalence between problem (2) and (4) we obtain a result originally proved by Merchant and Rao [ ].

<u>Corollary 1</u>: Problems (10) and (11) are equivalent.

In words the minimal distance problem has the same solution(s) whether we pick the city block or the Euclidean metric. Also this is the solution obtained by maximizing agreement over pairs in the social ranking; or minimizing disagreement over pairs reversed from individual to social ranking a la Dodgson. Many methods for implementing the constrained TMDR have been proposed: besides the solution of a quadratic assignment problem developed by Blin and Whinston ([3], [4]), a linear integer program can be formulated as suggested by Bowman and Colantoni [8], [9] while Merchant and Rao [20] set up an equivalent set-covering problem[1]. Before concluding this section, several points should be noted: (i) this approach is indeed a direct extension of majority decision rule (Lemma 5) - hence the name TMDR; and (ii) an alternative equivalent minimal distance formulation of the problem can be given; and (iii) TMDR meets all of Arrow's conditions except independence.

<u>Lemma 5</u>: Whenever simple majority voting yields a transitive ordering this is also the solution ordering to problem (2) (and (4), (10) and (11)

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[1] Further computational and algorithmic details can be found in [3] [8], [20], [21].

in view of Lemmas 3 and 4 and Corollary 1).

Proof: If majority voting does lead to a transitive ordering, say, the alphabetical order $a > b > c \dots > i > j > \dots m$ this means that each pair $v_{ij} > 1/2$ and $v_{ji} < 1/2$ for $i < j$ (assuming ℓ is odd for simplicity). Thus if the rows and columns of matrix V are also alphabetically ordered all elements above the diagonal are greater than their symmetric elements below the diagonal; which means that the sum of the above-diagonal entries is indeed maximal as required by equation (2). Q.E.D.

This result suggests the following interpretation of majority decision rule: it is also a minimum distance solution which picks the 'closest' vertex in the Euclidean or City-block sense. The cyclical majority phenominon stems from the fact that its solution set includes intransitive vertices (vertices of $\tilde{\mathcal{V}}$). Then transitivity constraints are needed to rule out these inadmissible solutions; the reduced solution set is the vertex set of $\mathcal{V}$. This metric interpretation suggests that other metrics, besides the Minkowski p metric for p = 1,2, may yield equivalent solutions. An example of this is the Hamming distance. Consider the problem

$$(12) \qquad \underset{P \in \mathcal{P}}{\text{Min}} \sum_{h=1}^{\ell} d(P_h, P)$$

where d is the Hamming distance over the space $\mathcal{P}$

$$d(\;) = \begin{cases} 1 & \text{if } P_h(i,j)=1 \text{ and } P(i,j)=0 \quad \text{or conversely} \\ 0 & \text{if } P_h(i,j)=P(i,j) \end{cases}$$

Thus d simply counts the number of pairs over which individual ranking P_h disagrees with social ranking P. Here the space $\tilde{\mathcal{V}}$ is the $\binom{m}{2}$-dimensional hypercube. Clearly problem (12) and (4) are equivalent; since in both cases we seek that social ranking which minimizes the number of individual disagreement over pairs (Dodgson's principle). The only difference is that we divide through by ℓ in (4) rather than taking the actual number of pair reversals as in 12. This, of course, leaves the maximum unchanged. And we can state

Corollary 2: Problems (2), (4), (10), (11), and (12) are all equivalent formulations of the constrained TMDR.

A complete characterization of constrained TMDR has been given by Levenglick [19]. Many necessary conditions of interest to social choice theorists can also be found. (See for instance, Merchant [ ], for a discussion of some of them). In terms of Arrow's conditions, Unrestricted Domain (U), Pareto (P), Independence (I), and Non-dictatorship (ND), it is clear that constrained TMDR meets the U and ND criteria. It is also clear that I cannot hold since the optimal social ordering is constructed by considering all pairs needed for a transitive ranking; specifically in the objective function we sum individual agreements over pairs rather than proceeding myopically one pair at a time, independently of the other pairs necessary to obtain a transitive ranking, as required by I. The only issue then is whether P holds for TMDR.

Lemma 5: Constrained TMDR meets the Pareto criterion

Proof: Assume P does not hold. That is assume

$$\exists\,(i,j) \in A \times A \qquad iP_h j \qquad \forall h$$

but jP*i where P* denotes the optimal ranking corresponding to the optimal permutation ρ^* in problem (2).

Let P** denote the same ranking as P* except for the (j,i) pair which is reversed: iP**j

We wish to show that $f(\rho^{**}) < f(\rho^{*})$ cannot hold so that by following the unanimity opinion ($iP_h j \ \forall h$) the score for the f function would be improved. Two mutually exclusive and jointly exhaustive cases must be considered.

Case 1: (j,i) are adjacent in ordering P*. Thus there is no other item say z between i and j. Setting iP**j we have

$$f(\rho^{**}) = f(\rho^{*}) + 1$$

or $f(\rho^{**}) > f(\rho^{*})$ as required for P to hold

Case 2: Without loss of generality consider a triple (j,z,i); and the two rankings P** and P* differ only on that triple

$$j \ P^* \ z \ P^* \ i$$

While $i \ P^{**} \ z \ P^{**} \ j$

Then we can decompose f into two parts

$$F_1 = \sum_{k,t \neq i,j,z} q_{kt} \, v_{\rho(k)\rho(t)}$$

$$F_2 = \sum_{w,s=i,j,z} q_{ws} \, v_{\rho(w)\rho(s)}$$

By construction of ρ^* and ρ^{**} F_1 is common to $f(\rho^*)$ and $f(\rho^{**})$

Thus all we need to show is that $F_2^{**} > F_2^{*}$

$$F_2^{**} = v_{iz} + v_{zj} + v_{ij}$$

$$F_2^{*} = v_{jz} + v_{zi} + v_{ji}$$

By assumption $v_{ij} = 1$ (or $v_{ji} = 0$)

Now
$$\left.\begin{aligned} v_{iz} &= (1 - v_{zi}) \\ v_{zj} &= (1 - v_{jz}) \end{aligned}\right\} \text{by property (4i) of } [\, v_{ij} \,]$$

Also recall property (4iii) of $[\, v_{ij} \,]$:

$$v_{iz} + v_{zj} \geqq v_{ij}$$

Then $\qquad v_{iz} + v_{zj} \geqq 1$

or $\qquad (1 - v_{zi}) + (1 - v_{jz}) \geqq 1$

or $\qquad \geqq v_{zi} + v_{jz}$ which implies

$$v_{iz} + v_{zj} \geqq v_{zi} + v_{jz}$$

(Equality is ruled out if ℓ is odd)

Thus $F_2^{**} \geqq F_2^{*} + 1 \Longleftrightarrow F_2^{**} > F_2^{*}$

and condition P holds for constrained TMDR

Q.E.D.

III. TMDR on Restricted Individual Preference Domains

We now assume that the individual preference rankings P_h are single-peaked. D. Black has shown that this is a sufficient condition for simple majority voting not to lead to a cyclical majority. A natural question is then to ask what additional special property of the voting matrix derives from single-peakedness to preserve transitivity in the group ranking. The intuitive rationale for single-peaked preferences usually refers to some underlying objective scale-for instance a dollars scale for budget appropriations - along which all alternatives are linearly ordered. It is then assumed that voters will follow that acale in their own ranking; specifically once a voter has settled on his top alternative say i, he is restricted to adhere to the objective order for any two elements located on the same side as i along that order. For instance he can only rank j over k but not the reverse. Thus if we were to graph his preferences against the objectively ordered alternatives we would find that the resulting graph is unimodal.

Single-Peakedness Condition (SP): Let $\bar{P}$ be the objective order on A. Then P_h is single-peaked if and only if

$$(13) \quad [\, i \bar{P} j \text{ and } j \bar{P} k \,] \Rightarrow [\, i P_h j \Rightarrow j P_h k \,] \qquad \forall (i,j,k)$$

Or equivalently

$$(13') \quad [\, (i \bar{P} j \text{ and } j \bar{P} k \,] \Rightarrow [\, k \bar{P}_h j \Rightarrow j P_h i \,]$$

Lemma 6: The relation matrix P_h for a SP preference order is always of the form

$$(14) \begin{cases} \text{(i) } P_h(ij) = 1 \Rightarrow P_h(t,k) = 1 & \text{for } k > j \text{ and } t > i \\ \text{(ii) } P_h(ij) = 0 \Rightarrow P_h(w,s) = 0 & \text{for } s < j \text{ and } w < i \end{cases}$$

Proof: From (13) it follows by transitivity that

$$i\,P_h\,k \quad \text{or} \quad p_h(i,k) = 1$$

Then by (13)

$$k\,P_h\,r \quad \text{or} \quad p_h(k\,r) = 1$$

Repeating this argument for all $k > j$ and $t > i$ establishes (14-i); while (13') and transitivity establishes (14-ii).

Q.E.D.

In words, equation (14) simply states that in a SP preference matrix whenever a 1 entry appears all entries to the right and below that entry are also constrained to be 1; and all elements above and to the left are 0. If we now examine the implications of this property for the voting matrix $[v_{ij}]$ the logic of Black's theorem becomes apparent.

Lemma 7: A convex linear combination of SP preference matrices is characterized by

$$(15) \quad \begin{cases} \text{(i) } v_{ij} > 1/2 & v_{tk} > 1 \text{ for } k > j \text{ and } t > i \\ \text{(ii) } v_{ij} < 1/2 & v_{ws} < 1/2 \text{ for } s < j \text{ and } w < i \end{cases}$$

Proof: First order the SP preference matrices according to the number of 1 entries they display (from highest to lowest) in row 1, row 2, etc... (By considering the order of the rows we avoid having to order in the same class the matrices with the same number of 1 entries).

Now consider $$V = \frac{1}{\ell} \sum_{h=1}^{\ell} P_h$$

Find the first entry $v_{ij} > 1/2$ in V. By the above construction all entries

v_{kt} for $k > i$ and $t > j$ are also $> 1/2$

Q.E.D.

In view of this property of the voting matrix, we have

Corollary 2 (Black) If all P_h are SP, then simple majority voting always yields a transitive SP ordering

Proof: Set $\underset{\rho \in \mathcal{L}}{\text{Max}}\ f(\rho) = \sum_{i,j} q_{ij}\, v_{\rho(i)\rho(j)}$

and let $f(\rho^*) = \text{Max}$

Then, from equation (15): $$v_{\rho^*(i)\,\rho^*(j)} = \begin{bmatrix} 0 & & >1/2 \\ & \ddots & \\ <1/2 & & 0 \end{bmatrix}$$

by the property of SP voting matrices (Lemma 7).

And by Lemma 5 we know this optimal permutation ρ^* is also the simple majority voting solution

Q.E.D.

Although this result has long been known, its derivation from the condition of equation (15) highlights the sufficiency of the condition for transitivity the social preference relation under simple majority voting. That a voting matrix V need not have property (15) for majority rule to lead to a transitive solution, is evident from the fact that for all non SP orderings of the row and columns of V, equation (15) will not hold; while it may still be possible to find some permutation ρ^* for which

$$[v_{\rho^*(i)\,\rho^*(j)}] = \begin{bmatrix} 0 & & >1/2 \\ & \ddots & \\ >1/2 & & 0 \end{bmatrix}$$

IV. Other Voting Rules as Assignment Problems

4.1 The quadratic case: If we recall the quadratic assignmet formulation of TMDR (equation 2)

$$\max_{\rho \in \mathcal{L}} f(\rho) = \sum_{i,j} q_{ij} v_{\rho(i)\rho(j)}$$

it is clear that many other voting rules can be generated by changing (i) the interaction matrix $[q_{ij}]$, or (ii) the voting matrix $[v_{ij}]$, or (iii) both. Besides some obvious suggestions for alternative voting rules, the assignment formalism provides a unifying way to compare the implicit operations which voting procedures effect on individual opinions. For instance, to what extent are these opinions linearly combined and how much of the pairwise individual preference information is being used by a given voting mechanism: these are some of the key questions which are readily answered by this formulation. We illustrate this point by briefly considering a few of the most often discussed voting procedures.

4.1 The Borda count assigns marks (ranks) 0,1,2,...,m-1 to each alternative in an individual order from worst to best. These marks are then summed across all voters and the resulting ordering of the sum of marks is the Borda ranking. The Borda winner is the top ranked item if only one alternative is to be chosen.

In the QA formulation the Borda winner is found by setting a different interaction matrix, namely:

$$(16) \qquad q_{ij} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ \vdots & 0 & 0 & \cdots & 0 \\ \vdots & & 0 & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \quad \left(\text{or } q_{ij} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 0 & & 0 \\ -1 & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ -1 & \cdots & \cdots & \cdots & 0 \end{bmatrix}\right.$$

if we allow indifference in individual orderings) and solving the standard QA problem (equation 2) with this interaction matrix. To see this, note: (1

(1) that if we compute the row sum in $\ell x [v_{ij}]$ - the "unnormalized voting matrix" - for every i, we obtain the Borda marks i.e. the sum of the individual marks; thus if, say, item a defeats all others, its individual mark is (m-1) but equivalently the first row of the individual preference matrix is (0 1 1...1), summing to (m-1). Similarly, in $\ell x[v_{ij}]$ the row sum $\ell \sum_j v_{ij}$ counts the number of times i is ranked over j in individual orders but, equivalently, the sum of the individual mards for i.

(2) Consequently, to find the Borda winner, that is the item with the highest sum of marks - we only need to consider the largest row sum score. From the above q_{ij} matrix (16) the row sum scores follow while maximizing of f determines the Borda winner. The full Borda ranking can be obtained sequentially by deleting the row and column corresponding to the Borda winner, the Borda second etc...in $[v_{ij}]$.

4.1.2 The <u>Condorcet winner</u> is that alternative, if any, which defeats every other alternative by a simple majority. In this case the QA information becomes

(17) $$\max_{\rho \in \mathcal{L}} f(\rho) = \sum_{i,j} q_{ij} \; \mu(v_{\rho(i)\rho(j)})$$

and

(18) $f(\rho^*) = m-1$ for ρ^* to be an optimal solution where

$$\text{where } q_{ij} = \begin{bmatrix} 0 & 1 & 1 \cdots 1 \\ \vdots & 0 & 0 \cdots 0 \\ \vdots & & \vdots \\ 0 & & 0 \end{bmatrix}$$

and

(19) $$\mu(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > 1/2 \\ 0 & \text{if } v_{ij} < 1/2 \end{cases}$$

4.1.3. The Copeland method is a direct extension of the Condorcet criterion; the Copeland winner is that alternative which defeats the maximal number of other alternatives by a simple majority. If it defeats all others this reduces to a Condorcet winner. The QA fromulation is the same as for Condorcet except that equation (18) is no longer required to hold

$$(17') \quad \max_{\rho \in \alpha} f(\rho) = \sum_{i,j} q_{ij} \; \mu(v_{\rho(i)\rho(j)})^0$$

The full Copeland ranking can also be obtained by sequential reduction of $[v_{ij}]$, eliminating the Copeland winner, the Copeland second etc...

4.1.4. Many other extensions are conceivable. For instance one could weigh the pairs (i,j) differentially - thus dropping the neutrality assumption embodied in the choice of all 1 entries in q_{ij} for TMDR, for instance. A system of differential weights in $[q_{ij}]$ is indeed possible. And even with uniform weights other entry patterns for q_{ij} could be used. Alternatively (or jointly), the voting matrix V can be transformed before solving the QA problem; for instance instead of using simple majority (μ function in (18)), 2/3 majority could be used.

4.2 The linear case: A completely different view consists in treating the individual preferences P_h as "positional" rather than pairwise preferences (Gardenfors [16]). This is the spirit of the Borda count and all scoring function methods (Young [ ]). Specifically, if we take the positional view of matrix preferences, we no longer use the relation matrix P_h to represent an individual preference ordering but the permutation matrix π_h. This is, also, a $\{0,1\}$ matrix which transforms the reference order of the alternative, say $\bar{P}$, into P_h upon postmultiplying A by π_h. For instance if $A = \{a,b,c\}$ and $b \; P_h \; a \; P_h \; c$

$$P_h = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{while} \qquad \pi_h = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(as (a,b,c). π_h = (b,a,c))

A positional voting matrix w is then defined as

$$(20) \qquad W = \frac{1}{\ell} \sum_{h=1}^{\ell} \pi_h$$

It is easy to see that the W matrix is doubly stochastic, that is:

$$(21) \qquad \begin{cases} \sum_{j=1}^{m} w_{ij} = 1 \\ \\ \sum_{i=1} w_{ij} = 1 \end{cases}$$

The well-known Birkhoff Von Neumann theorem states that the polytope of doubly stochastic matrices of dimension m has the set of permutation matrices π as its vertex set.

A natural voting rule studied elsewhere (Blin [ ]) is given by the following linear assignment problem

$$(22) \qquad \underset{\rho \in \mathcal{L}}{\text{Max}} \sum_{i} w_{i\,\rho(i)}$$

This is also a maximal agreement solution on positioning of the alternatives (rather than pairwise preferences). It is interesting to note that it can be shown that a minimal distance (in the Euclidean or city-block sense) solution concept is equivalent to problem (22) - just as it is in the quadratic case (Section II). In spite of its simplicity and rather attractive properties ([6]), the voting rule represented by equation (22) has apparently never been studied in the voting literature.

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