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ON NEIGHBORING CONSUMERS

by

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Abstract. A criterion is proposed for judging when different consumers have similar (transitive, locally nonsatiated) preferences. This criterion is less restrictive than others that have been studied, so consumers will more often be classified as similar. However similar consumers will still have similar demands. The space of preference relations is shown to be separable and metrizable, and a compactification is obtained without relaxing the requirements of local nonsatiation and transitive indifference.

1. Introduction

The purpose of this paper is to describe a criterion of similarity for consumers' preferences.[1] Beginning with Kannai (1970), such criteria have been used to determine when different consumers have similar demands and when different economies have similar equilibrium sets or cores. Kannai studied the convergence of the core, assuming consumers to have transitive, locally nonsatiated preferences. That class of preferences is also the focus of this paper. The criterion to be given here is less stringent than Kannai's, but it is still sufficiently restrictive that consumers will be counted as "neighboring" only if they would behave similarly in similar budget situations. Thus the new criterion allows one to deduce the proximity of consumers' demands in more general circumstances. The space of preference relations also has the same mathematical properties (separability, metrizability) under this criterion as under Kannai's.

Debreu (1969) gave a modification of Kannai's concept which is suitable even for nontransitive and satiated preference relations. Mertens (1970) suggested a weaker criterion for such relations, one which coincides with Kannai's when the relations to be compared are transitive and locally nonsatiated. This criterion is now a standard tool for equilibrium analysis; it counts as neighboring agents those agents whose preference relations, $\{(x,y) | x \succsim y\}$, are similar in the topology of closed convergence of sets. The usefulness of this extension of Kannai's criterion stems in part from the fact that one thereby obtains a compact space of preference relations (of course this is not the only reason for considering a more general class; the need for a framework including satiated relations is discussed by Mount and Reiter (1976). This point is made, e.g., by Grodal (1974, Remark 6). It is not necessary to make the same extension in order to obtain a compact space here. Compactness is obtained without dropping the conditions of local nonsatiation and transitive indifference. What is necessary is that one admit preference relations which are not lower-semicontinuous. Thus one may say that local nonsatiation and transitive indifference, but not lower-semicontinuity, are preserved in the limit under this convergence criterion, while the reverse is true under closed convergence.

Kannai (1970) established that closed convergence of locally nonsatiated transitive preferences is equivalent to uniform convergence on compacta of utility functions, in the sense that one can choose a utility function for each such preference relation in a homeomorphic way. A similar result is established in the author's paper (1986) for the criterion to be studied here. Of course the requisite topology for utility functions is coarser in this case (it is the topology of "subgraph convergence").

In contrast to the weakening proposed here, Chichilnisky (1977) has introduced a strengthening of the closed convergence topology. This was in response to the observation of Mount and Reiter (1976) that the closed convergence topology counts as neighboring agents some agents who are vastly dissimilar with respect to other natural criteria, criteria based for example on measurements of upper and lower contour sets. This objection applies only when one admits locally satiated agents, since locally insatiable agents who are counted as neighboring in the closed convergence topology will have similar upper-contour sets [see Mount and Reiter (1974)]. Thus the problem with which Chichilnisky was concerned will not be an issue in this paper.

Locally satiated agents are excluded here because the criterion cannot be extended in a fully satisfactory way to encompass them. This circumstance is discussed at the end of section 2.

The plan of the paper is as follows. In section 2 the criterion of similarity is defined, and an example is given to show that it is strictly less stringent than Kannai's. The example also illustrates how transitive indifference may not be preserved in the limit under the closed convergence topology. In section 3, demands are shown to vary in an upper-hemicontinuous way with changes in preferences, endowments and prices, just as they do when Kannai's criterion is used. In section 4, the space of preference relations is shown to be separable and metrizable. It also is shown that the space can be compactified by admitting preference relations which are not lower-semicontinuous, and an example is given to demonstrate the necessity of including such relations.

2. Definition of the Criterion

The commodity space will be a locally compact, second countable space E. A preference order is a pair $(X,\succsim)$ where $\succsim$ is a complete preorder on $X \subset E$ and X is closed. A preference order $(X,\succsim)$ is upper-semicontinuous if $\{x \in X | x \succsim y\}$ is closed for each $y \in X$, lower-semicontinuous if $\{x \in X | y \succsim x\}$ is closed for each $y \in X$, continuous if it is both upper and lower-semicontinuous, and locally nonsatiated if for each $x \in X$ and each open $G \subset E$ such that $x \in G$, there exists $y \in G$ such that $y \succ x$ (as usual "$x \succsim y$" means that $(x,y) \in \succsim$ and "$y \succ x$" means that $x,y \in X$ and $(x,y) \notin \succsim$).

Denote by **P** (resp. **Q**) the class of continuous (resp. upper-semicontinuous) locally nonsatiated preference orders.

Kannai suggests (1970, p. 797) that a "plausible requirement" for a topology on a class of preorders is that convergence of a sequence $(\succsim_n)$ to a preorder $\succsim$ entail the following: for each $x \succ y$ and each pair of sequences $x_n \to x$ and $y_n \to y$, one eventually has $x_n \succ_n y_n$. Without pretending that it is any more reasonable (but arguing that it may be useful simply because it is weaker) I propose the requirement that convergence imply: for each $x \succ y$ and each sequence $y_n \to y$, there exists a sequence $x_n \to x$ such that eventually $x_n \succ_n y_n$.

Proceeding more formally, choose a family of open, relatively compact sets B_i, $i = 1,2,\ldots$, which is a basis for the topology of E. Set

$$Q_{ij} = \{(X, \succsim) \in Q \mid \exists\, x \in B_i \text{ s.t. } x \succ \bar{B}_j\}$$

where the upper bar denotes closure and where "$x \succ \bar{B}_j$" means that $x \in X$ and $x \succ y$, $\forall y \in \bar{B}_j \cap X$. Set $Q_{0j} = \{(x,\succsim) \in Q \mid X \cap \bar{B}_j = \emptyset\}$. The family of sets $\{Q_{ij} \mid i = 0,1,2,\ldots;\ j = 1,2,3,\ldots\}$ is a subbase for a topology on **Q**. This topology will be denoted by τ.

The relative τ topology on P will be denoted by the same symbol. It has as a subbase the family of sets $P_{ij} = P \cap Q_{ij}$.

One corollary of the following is that the topology τ obtained would be the same for any choice of the basis $\{B_i\}$ for E, subject of course to the qualification that each B_i be relatively compact.

PROPOSITION 1. For each open $G \subset E$ and compact $K \subset E$, the set $\{(X,\succsim) \in Q \mid \exists\, x \in G \text{ s.t. } x \succ K\}$ is τ-open, as is the set $\{(X,\succsim) \in Q \mid X \cap K = \emptyset\}$.

<u>Proof</u>. Fix $(X,\succsim) \in Q$ and $x \in G$ satisfying $x \succ K$. Since X is closed and $\succsim$ is upper-semicontinuous, there exists for each $y \in K$ some index i such that $y \in B_i$ and $x \succ B_i$. Since E is regular [Engelking (1968, Thm. 3.6.1)] there is for each such i an index j such that $y \in B_j \subset \bar{B}_j \subset B_i$ [Engelking (1968, Thm. 1.5.3)]. Choose a finite subfamily of the collection of sets B_j whose union

contains K, say $\{B_j | j = 1,\ldots,n\}$. Let B_ℓ be an element of the given basis such that $x \in B_\ell \subset G$. Then

$$(X,\succsim) \in \bigcap_{j=1}^{n} \mathbf{Q}_{\ell j} \subset \{(X,\succsim) \in Q | \exists\ x \in G \text{ s.t. } x \succ K\}.$$

Thus the first type of set is open, and the same argument shows that the second type is as well. Q.E.D.

Since the collection of sets $\mathbf{Q}_{ij}$ is countable, the topological spaces $(\mathbf{Q},\tau)$ and $(\mathbf{P},\tau)$ are second countable. Convergence of sequences is characterized by closed convergence of consumption sets and the property previously stated. The notation "$X = \text{Lim } X_n$" used below means that X is the closed limit of the X_n; see Hildenbrand (1974). Hildenbrand's notation "Li X_n" and "Ls X_n" will also be followed.

PROPOSITION 2. A sequence $((X_n,\succsim))$ from $\mathbf{Q}$ is τ-convergent to a preference order $(X,\succsim) \in Q$ iff each of the following hold:

(a) $X = \text{Lim } X_n$

(b) For each x,y such that $x \succ y$ and each sequence $y_n \to y$ satisfying $y_n \in X_n$, $\forall n$, there exists a sequence $x_n \to x$ such that eventually $x_n \succ_n y_n$.

<u>Proof</u>. τ-convergence => (a): Let $G \subset E$ be any open set such that $X \cap G \neq \emptyset$. Choose $x \in X \cap G$. By local nonsatiation there exists $z \in G$ such that $z \succ x$. According to Proposition 1, there eventually exists $x_n \in G$

satisfying $x_n \succ \{x\}$. In particular $X_n \cap G \neq \emptyset$. This shows that $X \subset \text{Li } X_n$. The condition $\text{Ls } X_n \subset X$ follows immediately from the second part of Proposition 1.

τ-convergence => (b): By upper-semicontinuity there exists some B_i such that $y \in B_j$ and $x \succ B_j$. As noted in the proof of Proposition 1, E is regular so we can assume that $x \succ \bar{B}_j$. The τ-convergence implies that for each B_i such that $x \in B_i$ there eventually exists $x_n \in B_i$ satisfying $x_n \succ_n \bar{B}_j$. Hence eventually $x_n \succ_n y_n$. Since the basis $\{B_i\}$ is countable, one can certainly construct a sequence $x_n \to x$ such that eventually $x_n \succ_n y_n$.

(a) + (b) => τ=convergence: The condition $\text{Ls } X_n \subset X$ implies that, for any j such that $(x,\succsim) \in Q_{0j}$, one eventually has $(X_n,\succsim_n) \in Q_{0j}$. Now consider any $i,j \geq 1$ such that $(X,\succsim) \in Q_{ij}$. Since $X \subset \text{Li } X_n$ there exists N such that $X_n \cap B_i \neq \emptyset$ for $n \geq N$. Let I denote the set of integers n such that $(X_n,\succsim_n) \notin Q_{ij}$. If $n \in I$ and $n \geq N$, then it must be that $X_n \cap \bar{B}_j \neq \emptyset$. For each such n, choose $y_n \in X_n \cap \bar{B}_j$ satisfying $y_n \succsim_n z$, $\forall z \in X_n \cap \bar{B}_j$ (by upper-semicontinuity such a choice is possible). Suppose I is an infinite set. Then the sequence $(y_n, n \in I)$ has a subsequence $(y_n, n \in I')$ which has a limit $y \in \bar{B}_j$ (since compactness is equivalent to sequential compactness in regular, second countable spaces [Engelking (1968, Thm. 4.2.4 (Urysohn Metrization Theorem) and Exercise 4.3.E)]. By (a), $y \in X$ and moreover there exists a sequence $z_n \to y$ such that $z_n \in X_n$ for each integer n. Take $w_n = y_n$ for $n \in I'$ and $w_n = z_n$ for each integer $n \notin I'$. Since $(X,\succsim) \in Q_{ij}$ there exists $x \in B_i$ such that $x \succ y$. Hence by (b) there is a sequence $x_n \to x$ such that eventually $x_n \succ_n w_n$. For $n \in I'$ this implies that $x_n \succ_n \bar{B}_j$. Since the x_n are eventually in B_i this contradicts the definition of I. Hence I must be a finite set, i.e., the $(X_n,\succsim_n)$ are eventually in Q_{ij}. Q.E.D.

The concept of τ-convergence is evidently at least as weak as the criterion proposed by Kannai. It will now be seen that it is strictly weaker.

EXAMPLE 1. Let $E = \mathbb{R}^2$ and $X_n = X = \mathbb{R}_+^2$. Take $\succsim_n$ to be the preorder represented by the utility function $u_n(a,b) = \min\{(n+1)a + \frac{1}{2}b,\ a + b\}$ and take $\succsim$ to be represented by $u(a,b) = a + b$. See Figure 1.

To show that the preference orders are τ-convergent to $(X,\succsim)$, choose any $x = (a,b)$ and $y = (c,d)$ in $\mathbb{R}_+^2$ such that $u(x) > u(y)$. Let (y_n) be any sequence from $\mathbb{R}_+^2$ converging to y. Certainly $u(y) = \lim u(y_n) \geq \limsup u_n(y_n)$. If $a > 0$ then eventually $u_n(x) = u(x) > u(y)$, so Proposition 2 (b) is established by taking $x_n = x$. If $a = 0$, then we will obtain $u_n(x_n) = u(x)$ and hence the desired result if we set $x_n = (a_n,b_n)$ where $a_n = b\cdot(2n+1)^{-1}$ and $b_n = 2n\cdot b\cdot(2n+1)^{-1}$.

However the preference orders are not convergent to $(X,\succsim)$ under Kannai's criterion since, e.g., if $a = 0$, $b = 3$, $c = d = 1$, then $u(a,b) > u(c,d)$ but $u_n(a,b) < u_n(c,d)$, $\forall n$.

The sequence does converge in the closed convergence topology to a continuous, reflexive, negatively transitive relation on $\mathbb{R}_+^2$, say $\succsim_o$ [the existence of a subsequence convergent in this sense is demonstrated by Hildenbrand (1974, Thm. 1.2.1)]. This is the relation defined by $x \succsim_o y$ iff <u>either</u> $u(x) \geq u(y)$ <u>or</u> $u(x) \geq \frac{1}{2}u(y)$ and $y = (0,d)$ for some d. This relation does not belong to **Q** since the indifference relation $\sim_o$ (defined as usual by $x \sim_o y$ iff $x \succsim_o y$ and $y \succsim_o x$) is not transitive. For example, $(0,1) \sim_o (0,2)$ and $(0,2) \sim_o (0,4)$ but $(0,1) \not\sim_o (0,4)$.

[INSERT FIGURE 1 HERE]

The example illustrates a general principle: if a sequence $((X_n,\succsim_n))$ has a τ-limit $(X,\succsim) \in \mathbf{P}$ and a different limit $(X_o,\succsim_o)$ in the closed convergence topology (in which case it is necessarily true that $(X_o,\succsim_o) \notin \mathbf{P}$) then $X = X_o$ and $\succsim \subset \succsim_o$. The fact that $X = X_o$ follows from Proposition 2 and Appendix A(II) of Hildenbrand (1970). The inclusion $\succsim \subset \succsim_o$ follows from the fact that τ-convergence implies $\succsim \subset \text{Li} \succsim_n$ (to see this use local nonsatiation and Proposition 2). The condition $\succsim \subset \succsim_o$ means that if $x \succ_o y$ then $x \succ y$; on the other hand we may have $x \sim_o y$ but not $x \sim y$. There will be more (too much?) indifference under the relation $\succsim_o$, as the example also demonstrates.

The claim that $\succsim_o$ is the limit in the closed convergence topology but not in τ deserves some clarification. Of course it could not be the limit in τ since it is not in the space on which τ was defined. We could just as easily arrange matters so that it is not the limit in the closed convergence topology either, simply by restricting that topology to the space **P**. In that case we are back to Kannai's criterion. But then the sequence would not have any limit, nor even a cluster point. To obtain a compact space under Kannai's criterion, we need to count $\succsim_o$ as the limit; this is not necessary under the criterion defined by τ (see section 4).

The existence of the limit $\succsim_o \neq \succsim$ suggests there would be difficulty in extending the criterion given here to the entire space of continuous, reflexive, negatively transitive relations. If we were to attempt to do so in a way which would preserve the relation between this criterion and the Kannai-Mertens criterion -- namely the fact that this criterion is less restrictive -- then we would have to admit both $\succsim_o$ and $\succsim$ as limits of the sequence, i.e., the space would not be Hausdorff. This is a well-known corollary of the fact that the space is compact in the closed convergence topology, Hildenbrand (1974, Thm. 1.2.1).

There are Hausdorff extensions of this criterion to the space studied by Hildenbrand. For example one can take a metric ρ for $(\mathbf{P},\tau)$ and a metric δ for the closed convergence topology, each bounded by 1, and define $d((X,\succsim), (X',\succsim'))$ to equal $\rho((X,\succsim), (X',\succsim'))$ if both $(X,\succsim)$ and $(X',\succsim')$ belong to $\mathbf{P}$, to equal $\delta((X,\succsim), (X',\succsim'))$ if neither belongs to $\mathbf{P}$, and to equal 1 if one belongs to $\mathbf{P}$ and the other does not. This defines a metric on Hildenbrand's space [see, e.g., Engelking (1968, Thm. 4.2.1)] for which the relative topology on $\mathbf{P}$ obviously coincides with τ. This metric topology is neither weaker nor stronger than that of closed convergence -- the sequence furnished in the example is convergent in both topologies, but to the different limits already noted. Observe that the metric space is not connected, the subset $\mathbf{P}$ being both open and closed.

3. Upper-hemicontinuity of Demands

Here it will be shown that "similar agents behave similarly" when similarity of agents' preferences is understood as in the preceding section. Demands vary continuously with changes in preferences, endowments and prices in exactly the manner in which they do when the Kannai-Mertens criterion is used to define similarity of preferences [see Hildenbrand (1970, Appendix A)].

In the following we give the set M the relative product topology, equipping $\mathbf{P}$ with the topology τ and $\mathbb{R}^\ell \times \mathbb{R}^\ell$ with the usual topology.

THEOREM 1. Let $E = \mathbb{R}^\ell$ and let the subset M of $\mathbf{P} \times \mathbb{R}^\ell \times \mathbb{R}^\ell$ be such that for every $((X,\succsim),\omega,p) \in M$ the consumption set X is convex, the budget set $\gamma((X,\succsim),\omega,p) = \{x \in X | p\cdot x \leq p\cdot\omega\}$ is compact and $\inf p\cdot X < p\cdot\omega$. Then the demand correspondence ξ of M into $\mathbb{R}^\ell$ is upper-hemicontinuous -- $\xi((X,\succsim),\omega,p)$ is the set of $\succsim$-maximal elements in $\gamma((X,\succsim),\omega,p)$.

Proof. Since τ-convergence implies closed convergence of consumption sets, Hildenbrand's proof (1970, Appendix A (III)) applies to show that the budget set correspondence is upper-hemicontinuous. Therefore, as in Hildenbrand (1970, Appendix A (IV)), we need only show that ξ has a closed graph in $M \times \mathbb{R}^{\ell}$.

Let $((X_n, \succsim_n), \omega_n, p_n) \to ((X, \succsim), \omega, p)$ in M. Suppose $x_n \in \xi((X_n, \succsim_n), \omega_n, p_n)$ and $x_n \to x \in \mathbb{R}^{\ell}$. Then $x \in X$ and $p \cdot x \leq p \cdot \omega$. Consider $y \in X$ such that $p \cdot y \leq p \cdot \omega$. By the definition of M there is some $z \in X$ such that $p \cdot z < p \cdot \omega$. Let $y_k = \left(\frac{1}{k}\right) z + \left(1 - \frac{1}{k}\right) y$. By the definition of M, $y_k \in X$. Suppose $y_k \succ x$. Then by the τ-convergence there is a sequence $y_{kn} \to y_k$ as $n \to \infty$ such that eventually $y_{kn} \succ_n x_n$. But eventually $p_n \cdot y_{kn} \leq p_n \cdot \omega_n$, which would contradict $x_n \in \xi((X_n, \succsim_n), \omega_n, p_n)$. Hence $x \succsim y_k$. By the lower-semicontinuity of $\succsim$, $x \succsim y$. Q.E.D.

4. Properties of the Spaces of Preference Orders

The main result of this section is:

THEOREM 2. The space $(\mathbf{P}, \tau)$ is separable and metrizable, and the space $(\mathbf{Q}, \tau)$ is σ-compact, separable and metrizable.

The concept of σ-compactness is defined by Dugundji (1966, p. 240). It will actually be shown here that for any compact $K \subset E$, the set $\mathbf{Q}^K \equiv \{(X, \succsim) \in \mathbf{Q} \mid X \cap K \neq \emptyset\}$ is compact in the subspace topology. There is little loss in fixing some large K and taking $\mathbf{Q}^K$ rather than $\mathbf{Q}$ as the basic space of preferences. Hildenbrand (1974) follows this procedure to obtain a compactness result.

Before the theorem is proven, an example will be given to show that **P** does not have a compactness property. Compactness fails for **P** because it is not closed in **Q**. It seems reasonable to conjecture that it is a G_δ subset of **Q**. If so, then it would have the important property of being metrizable in a complete manner, Dugundji (1966, IX.7.2, XIV.2.4, XIV.8.3). However I have been unable to ascertain whether this is true.

EXAMPLE 2. Let $E = \mathbb{R}^2$ and $X_n = X = \mathbb{R}^2_+$. Let $\succsim_n$ be the preorder represented by the utility function

$$u_n(a,b) = \begin{cases} \dfrac{b+2^{-n}(a+b)}{1-a+2^{-n}} & \text{if } a+b < 1 \\ a+b & \text{if } a+b \geq 1 \end{cases}$$

Let $\succsim$ be the preorder represented by

$$u(a,b) = \begin{cases} \dfrac{b}{1-a} & \text{if } a+b < 1 \\ a+b & \text{if } a+b \geq 1 \end{cases}$$

The indifference curves for these functions are shown in Figure 2. It is straightforward to check that u_n is continuous and strictly monotone on $\mathbb{R}^2_+$. Thus $(X_n, \succsim_n) \in$ **P**. Similarly one can verify that u is upper-semicontinuous and monotone; however u fails lower-semicontinuity at (1,0) since $u(1-\varepsilon,0) = 0$ for every $0 < \varepsilon < 1$. Hence $(X,\succsim) \in \mathbf{Q}\setminus\mathbf{P}$. Yet the sequence $((X_n,\succsim_n))$ is τ-convergent to $(X,\succsim)$. In fact a strong form of Proposition 2(b) is satisfied: if $x \succ y$ and $y_n \to y$ in $\mathbb{R}^2_+$ then eventually $x \succ_n y_n$. This is true because the u_n are pointwise convergent to u and satisfy a one-sided form of continuous convergence -- we have $u(y) \geq \limsup u_n(y_n)$ for every sequence $y_n \to y$. The failure of the opposite condition, $u(y) \leq \liminf u_n(y_n)$, $\forall\, y_n \to y$, is obviously related to the failure of lower-semicontinuity of u.

In the closed convergence topology the preference orders $(X_n,\succsim_n)$ converge to the relation $\succsim_o$ on $\mathbb{R}^2_+$ defined by: if $x,y \neq (1,0)$ then $x \succsim_o y$ iff $u(x) \geq u(y)$; if $x = (a,b)$ such that $a + b > 1$ then $x \succ_o (1,0)$; and if $x = (a,b)$ such that $a + b \leq 1$ then $x \sim_o (1,0)$. Lower-semicontinuity of the limit relation at (1,0) is obtained, very roughly speaking, by assigning each point in the interval [0,1] as its utility level. Again we note the extent and nontransitivity of indifference under $\succsim_o$.

[INSERT FIGURE 2 HERE]

<u>Proof of the Theorem</u>. By construction the spaces are second countable and therefore separable. The space $(\mathbf{P},\tau)$ will inherit the metrizability of $(\mathbf{Q},\tau)$, so it is enough to show that $(\mathbf{Q},\tau)$ is σ-compact and metrizable. First we prove the metrizability.

Since $(\mathbf{Q},\tau)$ is second countable, the Urysohn Metrization Theorem [Engelking (1968, Thm. 4.2.4)] can be applied once we show that $(\mathbf{Q},\tau)$ is regular. The first step will be to prove that it is a T_1-space.

Let $(X,\succsim)$ and $(X',\succsim')$ be distinct elements of $\mathbf{Q}$. In view of Proposition 2(a) -- and the second countability of $(\mathbf{Q},\tau)$ -- the map $(X,\succsim) \to X$ from $\mathbf{Q}$ to the class of closed subsets of E, equipped with the closed convergence topology, is τ-continuous (see Hildenbrand (1974) for the definition of the closed convergence topology). The closed convergence topology is T_1. Hence if $X \neq X'$ there exists a τ-open set which contains $(X,\succsim)$ but not $(X',\succsim')$.

Suppose $X = X'$ but $\succsim \neq \succsim'$. Then there exists x, $y \in X$ such that either (i) $x \succsim y$ and $y \succ' x$, or (ii) $x \succsim' y$ and $y \succ x$. In case (i) by upper-semicontinuity there is an open set G such that $x \in G$ and $y \succ' G$. Taking $K = \{y\}$ we have that the set $\{(X'',\succsim'') \in \mathbf{Q} | \exists\ x \in G \text{ such that } x \succ'' K\}$. which by Proposition 1 is τ-open, does not include $(X',\succsim')$. But in case (i) it does include $(X,\succsim)$, by virtue of local nonsatiation. In case (ii) we rely on the upper-semicontinuity of $\succsim$ and the local compactness of E to obtain an open relatively compact set B such that $x \in B$ and $y \succ \bar{B}$ [see Engelking (1968, Thm. 3.6.2)]. By the local nonsatiation of $\succsim'$ there is some $z \in B$ such that $z \succ'$ y, and by the upper-semicontinuity of $\succsim'$ there is an open set H containing y such that $z \succ' H$. In particular there does not exist $w \in H$ such that $w \succ' \bar{B}$. Since we have already observed that $(X,\succsim)$ belongs to the (τ-open) set $\{(X'',\succsim'') \in \mathbf{Q} | \exists\ w \in H \text{ such that } w \succ'' \bar{B}\}$, the argument is complete.

Now to prove that $(\mathbf{Q},\tau)$ is regular, it suffices to take an arbitrary $(X,\succsim) \in \mathbf{Q}$ and a subbase element $\mathbf{Q}_{ij}$ containing $(X,\succsim)$, and to exhibit a neighborhood $\mathbf{U}$ of $(X,\succsim)$ such that $\mathbf{Q}_{ij}$ contains the τ-closure of $\mathbf{U}$ [Engelking (1968, Thm. 1.5.3)].

Suppose first that $i \geqslant 1$, so $(X,\succsim) \in \mathbf{Q}_{ij}$ means that there exists $x \in B_i$ with $x \succ \bar{B}_j$. Since E is regular there is a B_k in the given basis such that $x \in B_k \subset \bar{B}_k \subset B_i$ [Engelking (1968, Thm. 1.5.3)]. Also by upper-semicontinuity we can cover $\bar{B}_j$ with open sets H_α such that $x \succ H_\alpha$, $\forall\alpha$. Again using the regularity of E we can take this covering to be by basis elements B_ℓ, such that $x \succ \bar{B}_\ell$, $\forall\ell$. Choose a finite subcovering, say $\{B_\ell | \ell=1,\ldots,n\}$.

Setting $\mathbf{U} = \bigcap_{\ell=1}^{n} Q_{k\ell}$, we certainly have that $(X,\succsim) \in \mathbf{U}$. We claim moreover that the τ-closure of U is contained in $\mathbf{Q}_{ij}$. Clearly $\mathbf{U} \subset \mathbf{V} \subset Q_{ij}$ where

$$\mathbf{V} = \{(X',\succsim') \in Q \mid \exists y \in \bar{B}_k \text{ such that } y \succ' \bigcup_{\ell=1}^{n} B_\ell\}.$$

Also the complement of $\mathbf{V}$ is the set of $(X',\succsim')$ for which either $X' \cap \bar{B}_k = \emptyset$ or (using the fact that there is a $\succsim'$-maximal point in $\bar{B}_k$) for which there is some $x' \in \bigcup_{\ell=1}^{n} B_\ell$ such that $x' \succ' \bar{B}_k$. Hence by Proposition 1, the set $\mathbf{V}$ is τ-closed.

Now if i = 0 then $(X,\succsim) \in Q_{ij}$ means that $X \cap \bar{B}_j = \emptyset$. Since X is closed and $\bar{B}_j$ is compact, the local compactness of E implies that there is an open, relatively compact set G such that $\bar{B}_j \subset G$ and $X \cap \bar{G} = \emptyset$ [Engelking (1968), Thm. 3.6.2)]. Taking $\mathbf{U} = \{(X',\succsim') \mid X' \cap \bar{G} = \emptyset\}$, we have $(X,\succsim) \in \mathbf{U} \subset \mathbf{V} \subset Q_{ij}$ where $\mathbf{V} = \{(X',\succsim') \mid X' \cap G = \emptyset\}$. The continuity of the map $(X,\succsim) \to X$ implies that $\mathbf{V}$ is τ-closed [see Hildenbrand (1974, p.18)].

It remains only to prove the σ-compactness. First we note that E, being second countable and locally compact, can be written as $E = \bigcup_{\alpha=1}^{\infty} G_\alpha$ where each G_α is a relatively compact open set and where $\bar{G}_\alpha \subset G_{\alpha+1}$ for each α [Dugundji (1966, Thm. IX.7.2)]. Clearly $\mathbf{Q} = \bigcup_{\alpha=1}^{\infty} \mathbf{Q}_\alpha$ where $\mathbf{Q}_\alpha = \{(X,\succsim) \in Q \mid X \cap G_\alpha \neq \emptyset\}$. Each Q_α is τ-open (again use the continuity of the map $((X,\succsim) \to X)$. Moreover $\mathbf{Q}_\alpha$ is contained in $\mathbf{Q}'_\alpha \equiv \{(X,\succ) \mid X \cap \bar{G}_\alpha \neq \emptyset\}$ which in turn is contained in $\mathbf{Q}_{\alpha+1}$. Thus the result will follow from Dugundji (1966, Thm. IX.7.2) upon proving that each Q'_α is τ-compact. Since $(\mathbf{Q}'_\alpha,\tau)$ is metrizable, compactness is equivalent to sequential compactness [Engelking (1968, Exercise 4.3.E)].

Fix an α and write $K = \bar{G}_\alpha$. Choose a sequence $((X_n,\succsim_n))$ from Q such that $X_n \cap K \neq \emptyset$, $\forall n$. We can extract a subsequence such that the X_n have a closed limit $X \subset E$ satisfying $X \cap K \neq \emptyset$ [Hildenbrand (1974, Thm. I.B.2)]. We will continue to write this subsequence as $((X_n,\succsim_n))$.

For each B_i in the given basis for E, let $S_{in} = \{x \in X_n | x \succ_n B_i\}$. By local nonsatiation the complement in X_n of S_{in} is $\{x \in X_n | \exists y \in B_i$ such that $y \succ_n x\}$, which by upper-semicontinuity is open in X_n. Hence S_{in} is closed in X_n, equivalently closed in E.

We can extract a subsequence from $((X_n,\succsim_n))$ such that for each i, the sequence of sets S_{in} has a closed limit $S_i \subset E$, as $n \to \infty$ (S_i will possibly be empty). We write this subsequence now as $((X_n,\succsim_n))$.

Define a binary relation $\succsim$ on X by setting $x \succsim y$ iff, $\forall i$, $x \in B_i$ implies $y \notin S_i$. We will prove that $(X,\succsim) \in \mathbf{Q}$ and that it is the τ-limit of the $(X_n,\succsim_n)$.

REFLEXIVITY: If it were not the case that $x \succsim x$ then there would be some i such that $x \in B_i$ and $x \in S_i$. But $B_i \cap S_{in} = \emptyset$, $\forall n$, so $B_i \cap S_i = \emptyset$.

COMPLETENESS: Consider any x, $y \in X$ for which it is not the case that $x \succsim y$. Fix an index i such that $x \in B_i$ and $y \in S_i$. For any neighborhood B_j of y we certainly have $B_j \cap S_i \neq \emptyset$. This implies that eventually $B_j \cap S_{in} \neq \emptyset$. By the definition of the S_{in} it must be that $B_i \cap S_{jn} = \emptyset$. In the limit we obtain $B_i \cap S_j = \emptyset$. In particular $x \notin S_j$. Since this holds for each B_j containing y, we have $y \succsim x$.

TRANSITIVITY: Consider any $x,y,z \in X$ satisfying $y \succsim z$ and $z \succ x$. The latter condition means that there is some i such that $x \in B_i$ and $z \in S_i$. To show that $y \succ x$ it suffices to show that $y \in S_i$, which is equivalent to the condition that for each neighborhood B_j of y we eventually have $B_j \cap S_{in} \neq \emptyset$. Note that if $y \in B_j$ then the condition $y \succsim z$ implies that $z \notin S_j$. Since S_j is closed, and E is regular, there is an index k such that B_k includes z and $B_k \cap S_j = \emptyset$. Eventually we must have $B_k \cap S_{jn} = \emptyset$. Moreover since $z \in S_i \cap B_k$ it must also eventually be true that $B_k \cap S_{in} \neq \emptyset$, i.e., that there exists $v_n \in B_k$ satisfying $v_n \succ_n B_i$. But the condition $\bar{B}_k \cap S_{jn} = \emptyset$ implies that $v_n \not\succ_n B_j$; since $v_n \in X_n$, this is possible only if there is some $w_n \in B_j$ satisfying $w_n \succsim_n v_n \succ_n B_i$. We now have $B_j \cap S_{in} \neq \emptyset$, as desired.

UPPER-SEMICONTINUITY: Consider any $x, y \in X$ and any sequence $x_m \to x$ in X for which $x_m \succsim y$, $\forall m$. Choose any i such that $x \in B_i$. Eventually we have $x_m \in B_i$ so it must be that $y \notin S_i$. Hence $x \succsim y$.

LOCAL NONSATIATION: Choose any $x \in X$ and any neighborhood G of x. By the local compactness of E there are indices i,j such that $x \in B_i \subset \bar{B}_i \subset B_j \subset \bar{B}_j \subset G$ [Engelking (1968, Thm. 3.6.2)]. By the closed convergence of the sets X_n, we eventually have $X_n \cap B_i \neq \emptyset$. Since $\succsim_n$ is upper-semicontinuous there is a $\succsim_n$-maximal point in $X_n \cap \bar{B}_i$. By the local nonsatiation of $\succsim_n$ there must be some $z_n \in B_j$ which is $\succsim_n$-preferred to this point. Therefore, for sufficiently large n, $B_j \cap S_{in} \neq \emptyset$. This implies that $\bar{B}_j \cap S_i \neq \emptyset$. But if $y \in \bar{B}_j \cap S_i$ then $y \in G$ and, by definition, $y \succ x$.

CONVERGENCE: We need only verify Proposition 2(b). Let $x \succ y$. This means that there is some B_i such that $y \in B_i$ and $x \in S_i$. The latter condition implies that there is a sequence $x_n \to x$ such that, $\forall n$, $x_n \in S_{in}$, i.e., $x_n \succ_n B_i$. If we choose any sequence $y_n \to y$, we must eventually have $x_n \succ_n y_n$. Q.E.D.

FOOTNOTES

[1] This criterion was first mentioned in the author's 1986 paper in this journal. The present paper supplies proofs of properties mentioned in that paper as well as additional details.

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FIGURE 1

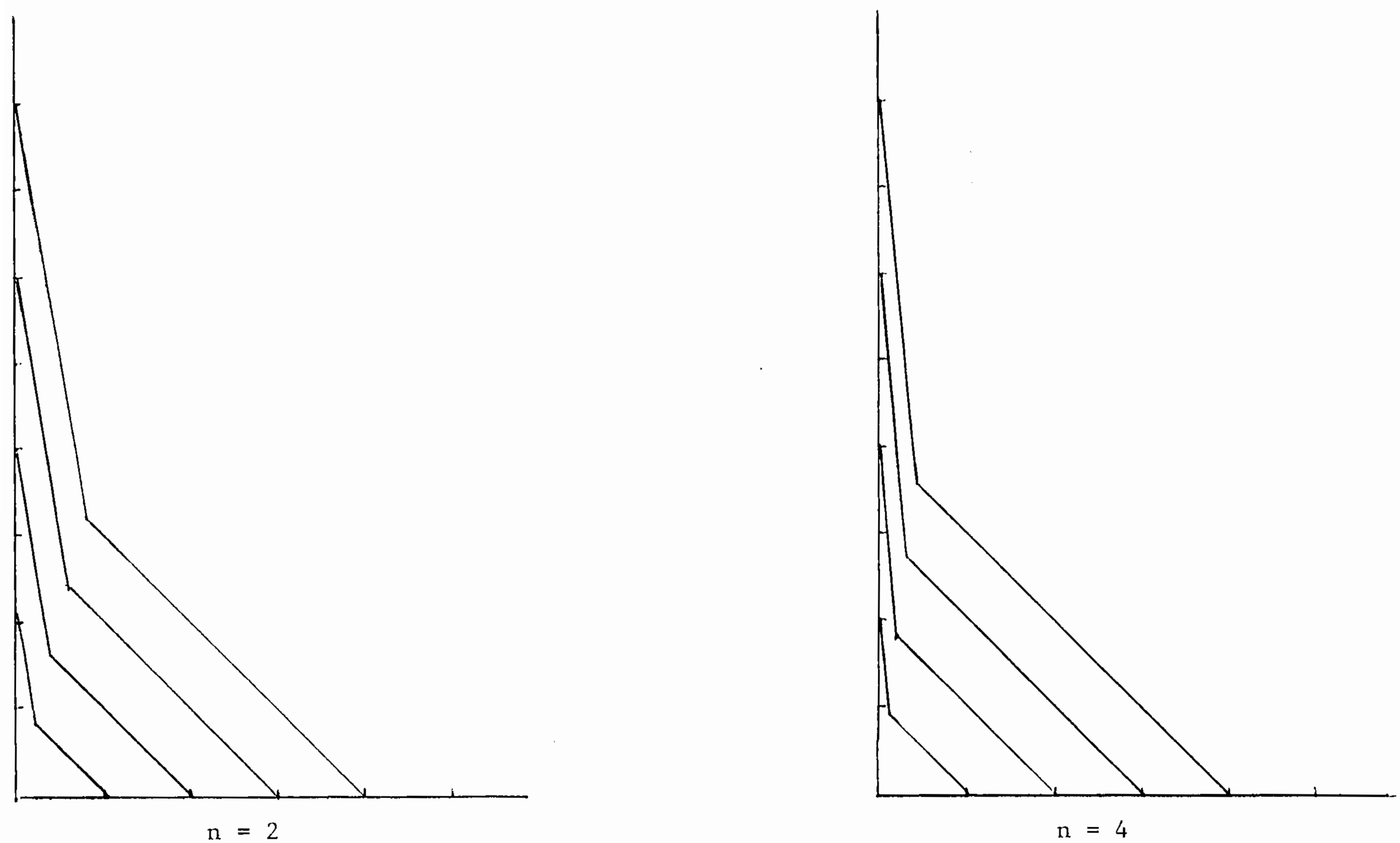
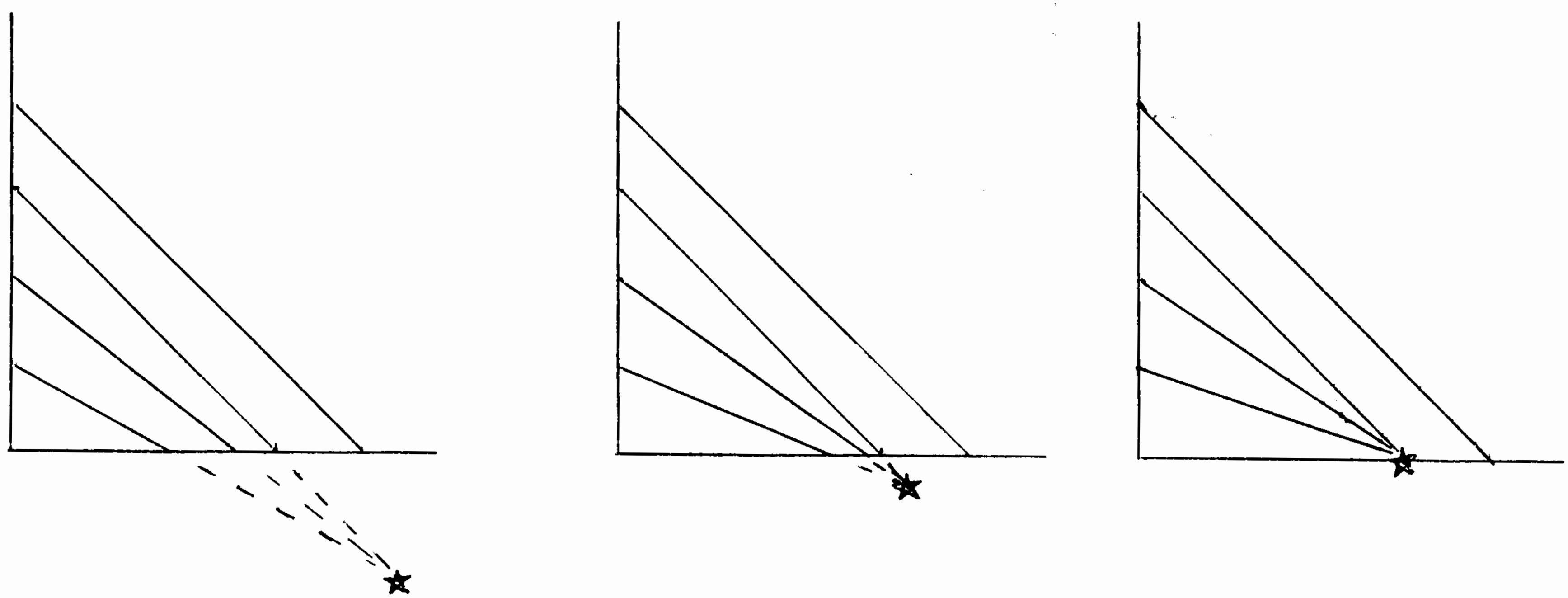


Figure 2

☆ $= (1+2^{-n}, -2^{-n})$



n = 1 n = 3 $n = \infty$