Good and Bad Investment: An Inquiry into the Causes of Credit Cycles

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Abstract

This paper presents dynamic general equilibrium models of imperfect credit markets, in which the economy fluctuates endogenously along its unique equilibrium path. During a recession, the net worth of the agents is too low and they cannot finance their trading activities. Much of the saving thus goes to business investment, which creates jobs, thereby making the next generation of the agents richer. As the economy booms and the net worth of the agents improves, they can eventually finance their trading activities, which do not create any job. As more credit is extended to trading at the expense of business investment, the economy plunges into a recession. The whole process repeats itself. Endogenous fluctuations occur because, as in ecological cycles driven by predator-prey interaction, good investment breeds bad investment, which destroys good investment.

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1. Introduction

This paper presents models of endogenous business cycles based on the following idea. Some investments are good, while others are bad. Good investments are like those in the business sector, which create jobs and purchase inputs, thereby generating much aggregate demand spillovers. On the other hand, some investments, like trading and speculation in the commodity and real estate markets, generate little aggregate demand spillovers, even though they may be highly profitable. Furthermore, some of these investments are relatively difficult to finance, because their default risk is high. During the recession, the agents are not rich enough to finance these investments. Much of the saving thus goes to the business sector, and many businesses are formed. This generates a large demand for labor and other inputs, which benefits other agents as well. As the economy expands, the agents become richer. With an improved net worth, the agents can finance their trading activities. Saving is now redirected from business investment to trading, which have little demand spillovers. At the peak of the boom, this change in the composition of credit and of investment causes a deterioration of the net worth, and the economy plunges into a recession. The whole process repeats itself. Endogenous fluctuations occur because good investment breeds bad investment, which destroys good investment, as in ecological cycles driven by predator-prey or host-parasite interactions.[2]

To capture this general idea in a concrete manner, the models developed below use many specific assumptions, made for the most part to simplify the analysis. Two key specifications of the model should be mentioned. First, the overlapping generations structure is used to model aggregate demand spillovers. More specifically, there are overlapping agents who live for two periods. Each generation earns and saves the wage income in their first period. They use their savings to finance the investment project. Because the project requires the minimum level of investment, they also have to borrow from the credit market. There are two types of investment, the Good and the Bad. When more credit is extended to the Good (business investment), more firms are set up and more jobs are created, which improves the net worth of the younger generation, and helping them to finance their investment projects. When credit is extended for the

[2]While the intuition behind fluctuations is similar with that of predator-prey cycles in biology, our models are mathematically quite different from what mathematical biologists call the predator-prey models (see, e.g., Murray 1990).

Bad (trading), it does not create any job for the young; the traders simply hoard and stockpile goods. Thus the investment in trading activities does not benefit the next generations of the agents. The second key feature is the borrowing constraint, which arises endogenously due to the risk of (potential) defaults.[3] The key condition for endogenous fluctuations is that the difficulty of enforcing repayment is large enough for the credit extended to trading that the agents cannot make the Bad investment when the net worth is low, but small enough that they can when it is high. This means that, in recessions, much of the saving goes to business investment, and as the economy booms, more saving is redirected from business investment to trading. One major advantage of using these specifications is that they make the model simple enough to allow for a global analysis of the nonlinear dynamical system governing the equilibrium trajectory in terms of a few key parameters.

It is commonly argued that the waves of economic expansions and contractions are caused by the changing nature of credit and investment at different phases of business cycles. At the peak of the boom, the argument goes, more credit is extended to finance "socially unproductive" activities, such as trading in the commodity, real estate, and stock markets. Such an expansion of credit fuels speculative manias, which destabilize the economy. (See Kindleberger 1996 for a review of the basic argument.) Central bankers indeed seem concerned that financial frenzies that emerge after a period of economic expansion might lead to misallocation of credit, thereby pushing the economy into a recession, and they often attempt to take precautionary measures to achieve a soft landing of the economy. The present paper does provide a theoretical argument for the view that changing compositions of credit and of investment are responsible for creating instability and fluctuations. Furthermore, the equilibrium dynamics display some features reminiscent of the popular argument.[4] It should be emphasized, however, that the present paper does not assume any irrationality on the side of the agents. Contrary to the popular argument,

[3] Macroeconomic models of imperfect credit markets based on imperfect enforcement of repayment and strategic defaults include Obstfeld and Rogoff (1996, Ch. 6.1 and 6.2), Kiyotaki and Moore (1997) and Aghion, Banerjee, and Piketty (1999). The specification here follows those of Matsuyama (2000a, 2000b).

[4] The idea explored here also has some vague resemblance to that of Hawtrey, who put (wholesale) traders at the center of his theory. He argued that the traders play a pivotal role at the turning point of business cycles, as their investment becomes sensitive to a small change in the interest rate, after their inventory stock has been built up. See, for example, Hawtrey (1913), which incidentally inspired the title of this paper.

instability is not caused by "irrational extension of credit" or "irrational exuberance" in the models developed below.[5]

Recent studies in macroeconomics of imperfect credit markets have emphasized the role of borrowing constraints in generating credit multipliers. Among the most influential are Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). Both studies, as well as many others, have shown how the borrowing constraint could magnify external shocks and create persistence in propagation of business cycles. In these models, a high (low) net worth of the agent stimulates (discourages) business investment, which contributes to a high (low) net worth of the agents. In the absence of exogenous shocks, no persistent fluctuation occurs in these models.[6] The present model, on the other hand, emphasizes a reversal mechanism, where a higher net worth of the agent leads to a decline in business investment, so that the economy perpetually alternates between periods of expansion and of contraction, even in the absence of any external disturbances. In other words, the credit multiplier mechanism explains why an economic expansion can be a slow, prolonged process, while the mechanism here explains why such an expansion is reversed. The key difference is that, in a credit multiplier model, an improved net worth of the agents redirects the savings away from the investment without aggregate demand spillovers towards the investment with aggregate demand spillovers, while in a credit reversal model, it redirects the savings in the opposite direction. Needless to say, these effects are not mutually exclusive, and can be usefully combined. Indeed, in section 5, the model is extended to allow for three types of investment, the Good (business investment), the Bad (trading), and the Ugly (storage), where the Ugly, like the Bad, does not generate any demand spillovers, but is not subject to any borrowing constraint. In this extended model, the credit multiplier mechanism is operative in recessions and the credit reversal mechanism is operative in booms. It will be shown

[5] Indeed, fluctuations are not at all driven by the expectations of the agents, whether they are rational or not. Contrary to most existing studies on endogenous business cycles, which rely on indeterminacy and self-fulfilling expectations, the equilibrium path is unique in this paper. Endogenous fluctuations occur because the unique steady state loses its stability.

[6] In one variation of their models, Kiyotaki and Moore (1997; Section III) demonstrated that the equilibrium dynamics display oscillatory convergence to the steady state, which is why they called their paper, "Credit Cycles." However, these oscillations occur because they added the assumption that the investment opportunity arrives stochastically to each agent. The borrowing constraints in all of their models work only to amplify the movement caused by shocks, instead of reversing it. In any case, in all of their models, the steady state is stable and any fluctuations will dissipate in the absence of exogenous shocks.

that, by combining the two effects, the extended model generates business fluctuations that are asymmetric; the economy experiences a long, slow process of recovery from a recession, followed by a rapid expansion, and, possibly after a period of high volatility, plunges to a recession.

The present paper is not the first to demonstrate endogenous credit cycles.[7] The mechanism explored here, however, differs significantly from those explored in the existing studies. In Azariadis and Smith (1998), the credit market cannot tell the investors from the savers. At a low level of the capital stock, the return to the investment is high, and the savers prefer lending; they have no incentive to misrepresent themselves. Thus, the equilibrium loan contract does not impose any limit on the amount that the investors can borrow. At a high level of the capital stock, the return to the investment is low. Instead of lending, the savers would be tempted to borrow, by misrepresenting themselves as the investors, and abscond with their loans. To prevent such frauds in the presence of asymmetric information, the equilibrium loan contract imposes the credit limit to make it unattractive for the savers, which reduces the volume of the credit extended to the true investors. In their framework, expansions stop when the credit constraints on business investment are tightened. In the present framework, expansions stop when the credit constraints on trading activities are loosened, which squeeze out business investment. Aghion, Banerjee and Piketty (1999) is also based on the separation of the savers and investors. In their model, the investment is always constrained by the wealth of the investors. When the investors own a small fraction of the aggregate wealth, the investment falls short of the saving, and the equilibrium interest rate is low, which helps to redistributes the wealth from the savers (i.e., the lenders) to the investors (i.e., the borrowers). When the investors own a large fraction of the wealth, the investment exceeds the saving, which pushes up the equilibrium interest rate, and hence redistributes the wealth from the investors to the savers. In the models developed below, all the agents have the same level of wealth, so that wealth distribution plays no role in generating cycles. Furthermore, in both Azariadis-Smith and Aghion-Banerjee-Piketty, the

[7]For a survey on endogenous business cycles in general, see Boldrin and Woodford (1990) and Guesnerie and Woodford (1992).

composition of the investment does not change over the cycles. Here, changing composition of the investment is the essential part of the story.[8]

The rest of the paper is organized as follows. Section 2 presents the model and derives the dynamical system that governs the equilibrium trajectory for the case where business investment never faces the borrowing constraint. Section 3 characterizes the equilibrium for a full set of parameter values, which enables us to identify the condition under which the steady state loses its stability and endogenous fluctuations occur. Some examples of chaotic behaviors are also presented at the end of this section. Section 4 shows that the results are robust, when the borrowing constraint on business investment is reintroduced. Section 5 develops a hybrid model, which incorporates the credit multiplier effect as well as the credit reversal effect. Section 6 concludes.

2. The Model.

Time is discrete and extends from zero to infinity (t = 0, 1, 2, ...). The economy is populated by overlapping generations of two-period lived agents. Each generation consists of a continuum of agents with unit mass. There is one final good, which is taken as the numeraire and can be either consumed or invested. Each agent supplies one unit of labor to the business sector in the first period, and consumes only in the second. Thus, the aggregate labor supply is $L_t = 1$, and the wage income, w_t, is also the net worth of the young at the end of period t. The young in period t need to allocate their net worth to finance their consumption in period t+1. The following options are available to them.

First, all the young agents can lend a part or all of the net worth in the competitive credit market, which earns the gross return equal to r_{t+1} per unit. If they lend the entire net worth, their second-period consumption is equal to $r_{t+1}w_t$. Second, some young agents have access to an investment project and may use a part or all of the net worth to finance it. There are two types of projects, both of which come in discrete units. Each young agent has access to at most one type of the project, and each young agent can manage at most one project. More specifically,

[8] In this respect, the models developed below are similar to the growth cycle model of Matsuyama (1999), in which the two types of investment, factor accumulation and innovation, alternate as the main engine of growth over the cycles.

The Good Investment:

A fraction μ_1 of the young are entrepreneurs, who know how to start a firm in the business sector. Setting up a firm requires one unit of the final good invested in period t. This enables these agents to produce $\phi(n_{t+1})$ units of the final good in period t+1 by employing n_{t+1} units of labor at the competitive wage rate, w_{t+1}. The production function satisfies $\phi(n) > 0$, $\phi'(n) > 0$ and $\phi''(n) < 0$ for all $n > 0$. Let us denote the gross profit from running a firm in period t+1 by π_{t+1}. Since $w_{t+1} = \phi'(n_{t+1})$ in equilibrium, the equilibrium gross profit can be expressed as an increasing function of the equilibrium employment, $\pi_{t+1} = \pi(n_{t+1}) \equiv \phi(n_{t+1}) - \phi'(n_{t+1})n_{t+1}$ with $\pi'(n_{t+1}) = -\phi''(n_{t+1})n_{t+1} > 0$.

If $w_t < 1$, these agents need to borrow by $1 - w_t > 0$ in the competitive credit market to start the project. If $w_t > 1$, they can start the project and lend by $w_t - 1 > 0$. In either case, the second-period consumption is equal to $\pi_{t+1} - r_{t+1}(1-w_t)$ if they start the project, which is greater than $r_{t+1}w_t$ (the second-period consumption if they simply lend the entire net worth in the credit market) if and only if

(1) $\pi_{t+1} \geq r_{t+1}$.

The entrepreneurs want to (or are at least willing to) set up firms if and only if the profitability condition, (1), holds.

The Bad Investment:

A fraction $\mu_2 \leq 1 - \mu_1$ of the young are traders. They have access to a project, which requires m units of the final good to be invested in period t and generates Rm units of the final good in period t+1. In other words, unlike the entrepreneurs, the traders simply hoard the final good for one period to earn the gross return equal to R per unit, without generating any employment. It is in this sense that we shall call their project the "bad" investment.

If $w_t < m$, these agents need to borrow by $m - w_t > 0$ to start the project. If $w_t > m$, they can start the project and lend by $w_t - m > 0$. Hence, their second-period consumption is equal to $Rm - r_{t+1}(m-w_t)$ as a trader, which is greater than $r_{t+1}w_t$ if and only if

(2) $R \geq r_{t+1}$.

The traders are willing to start their operation if and only if (2) holds.

Remarks: It is possible to give an alternative interpretation to the bad investment. The project done by these agents does not produce any final good; instead it provides these agents with the "private" benefits of Rm, which merely satisfies their ego. These alternative interpretations of the project make no difference in the formal analysis. Nevertheless, it should be pointed out that the key distinction between the good and bad is not the extent to which they are "productive," but the extent to which they improve the net worth of the other agents. Nor should one interpret the labor intensity of the production as the key distinction between the good and the bad investment. In the present setting, the agents obtain their wealth only through working in the first period, and as the workers, they are homogenous. In reality, of course, there are many sectors, the net worth of the agents can improve through profits and the agents can differ in skill as workers and in quality as entrepreneurs. In such a general setting, business investment in a labor intensive sector may not necessarily generate more aggregate demand spillovers nor improve the net worth of potential entrepreneurs. [9]

The Borrowing Constraints:

The credit market is competitive in the sense that both lenders and borrowers take the equilibrium rate, r_{t+1}, given. It is not competitive, however, in the sense that one may not be able to borrow any amount at the equilibrium rate. The borrowing limit exists because of the enforcement problem: the payment is made only when it is the borrower's interest to do so. More specifically, the entrepreneurs would refuse to honor its payment obligation, $r_{t+1}(1-w_t)$, if it is greater than the cost of default, which is taken to be a fraction λ_1 of the gross profit, i.e., $\lambda_1\pi_{t+1}$. Knowing this, the lender allows the entrepreneurs to borrow only up to $\lambda_1\pi_{t+1}/r_{t+1}$. Thus, they cannot start their businesses unless

(3) $w_t \geq 1 - \lambda_1\pi_{t+1}/r_{t+1}$.

[9] It should also be noted that the assumption that the good and bad investment are done by different agents is not crucial; one could alternatively assume that all the agents are homogenous and have access to the two types of investment. As long as we assume that no agent can invest both types of the projects simultaneously and the creditor can observe the type of the investment made by the borrower, the results would carry over, even though it would make the presentation more complicated.

The entrepreneurs set up their firms, only when both (1) and (3) are satisfied. Note that (3) implies (1) if $w_t \leq 1 - \lambda_1$ and that (1) implies (3) if $w_t \geq 1 - \lambda_1$. In other words, the profitability is a relevant constraint when $w_t > 1 - \lambda_1$, while the repayment problem is a relevant constraint when $w_t < 1 - \lambda_1$.

Likewise, the traders would refuse to honor the payment obligation, $r_{t+1}(m - w_t)$, if it is greater than the cost of default, which is taken to be a fraction λ_2 of the gross profit, i.e., $\lambda_2 Rm$. Knowing this, the lender would allow the traders to borrow only up to $\lambda_2 Rm/r_{t+1}$. Thus, they cannot start their operations unless

(4) $w_t \geq m[1 - \lambda_2 R/r_{t+1}]$.

The traders invest in their operations, only when both (2) and (4) are satisfied. Note that (4) implies (2) if $w_t \leq (1 - \lambda_2)m$ and that (2) implies (4) if $w_t \geq (1 - \lambda_2)m$. Again, the borrowing constraint (4) can be binding only if $w_t \leq (1 - \lambda_2)m$.

One interpretation of the default costs is that, in the event of default, the creditors would seize a fraction λ_i of the revenue and the borrowers would disappear with only $(1 - \lambda_i)$ fraction of the revenue in their pockets. More generally, λ_1 and λ_2 may be viewed as the effectiveness of the repayment enforcement in the two types of credits. If these parameters are equal to one, the borrowing constraints, (3) and (4), are never binding. If these parameters are equal to zero, the agents are never able to borrow. It should also be noted that the two types of investment differ in their repayment problems. We are mainly interested in situations where the repayment enforcement is severe for the credits extended to trading activities, but not so for those extended to business investment. That is, λ_1 is equal or sufficiently close to one, while λ_2 may be significantly smaller than one. The idea is that the default cost is high for business investment, because firms need to hire workers and to be operated in the formal sector, which leave enough of a paper trail of their activities, making it easy for the creditors to seize their revenue. On the other hand, the default cost is small in the case of trading, because it requires nothing but hoarding and stockpiling goods in a hidden place. For much of the following analysis, we will set $\lambda_1 = 1$ and drop the subscript from λ_2 and let $\lambda_2 = \lambda < 1$. This greatly minimizes the notational and algebraic burdens, without changing the results fundamentally. (It will be shown later, in section

4, that, for any fixed $\lambda_2 < 1$, the results are robust to a small reduction in λ_1 from $\lambda_1 = 1$. Allowing $\lambda_1 < 1$ would be crucial for the extension in section 5.)

Equilibrium Wage and Business Profit:

Let $k_{t+1} \leq \mu_1$ be the number of young entrepreneurs in period t that start their firms (hence equal to the aggregate business investment at the end of period t, as well as to the number of active firms in period t+1). Let $x_{t+1} \leq \mu_2$ be the number of young traders in period t that start their operations. (The aggregate investment they make is thus equal to mx_{t+1}.) Since only the firms hire workers, the labor market equilibrium in period t+1 is $n_{t+1}k_{t+1} = 1$, from which $n_{t+1} = 1/k_{t+1}$. Thus, the equilibrium wage rate and the business profit per firm in period t+1 may be expressed as functions of k_{t+1}:

(5) $w_{t+1} = \phi'(1/k_{t+1}) \equiv W(k_{t+1})$

(6) $\pi_{t+1} = \pi(1/k_{t+1}) = \phi(1/k_{t+1}) - \phi'(1/k_{t+1})/k_{t+1} \equiv \Pi(k_{t+1})$,

where $W'(k_{t+1}) > 0$ and $\Pi'(k_{t+1}) < 0$. A higher business investment means a high wage and a lower profit. Note that the investment in the business sector, the good investment, creates jobs and leads to a higher wage, thereby improving the net worth of the next generation of the agents. In contrast, trading, the bad investment, contributes nothing to the net worth of the next generation. Nevertheless, what is essential is that the former generates *more* aggregate demand spillovers than the latter.

It is straightforward to show that these functions satisfy

$\phi(1/k)k = k\Pi(k) + W(k)$,

$k\Pi'(k) + W'(k) = 0$,

as the identities. In addition, we make the following assumptions.

(A1) There exists $K > 0$, such that $W(K) = K$ and $W(k) > k$ for all $k \in (0, K)$.

(A2) $K < \mu_1$.

(A3) $\max_{k \in [0, K]} \{W(k) - k\} < m\mu_2$.

(A4) $\lim_{k \to +0} \Pi(k) = +\infty$.

For example, let $\phi(n) = (Kn)^{\beta}/\beta$, with $K < \mu_1$ and $0 < \beta < 1$. Then, (A1), (A2) and (A4) are all satisfied. (A3) is also satisfied if $K < (m\mu_2)/\beta(1-\beta)^{(1-\beta)/\beta}$. (A1) is introduced only to rule out an uninteresting case, where the dynamics of k_t would converge to zero in the long run. It will be shown later that, if $k_t \in (0, K]$, $k_s \in (0, K]$ for all $s > t$, so that K may be interpreted as the upper bound for the number of firms that the economy could ever sustain. Thus, (A2) means that the economy never runs out of the potential supply of the entrepreneurs. In other words, (A2) ensures that it is not the scarcity of the entrepreneurial talents, but the scarcity of the saving and of the credit that will drive the dynamics of business formation in this economy. (A3) may be interpreted similarly. It ensures that the aggregate investment in trading is potentially large enough, so that there are always some inactive traders in the steady state.[10] It turns out that dropping (A3) would not affect the results fundamentally, but would drastically increase the number of the cases that need to be examined. (A4) ensures that some entrepreneurs invest in equilibrium, $k_{t+1} > 0$.

The Investment Schedules:

Because we have set $\lambda_1 = 1$, the borrowing constraint for the entrepreneurs, (3), is never binding, whenever (1) holds, and (1) always holds because of (A4). If (1) holds with the strict inequality, all the entrepreneurs start firms. If (1) holds with the equality, they are indifferent. Therefore, the investment schedule by the entrepreneurs is given simply by the following complementarity slackness condition,

(7) $\quad 0 < k_{t+1} \leq \mu_1, \qquad \Pi(k_{t+1}) \geq r_{t+1},$

which is illustrated in Figures 1a through 1c. As shown below, (A1) and (A2) ensure that $k_{t+1} < \mu_1$ and $\Pi(k_{t+1}) = r_{t+1}$ in equilibrium. The investment demand schedule by the entrepreneurs is thus downward-sloping in the relevant range. At a higher interest rate, the gross profit must rise in

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[10]Note that (A2) and (A3) help to remove the unwanted implication of the assumption that each agent can manage at most one project. This assumption, which reduces the agent's investment choice to a zero-one decision, is made for the analytical simplicity. Both (A2) and (A3) would not be needed if the agents were allowed to invest at any scale, subject only the minimum investment requirement. It turns out, however, that such an alternative specification would make the model algebraic cumbersome. It should be also noted that these assumptions can be weakened significantly. (A2) can be replaced by $W(\min\{K, k_c\}) < \mu_1$ and (A3) by $W(k_{cc}) - k_{cc} < m\mu_2$, where k_c and k_{cc} are the values defined later. (A2) and (A3) are chosen simply because k_c and k_{cc} depends also on R and λ_2, hence the meanings of these alternative assumptions may not be immediately apparent to the reader.

order to keep business investment profitable, which means that the number of active firms must decline.

We now turn to the investment schedule by the traders. First, let us define $R(w_t) \equiv R/\max\{(1 - w_t/m)/\lambda, 1\}$, so that

$$R(W(k_t)) = \begin{cases} \lambda R/[1 - W(k_t)/m] & \text{if } k_t < k_\lambda, \\ R & \text{if } k_t \geq k_\lambda, \end{cases}$$

where k_λ is defined implicitly by $W(k_\lambda) \equiv (1 - \lambda)m$. Figure 2 illustrates the function, $R(W(k_t))$. If $r_{t+1} < R(W(k_t))$, both (2) and (4) are satisfied with the strict inequality, so that all the traders start the trading operation. If $r_{t+1} > R(W(k_t))$, at least one of the conditions is violated, so that no one starts the trading operation. A fraction of the traders starts their operation, if and only if $r_{t+1} = R(W(k_t))$. In words, $R(W(k_t))$ is the interest rate that the traders can afford and credibly commit to repay. Note that $R(W(k_t))$ is constant and equal to R for $k_t \geq k_\lambda$, when the profitability constraint, (2), is more stringent than the borrowing constraint, (4). On the other hand, it is increasing in k_t for $k_t < k_\lambda$, when the borrowing constraint, (4), is more stringent than the profitability constraint, (2). This is because, with a higher net worth, the traders need to borrow less, which means that they can credibly commit to repay at a higher interest rate. In other words, a higher net worth eases the borrowing constraint, so that the interest rate can go up without inducing the traders to default.

The investment by the traders may thus be expressed as

$$(8) \qquad mx_{t+1} \begin{cases} = m\mu_2 & \text{if } r_{t+1} < R(W(k_t)), \\ \in [0, m\mu_2] & \text{if } r_{t+1} = R(W(k_t)), \\ = 0 & \text{if } r_{t+1} > R(W(k_t)). \end{cases}$$

In each of Figures 1a through 1c, eq. (8) is illustrated as a step function, which graphs $W(k_t) - mx_{t+1}$.

The Credit Market Equilibrium:

The credit market equilibrium requires that the interest rate adjust so as to equate the aggregate investment and the aggregate saving, i.e., $k_{t+1} + mx_{t+1} = w_t$, or equivalently

(9) $k_{t+1} = W(k_t) - mx_{t+1}$.

Figures 1a through 1c illustrate three alternative cases, depending on the value of k_t.[11]

Figure 1a depicts the case for $k_t < k_c$, where k_c is defined uniquely by

$R(W(k_c)) \equiv \Pi(W(k_c))$.

This is the case where $W(k_t)$ is sufficiently low that $R(W(k_t)) < \Pi(W(k_t))$. Thus, the net worth of the traders is so low that they cannot finance their investment ($x_{t+1} = 0$) and all the savings are channeled into the investment in the business sector ($k_{t+1} = W(k_t) < \mu_1$). The equilibrium interest rate is too high for the traders ($r_{t+1} = \Pi(W(k_t)) > R(W(k_t))$).

Figure 1b depicts the case for $k_c < k_t < k_{cc}$, where k_{cc} ($> k_c$) is defined uniquely by

$R(W(k_{cc})) \equiv \Pi(W(k_{cc}) - m\mu_2)$

and satisfies $k_{cc} > k_c$. Thus, $\Pi(W(k_t)) < R(W(k_t)) < \Pi(W(k_t) - m\mu_2)$ in this case. The equilibrium interest rate is equal to $r_{t+1} = R(W(k_t)) = \Pi(k_{t+1}) = \Pi(W(k_t) - mx_{t+1})$ and $0 < x_{t+1} < \mu_2$. In this range, some, but not all, traders invest. An increase in k_t thus has the effect of further increasing the investment in trading. Its effect on business investment depends whether k_t is higher or less than k_λ. If $k_t > k_\lambda$, the borrowing constraint of the traders is not binding, so that the interest rate is fixed at $R(W(k_t)) = R$. Thus, the investment in the business sector remains constant at $\Pi^{-1}(R)$. On the other hand, if $k_t < k_\lambda$, the borrowing constraint for the traders is binding. Thus, $R(W(k_t))$ increases with k_t, because a higher net worth eases the borrowing constraint of the traders, so that they can borrow at a higher interest rate. As a result, the investment in the business sector is squeezed out.

Finally, Figure 1c depicts the case for $k_t > k_{cc}$. This is the case where $W(k_t)$ is sufficiently high that $R(W(k_t)) > \Pi(W(k_t) - m\mu_2) = r_{t+1}$, so that $x_{t+1} = \mu_2$ and $k_{t+1} = W(k_t) - m\mu_2$. Thus, the net worth is so high that all the traders invest. An increase in the saving translates to an increase in business investment. Hence, k_{t+1} increases with k_t in this range. This situation occurs as an unwanted by-product of the assumption that the traders can manage at most one trading operation, which was made to simplify the analysis of the trader's decision problem. Note,

[11] Figures 1a-1c are drawn under the assumption, $W(k_t) < \mu_1$, which ensures $k_{t+1} < \mu_1$ in equilibrium. This assumption will be verified later. These figures are also drawn such that $W(k_t) > m\mu_2$. In the cases of Figures 1a and 1b, this need not be the case, but it does not affect for the discussion in the text.

however, that we have imposed (A3) to ensure that $k_{t+1} = W(k_t) - m\mu_2 < k_t$ in this range, and hence that this situation would never occur in the steady state.

A Digression on Credit Rationing:

A remark should be made of the case, where $k_c < k_t < k_{cc}$ and $r_{t+1} = \Pi(k_{t+1}) = R(W(k_t))$. In this case, which is shown in Figure 1b, only a fraction of the traders starts their operation. When $k_t \geq k_\lambda$, $r_{t+1} = R$ holds in equilibrium, and (2) is thus satisfied with equality. Some traders invest while others do not, simply because they are indifferent. When $k_t < k_\lambda$, $r_{t+1} = \lambda R/[1 - W(k_t)/m] < R$, hence (4) is binding, while (2) is satisfied with strict inequality. In other words, all the traders strictly prefer borrowing to invest, rather than lending their net worth to others. Therefore, the equilibrium allocation necessarily involves credit rationing, where a fraction of the traders are denied the credit. Those who denied the credit cannot entice the potential lenders by raising the interest, because the lenders would know that the borrowers would default at a higher rate. It should be noted, however, that equilibrium credit rationing occurs in this model due to the homogeneity of the traders. The homogeneity means that, whenever some traders face the borrowing constraint, all the traders face the borrowing constraint, so that coin tosses or some random devices must be evoked to determine the allocation of the credit.[12] Suppose instead that the traders were heterogeneous in their observable characteristics. For example, suppose each young trader receives, in addition to the labor income, an endowment income, y, which is drawn from G, a cumulative distribution function with no mass point. Then, there would be a critical level of y, $Y(w_t, r_{t+1}) \equiv m(1 - \lambda R/r_{t+1}) - w_t$, such that only the traders whose endowment income exceed $Y(w_t, r_{t+1})$ would be able to finance their investment. This makes the aggregate investment in trading, $mx_{t+1} = m[1 - G(Y(w_t, r_{t+1}))]$, smoothly decreasing in r_{t+1}, and increasing in w_t. Thus, the borrowing constraint would be enough to determine the allocation of the credit, and credit

[12] While some authors use the term, "credit-rationing," whenever some credit limits exist, here it is used to describe the situation that the aggregate supply of credit falls short of the aggregate demand, so that some borrowers cannot borrow up to their credit limit. In other words, there is no credit rationing if every borrower can borrow up to its limit. In such a situation, their borrowing may be constraint by their net worth, which affects the credit limit, but not because they are credit-rationed. This is consistent with the following definition of credit rationing by Freixas and Rochet (1997, Ch.5), who attributed it to Baltensperger: "some borrower's demand for credit is turned down, even if this borrower is willing to pay all the price and nonprice elements of the loan contract."

rationing would not occur. What is essential for the following analysis is that, when the borrowing constraint is binding for marginal traders, an increase in the net worth of the traders increase the aggregate investment in trading, for a given interest rate. Therefore, it is the borrowing constraint, not the equilibrium credit rationing per se, that matters. The equilibrium credit rationing is nothing but an artifact of the homogeneity assumption, which is imposed to simplify the analysis.

The Equilibrium Trajectory:

As should be clear from Figures 1a-1c, $k_{t+1} = W(k_t)$ if and only if $k_t \leq k_c$; $\Pi(k_{t+1}) = R(W(k_t))$ if and only if $k_c \leq k_t \leq k_{cc}$; and $k_{t+1} = W(k_t) - m\mu_2$ if and only if $k_t \geq k_{cc}$. These observations can be summarized as follows:

$$(10) \qquad k_{t+1} = \Psi(k_t) \equiv \begin{cases} W(k_t) & \text{if } k_t \leq k_c, \\ \Pi^{-1}(\lambda R/[1 - W(k_t)/m]), & \text{if } k_c < k_t \leq \min\{k_\lambda, k_{cc}\}, \\ \Pi^{-1}(R), & \text{if } k_\lambda < k_t \leq k_{cc}, \\ W(k_t) - m\mu_2 & \text{if } k_t \geq k_{cc}. \end{cases}$$

Equation (10) determines k_{t+1} uniquely as a function of k_t. From (9) and the monotonicity of W, $k_t \in (0, K]$ implies that $k_{t+1} = \Psi(k_t) \in (0, W(k_t)] \subset (0, W(K)] = (0, K]$. Therefore, Ψ maps (0,K] into itself, and hence, for any $k_0 \in (0, K]$, this map defines a unique trajectory in (0, K]. Furthermore, $k_t \leq K$ and (A2) mean that $\mu_1 > K = W(K) \geq W(k_t)$, as has been assumed.

The equilibrium trajectory of the economy can thus be solved for by applying the map (10), Ψ, iteratively, starting with the initial condition, $k_0 \in (0, K]$. This completes the description of the model. We now turn to the characterization of the equilibrium dynamics.

3. The Dynamic Analysis.

It turns out that there are five generic cases of the equilibrium dynamics, as illustrated by Figure 3a through Figure 3e.[13] Figure 3a depicts the case, where the traders never become active, and $k_{t+1} = W(k_t)$ for all $k_t \in (0, K]$. Hence, k_t converges monotonically to $k^* = K$ for any

[13] Figure 3a through Figure 3e are drawn such that W(0) = 0 and W is concave. These need not be the case. (A1) assumes only that $W(k) > k$ for all $k \in (0, K]$ and $W(K) = K$.

$k_0 \in (0,K]$, from the monotonicity of W and (A1). The condition for this is $k_c \geq K$, or equivalently, $\Pi(K) \geq R(W(K)) = R(K)$, which can be further rewritten to

(11) $R \leq \Pi(K) \max\{(1 - K/m)/\lambda, 1\}$.

With a sufficiently small R (i.e., if trading is not very profitable), trading never competes with business investment for the credit. When $W(K) = K < m$, the condition (11) is also met when λ is sufficiently small for any R. This is because the traders must borrow to start their operations even when the net worth reaches its highest possible value. If λ is sufficiently small, they can never borrow, and hence they can never invest, and hence all the saving goes to business investment.

In the other four cases, (11) is violated, so that $k_c < K$. Thus, some traders are active (i.e., $x_{t+1} > 0$) when $k_t \in (k_c, K]$. Figure 3b depicts the case, where $k_\lambda \leq k_c$ or equivalently, $W(k_c) \geq (1 - \lambda)m$, which can be rewritten as to

(12) $R \leq \Pi((1 - \lambda)m)$.

Under this condition, $W(k_t) > (1 - \lambda)m$ and $R(W(k_t)) = R$ for all $k_t > k_c$. This means that the borrowing constraint is not binding for the traders, whenever they are active. Thus, from eq. (10),

$$(13) \quad k_{t+1} = \Psi(k_t) = \begin{cases} W(k_t), & \text{if } k_t \leq k_c \\ \Pi^{-1}(R) = W(k_c), & \text{if } k_c < k_t \leq \min\{k_{cc}, K\} \\ W(k_t) - m\mu_2. & \text{if } k_{cc} < k_t \leq K. \end{cases}$$

As shown in Figure 3b, the map has a flat segment, over $(k_c, \min\{k_{cc}, K\})$, but it is strictly increasing elsewhere. Furthermore, (A3) ensures $k_{cc} > W(k_{cc}) - m\mu_2$, so that the steady state is located at the flat segment.[14] The dynamics of k_t hence converges monotonically to the unique steady state, $k^* = \Pi^{-1}(R) = W(k_c)$. As the business sector expands, the return for business investment declines and the net worth improves, and eventually the traders start investing. However, because the traders do not face the binding borrowing constraint, the equilibrium interest rate is always equal to R. Thus, business investment remains constant at $\Pi^{-1}(R)$ and an increase in trading activities does not reduce business investment.

In the three cases depicted by Figure 3c through 3e, $W(k_c) < (1 - \lambda)m$, or $k_c < k_\lambda$, so that there is an interval, $k_t \in (k_c, k_\lambda)$, in which the borrowing constraint is more stringent than the profitability constraint for the traders. For $k_c < k_t < \min\{k_\lambda, k_{cc}, K\}$, the borrowing constraint is binding, and

(14) $k_{t+1} = \Psi(k_t) = \Pi^{-1}(\lambda R/[1 - W(k_t)/m])$,

which is decreasing in k_t. Thus, with the failure of both (11) and (12), the map has a downward sloping segment. For $k_\lambda < k_t < \min\{k_{cc}, K\}$, the borrowing constraint is not binding, and hence $k_{t+1} = \Psi(k_t) = \Pi^{-1}(R)$, so that $\Psi(k_t)$ is constant.

The reason why an increase in k_t leads to a lower k_{t+1} when the traders face the binding borrowing constraint should be clear. A higher k_t, by improving the net worth of the traders, eases their borrowing constraint. This drives up the equilibrium interest rate. To keep the investment in the business sector profitable, the number of active firms in the business sector must decline. Thus, more saving is channeled into the investment in trading at the expense of the investment in the business sector.

Figure 3c depicts the case where the borrowing constraint for trading is not binding in the steady state. That is, the map intersects with the 45° line at a flat segment, i.e., over the interval, $(k_\lambda, \min\{k_{cc}, K\})$. The condition for this is $k_\lambda \leq k^* = \Pi^{-1}(R) < k_{cc}$. Since (A3) ensures $k^* < k_{cc}$, this occurs whenever $k_\lambda \leq \Pi^{-1}(R)$, or equivalently, $W(\Pi^{-1}(R)) \geq (1 - \lambda)m$, which can be further rewritten to

(15) $R \leq \Pi(W^{-1}((1 - \lambda)m))$.

When (15) holds but (11) and (12) are violated, the dynamics of k_t converges to $k^* = \Pi^{-1}(R) < W(k_c)$, as illustrated in Figure 3c. The dynamics is not, however, globally monotone. Starting from $k_0 < k_\lambda$, the dynamics of k_t generally overshoots k^* and approaches k^* from above.[15]

For the cases depicted by Figures 3d and 3e, (11) and (15) are both violated, which also implies the violation of (12).[16] Thus, the map intersects with the 45° line at the downward sloping part, $(k_c, \min\{k_\lambda, k_{cc}, K\})$. Therefore, the traders face the binding borrowing constraint in a neighborhood of the steady state. By setting $k_t = k_{t+1} = k^*$ in (14), the steady state is given by

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[14]In both Figures 3b and 3c, $k_{cc} > K$. This need not be the case, nor is it essential for the discussion in the text.

[15] The qualified "generally" is needed, because the equilibrium trajectory is monotone, if $k_0 \in \{W^{-T}(k^*) \mid T = 0, 1, 2, ...\}$, which is at most countable and hence of measure zero.

[16] Figures 3d and 3e are drawn such that $k_\lambda < K$. This need not be the case, nor is it essential for the discussion in the text.

(16) $\Pi(k^*)[1 - W(k^*)/m] = \lambda R$.

Figure 3d and Figure 3e differ in the stability of the steady state, which depends on the slope of the map at k*. By Differentiating (14) and then setting $k_t = k_{t+1} = k^*$ yield,

$$\Psi'(k^*) = W'(k^*)\Pi(k^*)/\Pi'(k^*)[m - W(k^*)] = -k^*\Pi(k^*)/[m - W(k^*)],$$

where use has been made of (16) and $W'(k^*)+k^*\Pi'(k^*) = 0$. From $k^*\Pi(k^*) + W(k^*) = k^*\phi(1/k^*)$, $|\Psi'(k^*)| < 1$ if and only if

(17) $k^*\phi(1/k^*) < m$.

Note that the LHS of (17) is increasing in k*, while the LHS of (16) is decreasing in k*. Hence, (17) can be rewritten to

(18) $\lambda R > \Pi(h(m))[1 - W(h(m))/m]$,

where h(m) is defined implicitly by $h\phi(1/h) \equiv m$. This case is illustrated in Figure 3d. When (18) holds, the steady state, k*, is asymptotically stable; the convergence is locally oscillatory.

On the other hand, if

(19) $\lambda R < \Pi(h(m))[1 - W(h(m))/m]$,

then $|\Psi'(k^*)| > 1$ and hence the steady state, k*, is unstable, as illustrated in Figure 3e. For any initial condition, the equilibrium trajectory will eventually be trapped in the interval, $I \equiv [\max\{\Psi(W(k_c)), \Psi(\min\{k_\lambda, k_{cc}\})\}, W(k_c)]$, as illustrated by the box in Figure 3e.[17] Furthermore, if $k_\lambda \geq \min\{k_{cc}, K\}$, k_t fluctuates indefinitely except for a countable set of initial conditions. If $k_\lambda < \min\{k_{cc}, K\}$, k_t fluctuates indefinitely except for a countable set of initial conditions for a generic subset of the parameter values satisfying (19) and violating (11) and (15).[18] In other words, the equilibrium dynamics exhibit permanent endogenous fluctuations almost surely.

To summarize,

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[17] In Figure 3e, $k_\lambda < W(k_c) < K < k_{cc}$. Hence, $I = [\Psi(k_\lambda), W(k_c)] = [\Pi^{-1}(R), W(k_c)]$.

[18] To see this, let $C \subset (0, K]$ be the set of initial conditions for which k_t converges. Let $k_\infty = \lim_{t\to\infty} \Psi^t(k_0)$ be the limit point for $k_0 \in C$. From the continuity of Ψ, $\Psi(k_\infty) = \lim_{t\to\infty} \Psi(k_t) = \lim_{t\to\infty} k_{t+1} = k_\infty$. Hence, $k_\infty = k^*$. Since k* is unstable, k_t cannot approach it asymptotically. It must be mapped to k* in a finite iteration. That is, there exits T such that $\Psi^T(k_0) = k^*$, or $C = \{\Psi^{-T}(k_0) | T = 0,1,2,...\}$. If $k_\lambda \geq \min\{k_{cc}, K\}$, the map has no flat segment and hence the preimage of Ψ is finite and hence C is at most countable. If $k_\lambda < \min\{k_{cc}, K\}$, the map has a flat segment, at which it is equal to $\Pi^{-1}(R)$. Thus, C is at most countable unless $\Pi^{-1}(R) \in \{\Psi^{-T}(k_0) | T = 0,1,2,...\}$, which occurs only for a nongeneric set of parameter values. (As clear from this proof, it is easy to show that, even when $k_\lambda < \min\{k_{cc}, K\}$, if $W(k_c) < \min\{k_\lambda, k_{cc}\}$, the flat segment does not belong to I. Hence, if we restrict the initial condition in I, k_t fluctuates indefinitely for almost initial conditions in I for all the parameter values satisfying (19) and violate (11) and (15).)

<u>Proposition 1.</u> Let $\lambda_1 = 1$ and $\lambda_2 = \lambda \in (0,1)$. Then,

A. Let $R \leq \Pi(K)\max\{(1 - K/m)/\lambda, 1\}$. Then, $x_{t+1} = 0$ for all $t \geq 0$ and the dynamics of k_t converges monotonically to the unique steady state, K.

B. Let $\Pi(K) < R \leq \Pi((1 - \lambda)m)$. Then, the dynamics of k_t converges monotonically to the unique steady state, $k^* = \Pi^{-1}(R) = W(k_c)$. Some traders eventually become active and never face the binding borrowing constraint.

C. Let $\Pi((1 - \lambda)m) < R \leq \Pi(W^{-1}((1 - \lambda)m)))$. Then, the dynamics of k_t converges to the unique steady state, $k^* = \Pi^{-1}(R) < W(k_c)$. Some traders are active and do not face the binding borrowing constraint in the steady state.

D. Let $R > \Pi(W^{-1}((1 - \lambda)m)))$, $\Pi(h(m))[1 - W(h(m))/m]/\lambda$. Then, the dynamics of k has the unique steady state, $k^* \in (k_c, \min\{k_\lambda, k_{cc}, K\})$, satisfying $\Pi(k^*)[1 - W(k^*)/m] = \lambda R$. The traders face the binding borrowing constraint in the steady state. The steady state is asymptotically stable. The convergence is locally oscillatory.

E. Let $\Pi(K)(1 - K/m)/\lambda$, $\Pi(W^{-1}((1 - \lambda)m))) < R < \Pi(h(m))[1 - W(h(m))/m]/\lambda$. Then, the dynamics of k has the unique steady state, $k^* \in (k_c, \min\{k_\lambda, k_{cc}, K\})$, satisfying $\Pi(k^*)[1 - W(k^*)/m] = \lambda R$. The traders face the binding borrowing constraint in the steady state. The steady state is unstable. Every equilibrium trajectory will be eventually trapped in the interval, $I \equiv [\max\{\Psi(W(k_c)), \Psi(\min\{k_\lambda, k_{cc}\})\}, W(k_c)]$. Furthermore, the equilibrium dynamics exhibits permanent, endogenous fluctuations almost surely.

Figures 4a through 4c illustrate Proposition 1 in terms of the three parameters that characterize the trading operation, m, λ, and R. In Figure 4a, $m < K$; In Figure 4b, $K < m < K\phi(1/K)$; and in Figure 4c for $m > K\phi(1/K)$.[19] In these figures, the parameter space, (λ, R), is divided into four or five regions, where Region A satisfies the conditions given in Proposition 1A,

[19] Note $K\phi(1/K) = K\Pi(K) + W(K) > W(K) = K$ for any K, which verifies the existence of the case depicted in Figure 4b. To see that none of these parameter configurations contradicts with the assumptions made so far, recall the example, $\phi(n) = (Kn)^\beta/\beta$, with $0 < \beta < 1$, $K < \mu_1$, and $K < (m\mu_2)/\beta(1-\beta)^{(1-\beta)/\beta}$, which satisfy (A1) through (A4). With this functional form, $K\phi(1/K) = K/\beta$. Thus, if $1-\mu_1 \geq \mu_2 > \beta(1-\beta)^{(1-\beta)/\beta}$, Figure 4a applies for $m < K < \min\{\mu_1, (m\mu_2)/\beta(1-\beta)^{(1-\beta)/\beta}\}$, Figure 4b for $\beta m < K < \min\{\mu_1, m\}$, and Figure 4c for $K < \min\{\mu_1, \beta m\}$.

Region B satisfies those given in Proposition 1B, etc. The borders between B and C and those between C and D in Figures 4a through 4c are asymptotic to $\lambda = 1$. The borders between D and E in Figures 4a and 4b, the border between A and E in Figure 4b and the border between A and D in Figure 4c are hyperbolae and asymptotic to $\lambda = 0$.

Figure 4a depicts the case, $m < K$, hence the minimum investment is sufficiently small that, if all the saving were to go to the business sector, the traders would eventually become rich enough to be able to finance their investment without borrowing from the credit market. In this case, if $R < \Pi(W^{-1}(m))$, they do not face the binding borrowing constraint in the steady state, regardless of λ. If $R > \Pi(W^{-1}(m))$, however, the economy is in either Region D or Region E for a sufficiently small λ (i.e., $0 < \lambda < 1 - W(\Pi^{-1}(R))/m$), and the traders face the binding borrowing constraint in the steady state. Starting in a neighborhood of the steady state, an improvement in the net worth eases their borrowing constraint, which pushes the equilibrium interest rate, thereby causing a decline in business investment, which reduces the net worth of the agents in the next period. Indeed, a sufficiently small λ ensures that the economy is in Region E, where this effect is so strong that the steady state becomes unstable, and generates endogenous fluctuations. (Technically speaking, as the economy crosses $\lambda R = \Pi(h(m))[1 - W(h(m))/m]$ from Region D to Region E, the dynamical system experiences a *flip bifurcation*.)

This does not necessarily mean that fluctuations would be more likely to occur with a greater imperfection of the credit market. Figure 4b depicts the case for an intermediate value of m, the case where the minimum investment is sufficiently high that the agents need to borrow from the credit market to invest in trading ($m > K$). Therefore, trading never occurs for a sufficiently small λ. For any $R > \Pi(K)$, Region D and Region E exist for an intermediate range of λ. That is, the credit market imperfection must be sufficiently high, but cannot be too severe for endogenous fluctuations to occur. In Figure 4c, when the minimum investment is very high, Region E no longer exists. The following corollary of Proposition 1 summarizes the condition for endogenous fluctuations.

<u>Corollary 1.</u>

Suppose $0 < m < K$. For any $R > \Pi(W^{-1}(m))$, endogenous fluctuations occur (almost surely) for a sufficiently small $\lambda > 0$. Suppose $K < m < K\phi(1/K)$. For any $R > \Pi(K)$, endogenous fluctuations occur (almost surely) for an intermediate value of λ.

Region D is also of some interest, because the local convergence toward the steady state is oscillatory, and the transitional dynamics is cyclical. If the economy is hit by recurrent shocks, the equilibrium dynamics exhibit considerable fluctuations.[20] A quick look at Figure 4a through 4c, verifies that a sufficiently high R ensures that the economy is in Region D. Thus,

<u>Corollary 2</u>.

For any $\lambda \in (0,1)$, the dynamics around the steady state is oscillatory for a sufficiently high R.

The intuition behind this result is easy to grasp. Whenever the repayment enforcement is imperfect, the trader would face the binding borrowing constraint around the steady state, if the traders are sufficiently eager to invest, i.e., if the trading operation is sufficiently productive.

Propositions 1D and 1E give the conditions under which the model generates oscillatory convergence and endogenous fluctuations, respectively. Without a specific functional form, however, little more can be said about the nature of global dynamics. Therefore, let us turn to some examples.

Example 1:

Let $\phi(n)= 2(Kn)^{1/2}$, with $K < \mu_1$, $4m\mu_2$, which satisfies (A1) through (A4). If $R > K/(1-\lambda)m$, and $R > (1- K/m)/\lambda$, the economy is either in Region D (for $\lambda R > K/m$) or in Region E (for $\lambda R < K/m$). Furthermore, in order to avoid a taxonomical exposition, let us focus on the case, where $W(k_c) < \min \{k_\lambda, k_{cc}\}$ so that the map is strictly decreasing in $(k_c, W(k_c)]$.[21] Some algebra can

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[20] In addition, it is possible that there may be endogenous fluctuations in Region D. When the parameters satisfy the conditions given in Proposition 1D, we do know that the local dynamics converges, but little can be said of the nature of global dynamics. For example, there are (unstable) period-2 cycles in the neighborhood of k* near the boundary on the side of Region D, if the flip bifurcation that occurs at the boundary of D and E is of *subcritical* type: see Guckenheimer and Holmes (1983, Theorem 3.5.1).

[21] For example, $K < (1-\lambda)m(1-\lambda+\lambda R)$ ensures $k_\lambda > W(k_c)$; $K > mR^2(1-\lambda-\mu_2)$ ensures $k_{cc} > k_\lambda$, hence $k_{cc} > W(k_c)$.

show that, by defining $z_t \equiv (k_t/K)^{1/2}$, the equilibrium dynamics over this range can be expressed by the map: ψ: $(0, \psi(z_c)] \to (0, z_c]$, defined by

$$z_{t+1} = \psi(z_t) \equiv \min\{z_t^{1/2}, [1-(K/m)z_t]/(\lambda R)\},$$

where $z_c \equiv (k_c/K)^{1/2} < 1$, which satisfies $\psi(z_c) = z_c^{1/2} = [1-(K/m)z_c]/(\lambda R)$. The map is unimodal: it is strictly increasing in $(0, z_c)$ and strictly decreasing in $I_- \equiv (z_c, \psi(z_c)]$. Furthermore, the slope is constant in I_-. In Region D, where $\lambda R > K/m$, the slope in I_- is less than one in absolute value. Therefore, the economy converges to the steady state, $z^* = 1/(\lambda R + K/m) \in I_-$, for any initial condition. In Region E, the case illustrated in Figure 5a, the slope in I_- is greater than one in absolute value. This means that, if $z_t \neq z^*$, the equilibrium trajectory will escape I_- after a finite iteration. However, it will never leave $I = [\psi^2(z_c), \psi(z_c)]$, because the map is strictly increasing in $I_+ \equiv [\psi^2(z_c), z_c)$. Therefore, the equilibrium trajectory visits both I_+ and I_- infinitely often, for almost all initial conditions in I (i.e., except for a countable set of initial conditions in I, for which the equilibrium trajectory is mapped into z* in a finite iteration). Furthermore, if $\lambda R > 2(1 - K/4m)$, then $z_c < 1/4$, which ensures that the slope of the map is strictly greater than one in absolute value anywhere in $I_+ \cup I_-$.[22] This means that there are period cycles of every period length, all of which are unstable, and the equilibrium trajectory does not converge to any periodic cycle for almost all initial conditions. In short, the map is chaotic.[23]

In the previous example, the functional form is chosen so that the slope of the map is constant when $z_t > z_c$. This guarantees that there exist no periodic cycles that stay entirely above z_c. In the next example, the functional form is chosen so that the slope of the map is constant also when $z_t < z_c$.

Example 2:

[22] For example, $\mu_1 = 0.2$, $\mu_2 = 0.8$, $K = 0.1$, $m = 0.05$, $\lambda = 0.25$, $R = 7.8$ satisfy the last condition, as well as all the other conditions imposed earlier.

[23] See, for example, Devaney (1987, Chapter 1.6 and 1.7). The set of initial conditions for which the trajectory is eventually periodic is a Cantor set, i.e., it is uncountable, but contains no interior or isolated points. Furthermore, this chaotic map is structually stable. (For introductions to the chaotic dynamical system written for economists, see Grandmont 1986 and Baumol and Benhabib 1989).

Let $\phi(n) = 2(Kn)^{1/2}$ if $n \le 1/k_c$; $= 2(z_c)^{-1/2} + \log(k_c n)/z_c$, if $n > 1/k_c$, which satisfies (A1) through (A4) with $K < \mu_1$, $4m\mu_2$. As in Example 1, let $R > K/(1-\lambda)m$, and $R > (1- K/m)/\lambda$, so that the economy is either in Region D (for $\lambda R > K/m$) or in Region E (for $\lambda R < K/m$), and impose the same restrictions on the parameters to ensure $W(k_c) < \min\{k_\lambda,\ k_{cc}\}$. Then, the dynamics is now given by

$$z_{t+1} = \psi(z_t) \equiv \min\{(z_c)^{-1/2} z_t\, , [1 - (K/m)z_t]/(\lambda R)\},$$

on $(0, \psi(z_c)]$, as illustrated in Figure 5b. This map differs from Example 1 in that the slope of the map is constant in $(0, z_c)$, which is greater than one because $z_c < 1$. Therefore, for $\lambda R < K/m$, the slope of the map is greater than one in absolute value anywhere in $I_+ \cup I_-$. Thus, with this functional form, the map is chaotic whenever the parameters satisfy the conditions in Proposition 1E.

4. Reintroducing the Borrowing Constraint in the Business Sector

So far, we have analyzed the equilibrium trajectory under the assumption that $\lambda_1 = 1 > \lambda_2 = \lambda$. We are now going to show that, for any $\lambda_2 = \lambda < 1$, a small reduction in λ_1 from $\lambda_1 = 1$ would not affect the equilibrium trajectory.

When $\lambda_1 < 1$, the entrepreneurs start firms when both (1) and (3) are satisfied. (A4) ensures that some entrepreneurs are active, $k_{t+1} > 0$, hence both (1) and (3) hold in equilibrium. Furthermore, $k_t \le K$ ensures that $k_{t+1} = W(k_t) - mx_{t+1} \le W(k_t) \le W(K) = K < \mu_1$. Therefore, at least (1) or (3) must be binding, hence

(20) $\Pi(k_{t+1})/\max\{[1 - W(k_t)]/\lambda_1, 1\} = r_{t+1}$.

The credit market equilibrium is given by (8), (9) and (20). It is easy to see that, given k_t, these equations jointly determine k_{t+1} uniquely.

Let us find the condition under which the map given in eq. (10) solves the credit market equilibrium, determined by (8), (9), and (20). First, for any $k_t \ge k_c$, eq. (10) solves the credit market equilibrium if and only if the entrepreneurs do not face the binding borrowing constraint, that is, when (20) is $\Pi(k_{t+1}) = r_{t+1,}$ i.e., $W(k_t) \ge 1 - \lambda_1$ for all $k_t \ge k_c$. The condition for this is $\lambda_1 \ge$

$1 - W(k_c)$. Then, in order for (10) to be the equilibrium, it suffices to show that $x_{t+1} = 0$ and $k_{t+1} = W(k_t)$ solve (8), (9) and (20) for $k_t < k_c$. This condition is given by

$$(21) \quad R/\max\{[1 - W(k_t)/m]/\lambda_2, 1\} \leq \begin{cases} \lambda_1\Pi(W(k_t))/[1 - W(k_t)] & \text{if } k_t < k_{\lambda 1} \\ \Pi(W(k_t)) & \text{if } k_{\lambda 1} \leq k_t < k_c, \end{cases}$$

where $k_{\lambda 1}$ is defined implicitly by $W(k_{\lambda 1}) \equiv 1 - \lambda_1$ and satisfies $k_{\lambda 1} < k_c$. Eq. (21) is illustrated by Figure 6a (for $k_c < k_\lambda$) and Figure 6b (for $k_c > k_\lambda$). By definition of k_c, the LHS of (21) is strictly less than $\Pi(W(k_t))$ for all $k_t < k_c$. Since the RHS of (21) converges to $\Pi(W(k_t))$, as λ_1 approaches one, there exists $\lambda_1' < 1$ such that eq. (21) holds for $\lambda_1 \in [\lambda_1', 1]$. Since the LHS of (21) weakly increases with λ_2, the lowest value of λ_1 for which (21) holds, λ_1', is weakly increasing in λ_2. It is also easy to see that (21) is violated for a sufficiently small λ_1, hence, $\lambda_1' > 0$. Furthermore, for any $\lambda_1 > 0$, (21) holds for a sufficiently small $\lambda_2 > 0$. Thus, λ_1' approaches zero with λ_2. One can thus conclude

<u>Proposition 2.</u>

For any $\lambda_2 = \lambda \in (0,1)$, there exists $\Lambda(\lambda_2) \in (0,1)$, such that, for $\lambda_1 \in [\Lambda(\lambda_2), 1]$, the equilibrium dynamics is independent of λ_1.[24] Λ is nondecreasing in λ_2 and satisfies $\Lambda(\lambda_2) \geq 1 - W(k_c)$, and $\lim_{\lambda_2 \to 0} \Lambda(\lambda_2) = 0$.

Proposition 2 thus means that the analysis need not be changed, as long as λ_1 is sufficiently high. In particular, Proposition 1, their corollaries, as well as Examples 1 and 2 are all unaffected.

Even with a weaker condition on λ_1, the possibility of endogenous fluctuations survives. When $\lambda_1 < \Lambda(\lambda_2)$, the map depends on λ_1, but shifts continuously as λ_1 changes. Therefore, as long as the reduction is small enough, k* is unaffected and remains the only steady state of the map. Therefore, as long as $\lambda_2 = \lambda$ satisfies the condition given in Proposition 1E, the map generates endogenous fluctuations, because its unique steady state is unstable.

[24] The function, Λ, also depends on other parameters of the model, m, R, K, as well as the functional form of ϕ.

The above analysis thus shows that the key mechanism in generating endogenous fluctuations is that an improved economic condition eases the borrowing constraints for the bad investment more than those for the good investment, so that the saving is channeled into the former at the expense of the latter. The assumption made earlier that the good investment faces no borrowing constraint itself is not crucial for the results obtained so far.

The analysis in this section, nevertheless, provides a caution when interpreting Proposition 1. In Proposition 1, it is shown that a sufficiently high $\lambda_2 = \lambda$ eliminates the possibility of endogenous fluctuations. From this, one should not jump to the conclusion that endogenous fluctuations would disappear with an improvement of the credit market. When $\lambda_1 < 1$, an improvement could mean not only a higher λ_2 but also a higher λ_1, so that the unique steady state could very well remain unstable.

5. The Good, The Bad and The Ugly: Introducing Credit Multiplier

This section presents an extension of the above model, which serves two purposes. First, recent studies in macroeconomics, such as Bernanke and Gertler (1989) and Kiyotaki and Moore (1997), have emphasized the role of imperfect credit markets in propagation mechanisms of business cycles. In particular, they stressed a *credit multiplier* effect. An increase in the net worth stimulates business investment by easing the borrowing constraint of the entrepreneurs, which further improves their net worth, leading to more business investment. This introduces *persistence* into the system. The model developed above has no such a credit multiplier effect.[25] Quite the contrary, the mechanism identified may be called a *credit reversal* effect, because an increase in the net worth stimulates trading at the expense of business investment, leading to a deterioration of the net worth. This introduces *instability* into the system. This does not mean, however, that these two mechanisms are mutually exclusive. Combining the two is not only feasible but also useful because it adds some realism to the equilibrium dynamics. In the model shown below, both multiplier and reversal effects of imperfect credit markets are present and the

[25] In the model above, an increase in the net worth leads to an increase in business investment when $k_t < k_c$. This occurs because an increase in the net worth leads to an increase in the aggregate savings, all of which are used to finance the investment in the business sector. The aggregate investment in the business sector is independent of whether the entrepreneurs face the borrowing constraint. Therefore, it should not be interpreted as the credit multiplier effect.

equilibrium dynamics exhibit persistence at a low level of economic activities and instability at a high level.

Second, in the model developed above, the only alternative to business investment, trading, not only generates less aggregate demand spillovers than the other, but also faces tighter borrowing constraints. This might give the reader a false impression that it would be necessary to have these two features, less spillovers and tighter borrowing constraints, must go together to have instability and fluctuations. The extension presented below will show that this need not be the case, by adding another investment opportunity, which generates less spillovers and face less borrowing constraints. What is needed for endogenous fluctuations is that *some* projects that have less spillovers than business investment are high profitable, and yet can be financed only at at a high level of economic activities.

The model discussed in the last section is now modified to allow the young agents to have access to a storage technology, which transforms one unit of the final good in period t into ρ units of the final good in period t+1. The storage technology is divisible and available to all the young, even when their net worth is low. It is assumed that the gross rate of return on storage satisfies $\rho \in (\lambda_2 R, R)$. This restriction ensures that storage dominates trading when the net worth is low, while trading dominates storage when the net worth is high. That is, the economy now has the following three types of the investment: i) *The Good* (Business Investment), which creates jobs and is relatively easy to finance; ii) *The Bad* (Trading), which is profitable but creates no jobs and is difficult to finance; and iii) *The Ugly* (Storage), which is unprofitable and creates no jobs, but has no problem of financing.

Let s_t be the total units of the final good invested in storage at the end of period t. Then, the credit market equilibrium condition is now given by

$$(8)\qquad mx_{t+1} \begin{cases} = m\mu_2 & \text{if } r_{t+1} < R(W(k_t)), \\ \in [0, m\mu_2] & \text{if } r_{t+1} = R(W(k_t)), \\ = 0 & \text{if } r_{t+1} > R(W(k_t)). \end{cases}$$

$$(20)\qquad \Pi(k_{t+1})/\max\{[1 - W(k_t)]/\lambda_1, 1\} = r_{t+1}.$$

$$(22)\qquad s_t \begin{cases} = 0, & \text{if } r_{t+1} > \rho \\ \geq 0, & \text{if } r_{t+1} = \rho \end{cases}$$

$= \infty$,   if $r_{t+1} < \rho$

(23) $k_{t+1} = W(k_t) - mx_{t+1} - s_t$.

Here, (8) and (20) are reproduced for easy reference. Introducing the storage technology does not make any difference in the range where $r_{t+1} > \rho$. If the storage technology is used in equilibrium, the equilibrium interest rate must be $r_{t+1} = \rho$.

Characterizing the credit market equilibrium and the equilibrium trajectory determined by (8), (20), (22) and (23) for a full set of parameter values require one to go through a large number of cases. Furthermore, in many of these cases, the presence of the storage technology does not affect the properties of the equilibrium dynamics fundamentally. In what follows, let us focus on a representative case, in which the introduction of the storage technology creates some important changes. More specifically, let us consider the case, where the following conditions hold. First, R and $\lambda_2 = \lambda$ satisfy the conditions given in Proposition 1E. This ensures that $k_c < k^* < k_\lambda$. Second, ρ is not too low nor too high so that $k_c < k_\rho < k^*$, where k_ρ is implicitly defined by $R(W(k_\rho)) \equiv \rho$. Third, λ_1 is large enough that $k_{\lambda 1} < k_\rho$, and small enough that the RHS of (21) is greater than ρ for $k_t < k'$ and smaller than ρ for $k_t > k'$. (It is feasible to find such λ_1 because $k_c < k_\rho$.) These conditions are illustrated in Figure 7.[26]

Then, for $k_t < k'$, the business profit is so high that all the saving goes to the investment in the business sector, and $x_{t+1} = s_t = 0$. For $k' < k_t < k_\rho$, some saving goes to the storage, $s_t > 0$, and hence $r_{t+1} = \rho > R(W(k_t))$, and the trading remains inactive, $x_{t+1} = 0$. Within this range, the borrowing constraint is binding for the entrepreneurs when $k' < k_t < k_{\lambda 1}$, and the profitability constraint is binding for the entrepreneurs when $k_{\lambda 1} < k_t < k_\rho$. For $k_\rho < k_t < \min\{k_\lambda, k_{cc}, K\}$, the storage technology is not used, $s_t = 0$. The entrepreneurs, whose borrowing constraint is not binding, compete for the credit with the traders who become active, and face the binding borrowing constraint, and the interest is given by $r_{t+1} = R(W(k_t)) > \rho$. The unstable steady state, k^*, shown in Proposition 1E, is located in this range.

The equilibrium dynamics is thus governed by the following map:

$$W(k_t), \qquad \text{if } k_t \le k',$$

$$\Pi^{-1}(\rho[1 - W(k_t)]/\lambda_1), \qquad \text{if } k' < k_t \le k_{\lambda 1},$$

(24) $k_{t+1} = \Psi(k_t) \equiv$

$$\begin{cases} \Pi^{-1}(\rho), & \text{if } k_{\lambda 1} < k_t \leq k_\rho. \\ \Pi^{-1}(\lambda_2 R/[1 - W(k_t)/m]), & \text{if } k_\rho < k_t \leq \min\{k_\lambda, k_{cc}\}, \\ \Pi^{-1}(R), & \text{if } k_\lambda < k_t \leq k_{cc}, \\ W(k_t) - m\mu_2. & \text{if } k_t \geq k_{cc}, \end{cases}$$

where k' is given implicitly by $\lambda_1\Pi(W(k'))/[1 - W(k')] \equiv \rho$. Eq. (24) differs from (10) for $k' < k_t < k_\rho$, where some saving go to the storage technology and the interest rate is fixed at ρ. In particular, for $k' < k_t < k_{\lambda 1}$, the investment in the business sector is determined by the borrowing constraint,

(25) $W(k_t) = 1 - \lambda_1\Pi(k_{t+1})/\rho$.

In this range, an increase in the net worth, $W(k_t)$, eases the borrowing constraint of the entrepreneurs, so that their investment demand goes up. Instead of pushing the equilibrium interest rate, the rise in the investment demand in the business sector is financed by redirecting the savings from storage. Intuitively enough, an increase in ρ/λ_1 shifts down the map in this range. The presence of the ugly investment thus reduces the good investment, which slows down expansion processes. Unlike the bad investment, however, the ugly investment does not destroy the good investment. And a higher business investment today leads to a higher business investment tomorrow. This mechanism is essentially identical with the one studied by Bernanke and Gertler (1989).

The crucial feature of the dynamics governed by (24) is that the credit multiplier effect is operative at a lower level of activities, while the credit reversal effect is operative at a higher level, including in the neighborhood of the unstable steady state, k*. In this sense, this model is a hybrid of the model developed earlier and of a credit multiplier model à la Bernanke-Gertler.

Figure 8 illustrates the map (24) under additional restrictions, $\Psi(k_\rho) = \Pi^{-1}(\rho) \leq \min\{k_\lambda, k_{cc}\}$ and $k_{\lambda 1} > \Psi^2(k_\rho) = \Psi(\Pi^{-1}(\rho))$. The first restriction ensures that some traders remain inactive at $\Psi(k_\rho)$. This means that the trapping interval is given by $I \equiv [\Psi^2(k_\rho), \Psi(k_\rho)] = [\Psi(\Pi^{-1}(\rho)), \Pi^{-1}(\rho)]$.[27] The second restriction ensures that the trapping interval, I, overlaps with $(k', k_{\lambda 1})$, i.e., the range over which the credit multiplier effect is operative. Let us fix ρ and change λ_1. As λ_1 is

[26] In Figure 7, $k_{\lambda 1} < k_c$. This need not be the case, nor is it essential for the discussion in the text.

reduced, $k_{\lambda 1}$ increases from $\Psi^2(k_\rho)$ to k_ρ, and at the same time, the map shifts down below $k_{\lambda 1}$. Clearly, the map has the unique steady state, k*, as long as λ_1 is not too small (or $k_{\lambda 1}$ is sufficiently close to $\Psi^2(k_\rho)$). As λ_1 is made smaller (and $k_{\lambda 1}$ approaches k_ρ), the equilibrium dynamics may have additional steady states in $(k', k_{\lambda 1})$.[28] The following proposition gives the exact condition under which that happens.

<u>Proposition 3</u>. Let k* be the (unstable) steady state in Proposition 1E.

A. If $\lambda_1 < 1 - W(h(1))$ and $\lambda_1 < \rho h(1)$, the equilibrium dynamics governed by (24) has, in addition to k*, two other steady states, $k_1^{**}, k_2^{**} \in (k', k_{\lambda 1})$. They satisfy $k_1^{**} < h(1) < k_2^{**}$, and k_1^{**} is stable and k_2^{**} is unstable.

B. If $\lambda_1 < 1 - W(h(1))$ and $\lambda_1 = \rho h(1)$, the equilibrium dynamics governed by (24) has, in addition to k*, another steady state, $k^{**} = h(1) \in (k', k_{\lambda 1})$, which is stable from below and unstable from above.

C. Otherwise, k* is the unique steady state of (24).

<u>Proof</u>. See Appendix.

If $\lambda_1 > 1 - W(h(1))$ or $\lambda_1 > \rho h(1)$, neither condition given in Proposition 3A or 3B hold, endogenous fluctuations clearly survive, because the map has a unique steady state, k*, which is unstable. Even if $\lambda_1 < 1 - W(h(1))$ and $\lambda_1 \leq \rho h(1)$, the equilibrium dynamics may still exhibit endogenous fluctuations in $I \equiv [\Psi^2(k_\rho), \Psi(k_\rho)]$. This is because, if $h(1) < \Psi^2(k_\rho)$, $k_2^{**} < \Psi^2(k_\rho)$ as long as λ_1 is not too much lower than $\rho h(1)$, and hence the map has a unique steady state in I, k*, which is unstable, and, for any initial condition in I, the equilibrium trajectory never leaves I.

The above argument indicates that, as long as λ_1 is not too small (or ρ is not too large), the introduction of the credit multiplier effect does not affect the result that the borrowing-constrained investment in trading generates endogenous fluctuations. This does not mean, however, that the credit multiplier effect has little effects on the nature of fluctuations. The

[27] Note that this restriction is weaker than the restriction, $W(k_c) \leq \min\{k_\lambda, k_{cc}\}$, because $k_c < k_\rho$ implies $\Pi(W(k_c)) = R(W(k_c)) < R(W(k_\rho)) = \rho$, hence $W(k_c) > \Pi^{-1}(\rho)$.

introduction of the credit multiplier effect, by shifting down the map below $k_{\lambda 1}$, can slow down an economic expansion, thereby creating asymmetry in business cycles. This is most clearly illustrated by Figure 9, which depicts the case where $\Psi^2(k_\rho) < h(1) < k_{\lambda 1}$. If $\lambda_1 = \rho h(1)$, as indicated in Proposition 3B, the map is tangent to the 45° line at h(1), which creates an additional steady state, k** = h(1). It is stable from below but unstable from above, and there are *homoclinic orbits*, which leave from k**, and converges to k** from below. Starting from this situation, let λ_1 go up slightly. As indicated in Proposition 3A, such a change in the parameter value makes the steady state, k**, disappear, and the map is left with the unique steady state, k*, in its downward-sloping segment, which is unstable. (Technically speaking, this is known as a *saddle-node* or *tangent bifurcation*.) The credit multiplier effect is responsible for the segment, where the map is increasing and stays above but very close to the 45° line. Thus, the equilibrium dynamics display *intermittency*, as a tangent bifurcation eliminates the tangent point, k**, and its homoclinic orbits. The equilibrium trajectory occasionally has to travel through the narrow corridor. The trajectory stays in the neighborhood of h(1) for possibly long time, as the economy's business sector expands gradually. Then, the economy starts accelerating through the credit multiplier effect. At the peak, the traders start investing. Then, the economy plunges into a recession (possibly after going through a period of high volatility, as the trajectory oscillates around k*). Then, at the bottom, the economy begins its slow and long process of expansion. The map depicted in Figure 9 is said to display intermittency, because its dynamic behavior is characterized by a relatively long (but seemingly random) periods of small movements punctuated by intermittent periods of violent movements.[29]

Example 3.

As in Examples 1 and 2, let $\phi(n) = 2(Kn)^{1/2}$ with $K < \mu_1, 4m\mu_2$, and impose the same restrictions to ensure $W(k_c) \leq \min\{k_\lambda, k_{cc}\}$. This guarantees $\Psi(k_\rho) = \Pi^{-1}(\rho) < W(k_c) \leq \min\{k_\lambda, k_{cc}\}$. As seen

[28] Since $k_\rho < k^* < \Pi^{-1}(\rho)$, the map does not intersect with the 45° line in $[k_{\lambda 1}, k_\rho)$.

[29] What is significant here is that the introduction of the credit multiplier effect can create the intermittency, regardless of the functional form of ϕ. Even without the credit multiplier effect, one can always choose a functional form of ϕ, so as to make the function $W(k) = \Psi(k)$ come close to the 45° line below k_c to generate the intermittency phenomenon. In this sense, the presence of the credit multiplier effect is not necessary for the intermittency. It simply makes it more plausible.

in Examples 1 and 2, $1 - K/m < \lambda_2 R < K/m$, and $R > K/(1-\lambda_2)m$ ensure that the conditions in Proposition 1E are satisfied. Let us choose ρ such that $K/m\rho^2 < (1 - \lambda_2 R/\rho) < K/m\rho$ (this is feasible because $\lambda_2 R + K/m > 1$) and λ_1 such that $K/\rho^2 < 1 - \lambda_1 < m(1 - \lambda_2 R/\rho)$. Then, (24) can be rewritten in the relevant range as

$$
(26) \qquad z_{t+1} = \psi(z_t) \equiv \begin{cases} (z_t)^{1/2}, & \text{if } z_t \leq z', \\ \lambda_1/\rho(1 - Kz_t), & \text{if } z' < z_t \leq z_{\lambda 1}, \\ 1/\rho, & \text{if } z_{\lambda 1} < z_t \leq z_\rho. \\ [1 - (K/m)z_t]/(\lambda_2 R), & \text{if } z_\rho < z_t \leq 1/\rho, \end{cases}
$$

where $z_t \equiv (k_t/K)^{1/2}$ and $z' \equiv (k'/K)^{1/2}$, $z_{\lambda 1} \equiv (k_{\lambda 1}/K)^{1/2}$, and $z_\rho \equiv (k_\rho/K)^{1/2}$ satisfy $(z')^{1/2} = \lambda_1/\rho(1 - Kz')$, $\lambda_1 = 1 - Kz_{\lambda 1}$, and $\lambda_2 R/\rho = [1 - (K/m)z_\rho]$, and $z' < z_{\lambda 1} < z_\rho < z^* = 1/(\lambda_2 R + K/m) < 1/\rho < 1$.

Let $\lambda_1 < 1/2$, or equivalently $z_{\lambda 1} > (h(1)/K)^{1/2} = 1/2K$. If $\lambda_1 \geq \rho/4K$, z^* is the unique steady state of (26). If $\lambda_1 < \rho/4K$, $z_1^{**} \equiv [1 - (1 - 4\lambda_1 K/\rho)^{1/2}]/2K$ and $z_2^{**} \equiv [1 + (1 - 4\lambda_1 K/\rho)^{1/2}]/2K$ are two additional steady states of (26). They satisfy $z_1^{**} < (h(1)/K)^{1/2} < z_2^{**}$. If $1/2K < \psi^2(z_\rho) = [1 - (K/m\rho)]/(\lambda_2 R)$, then z^* remains the unique steady state in $I \equiv [\psi^2(z_\rho), z_\rho]$, for all $\lambda_1 > \lambda_{1min}$, where λ_{1min} is defined by $[1 + (1 - 4\lambda_{1min} K/\rho)^{1/2}]/2K \equiv [1 - (K/m\rho)]/(\lambda_2 R)$. If $1/2K > \psi^2(z_\rho) = [1 - (K/m\rho)]/(\lambda_2 R)$, then a tangent bifurcation occurs at $\lambda_1 = \rho/4K$, and intermittency phenomena emerge for $\lambda_1 > \rho/4K$.

6. Concluding Remarks

This paper has presented dynamic general equilibrium models of imperfect credit markets, in which the economy fluctuates endogenously along its unique equilibrium path. During recessions, the agents are too poor to be able to finance their trading activities. Much of the saving thus goes to business investment, which creates jobs, thereby making the next generation of the agents richer. As the economy booms and the net worth of the agents improves, they can eventually finance their trading activities, which do not create any job. As more credit is extended to trading at the expense of business investment, the economy plunges into a recession. The whole process repeats itself. Endogenous fluctuations occur because, as in ecological cycles driven by predator-prey or host-parasite interactions, good investment breeds bad investment, which destroys good investment. When this credit reversal mechanism is combined with the credit multiplier mechanism studied in the literature, the model generates asymmetry in business cycles. That is, the economy experiences a long and slow process of recovery, followed by a rapid expansion, and then, possibly after periods of high volatility, it plunges into a recession.

Appendix: Proof of Proposition 3.

Because the introduction of the storage technology changes the map only for (k', k_ρ), and since $k_\rho < k^* < \Pi^{-1}(\rho)$ implies $\Psi(k_t) > k_t$ in $[k_{\lambda 1}, k_\rho)$, the dynamical system, (24), could have additional steady states only in $(k', k_{\lambda 1})$, where it is given by

(*) $\quad k_{t+1} = \Psi(k_t) = \Pi^{-1}(\rho[1 - W(k_t)]/\lambda_1)$.

By differentiating (*) and then setting $k_t = k_{t+1} = k^{**}$, the slope of the map at a steady state in this range is equal to $\Psi'(k^{**}) = -\rho W'(k^{**})/\Pi'(k^{**})\lambda_1 = \rho k^{**}/\lambda_1$, which is increasing in k^{**}. Since Ψ is continuous, and $\Psi(k') > k'$ and $\Psi(k_{\lambda 1}) > k_{\lambda 1}$ hold, this means that either

i) the map intersects with the 45° line twice at k_1^{**} and $k_2^{**} > k_1^{**}$;

ii) it is tangent to the 45° line at a single point, $k^{**} \in (k', k_{\lambda 1})$ and $\Psi(k_t) > k_t$ in $(k', k_{\lambda 1})/\{k^{**}\}$; or

iii) $\Psi(k_t) > k_t$ in $(k', k_{\lambda 1})$.

Consider the case of ii). Then, $\rho k^{**}/\lambda_1 = 1$ and $k^{**} = \Pi^{-1}(\rho[1 - W(k^{**})]/\lambda_1)$, which imply that $\Pi(k^{**})k^{**} + W(k^{**}) = \phi(1/k^{**})k^{**} = 1$, or $k^{**} = h(1) = \lambda_1/\rho$. Thus, $\lambda_1 = \rho h(1)$ implies that (*) is tangent to the 45° line at $k^{**} = h(1)$. Furthermore, $h(1) = \Psi(h(1)) < W(h(1))$ implies that $\lambda_1\Pi(W(h(1)))/[1 - W(h(1))] < \lambda_1\Pi(h(1))/[1 - W(h(1))] = \lambda_1/h(1) = \rho = \lambda_1\Pi(W(k'))/[1 - W(k')]$, or equivalently, $k^{**} = h(1) > k'$, and $\lambda_1 < 1 - W(h(1))$ implies that $k^{**} = h(1) < k_{\lambda 1}$. This proves Proposition 3B. The case of i) can always be obtained by increasing ρ from the case of i), which shifts down the map to create a stable steady state at $k_1^{**} < h(1)$ and an unstable steady state at $k_2^{**} > h(1)$. This proves Proposition 3A. Otherwise, iii) must hold, i.e., the map must lie above 45° line over the entire range, in $(k', k_{\lambda 1})$, which completes the proof of Proposition 3.

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