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EQUILIBRIUM WITH EXTERNALITIES,

COMMODITY TAXATION, AND LUMP SUM TRANSFERS

by

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# ABSTRACT

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Commodity Taxation, and Lump Sum Transfers

In this paper we investigate sufficient conditions for the
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authority and externalities.  The theorem extends a result of
Sontheimer.  It verifies the consistency of competitive behavior
in economies with taxation and externalities.

The model is sufficiently general to include the possibility
of public goods, commodity taxation, income taxation, government demand,
or any subset of these.

1.  INTRODUCTION

In this paper we investigate sufficient conditions for the existence of competitive equilibrium in economies with a taxing authority and externalities. The theorem extends a result of Sontheimer [13] (and is related to issues explored in Diamond-Mirrlees [5], Shoven [12] and Mantel [8]). It verifies the consistency of competitive behavior in economies with taxation and external-ities. [2/] Thus the result adds meaning to the normative theorems which relate taxation, externalities, and competitive behavior.

The model that we explore extends the standard framework for studying externalities and taxation. It is sufficiently general to include the possi-bility of public goods, commodity taxation, income taxation, government demand, or any subset of these. Preferences and the technology of firms depend on the state of the economy. Individuals hold wealth in the form of initial en-dowments of commodity and ownership shares in firms. The government both taxes and subsidizes the purchase of commodities. We allow for the possibility that economic agents face different prices for commodities. The extent to which these prices differ from the tax free prices defines the tax (or subsidy) on each commodity imposed on each agent. Furthermore, taxes are allowed to vary with allocations and tax free prices. Finally, the government redistributes the net revenue (possibly negative) associated with its actions to consumers.

The now classical literature on externalities and corrective taxation serves nicely to provide motivation for the analysis. In that framework it is useful to dichotomize the function of government. First, the government effects the competitive mediation of the externalities by altering the value of

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[2/] For economies with externalities alone the existence of equilibrium was established by McKenzie [10], and more recently by Arrow and Hahn [2].

actions. Its purpose is to make visible real costs and benefits which exist for the economy as a whole but are not visible to the agents who cause them. In this function the government helps to bring about the efficient allocation of resources. Second, any government which taxes is intimately involved in the redistribution of income. This is because taxation effects commodity prices (including factor prices), and also because the proceeds of taxation must be distributed. Thus, an essential problem for a government engaged in corrective taxation is to determine which allocation, each associated with a different distribution of income, it wishes to choose. The particular schedule it chooses must necessarily depend on its preferences over distributions of income as well as the exact specification of the externality. However, it is well known that with externalities, the design of a tax system consistent with the Pareto efficient allocation of resources requires precise knowledge of the technologies (including externalities) and preferences of all agents. Here we do not assume that governments have exact knowledge of the "data" of the economy. Even without exact knowledge they can (and do) purposefully effect the alloca- tion of resources through the use of corrective taxation. The ambition level of a government may be better expressed as "to make some social costs and bene- fits visible in markets and to regulate the distribution of income rather than "to achieve Pareto optimality". From this perspective a government will combine its conception of the true preferences and technology of society with some know- ledge of prices and the distribution of income to obtain a schedule of taxes and a plan for the distribution of tax revenue. The main purpose of this paper is to investigate conditions under which competitive behavior is consistent with the presence of a government which behaves in this way. More precisely we provide conditions for the existence of competitive equilibrium for economies with very general externalities and with governments that impose taxes (not necessarily optimal taxes) which depend on both the allocation of commodities and commodity

prices.

Before entering into the substance of the argument a few words concerning technique may be in order. This is especially true since the methods we employ have not (to our knowledge) previously been applied to the problem at hand. In our mind they provide some of the raison d'etre of the paper. It is felt that the most natural technique for proving the existence of equilibrium with externalities and commodity taxation is the very beautiful theorem of G. Debreu [3] on equilibrium in a generalized game. Unfortunately this theorem does not directly apply because of the possible emptiness of the budget correspondence (when it is defined in the natural way). To overcome this problem we have replaced the original economy with a new economy which has the property that an equilibrium of the new economy is an equilibrium of the old economy. For the new economy the natural budget correspondence is always nonempty; however, preferences are no longer orders. An extension of the Debreu theorem to the case of nonordered preferences [11] is used to prove the existence of equilibrium for the new economy. Our extension of the Debreu theorem was motivated by recent results due to Mas-Colell [9] and Gale - Mas-Colell [6] on the existence of competitive equilibrium without ordered preferences.

## II. THE MODEL

In this section we will present the definition of an economy), state the main theorem, and discuss the conditions of the theorem.

<u>Definition 1</u>    By an economy $\mathcal{E} = (X_i, \omega_i, P_i, Y_j, \mathcal{Y}_j, \phi_i, \mu_i, \psi_j)$ is meant:

<b>For each consumer</b> i = 1,2,...,n, a consumption set $X_i \subset R^\ell$ $(X = \prod_{i=1}^{n} X_i)$.

For each firm j = 1,2,...,m, a production set $Y_j \subset R^\ell$ $(Y = \prod_{j=1}^{m} Y_j)$.

A set $\dot{\Omega} = R_+^\ell \setminus \{0\}$ of price vectors, and a tax authority described by

$$\phi_i : X \times Y \times \dot{\Omega} \to \dot{\Omega} \qquad i = 1,2,...,n$$
$$\mu_i : X \times Y \times \dot{\Omega} \to R \qquad i = 1,2,...,n$$
$$\psi_j : X \times Y \times \dot{\Omega} \to \dot{\Omega} \qquad j = 1,2,...,m.$$

Each consumer i=1,2,...,n has an initial endowment vector $\omega_i \in R_+^\ell$ and a preference correspondence $P_i : X \times Y \times \dot{\Omega} \gg R_+^\ell$. At a state $(x,y,p) \in X \times Y \times \dot{\Omega}$, $P_i(x,y,p) \cap X_i$ is interpreted to be the set of all $z \in X_i$ which agent i prefers to $x_i$ (the ith component vector of x). ($P_i$ is representable by a utility function if there exists a $U_i : R^\ell \times X \times Y \times \dot{\Omega} \to R$ such that for all states (x,y,p), $X_i \cap P_i(x,y,p) = \{z \in X_i : U_i(z,x,y,p) > U_i(x_i,x,y,p)\}$.) At a state $(x,y,p) \in X \times Y \times \dot{\Omega}$, agent i faces the price vector $\phi_i(x,y,p)$, to be interpreted as including the commodity taxes and subsidies the tax authority would wish to prevail at that state, and receives a (scalar) lump sum subsidy $\mu_i(x,y,p)$ (possibly negative), which for notational simplicity is assumed to also include profit dividends.[3] Thus the agent's wealth in state (x,y,p) is $\phi_i(x,y,p)\omega_i + \mu_i(x,y,p)$, and the cost of a consumption vector $z_i \in X_i$ is $\phi_i(x,y,p)z_i$.

Each firm j = 1,2,...,m has a production correspondence $\mathcal{Y}_j : X \times Y \gg Y_j$.

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3/ For the case of an <u>ad valorem</u> tax $\phi_i$ is of the form Ap where A is a diagonal matrix, and for the case of a specific tax $\phi_i$ is of the form p plus a nonnegative vector.

At a state $(x,y,p) \in X \times Y \times \dot{\Omega}$, the set $\mathcal{Y}_j(x,y)$ is composed of all feasible production vectors for firm $j$. At this state, the firm faces the price vector $\Psi_j(x,y,p)$, which includes commodity taxes and subsidies and the corporation profit tax; so the firm's net profit at a production $z_j \in \mathcal{Y}_j(x,y)$ is $\Psi_j(x,y,p)z_j$.

<u>Definition 2</u>  An equilibrium for $\mathcal{E}$ is a point $(x^*,y^*,p^*) \in X \times Y \times \dot{\Omega}$ such that:

E 1)  $\varphi_i(x^*,y^*,p^*)x_i^* = \varphi_i(x^*,y^*,p^*)\omega_i + \mu_i(x^*,y^*,p^*)$   $i = 1,2,\ldots,n$,

E 2)  $\sum_i x_i^* \leqq \sum_j y_j^* + \sum_i \omega_i$ and $p^*(\sum_i x_i^* - \sum_j y_j^* - \sum_i \omega_i) = 0$,

E 3)  $z_i \in P_i(x^*,y^*,p^*) \cap X_i$ implies

   $\varphi_i(x^*,y^*,p^*)(z_i - \omega_i) > \mu_i(x^*,y^*,p^*)$, $i = 1,2,\ldots,n$, and

E 4)  $y_j^*$ maximizes $\Psi_j(x^*,y^*,p^*)z_j$ over $z_j \in \mathcal{Y}_j(x^*,y^*)$.

<u>Theorem.</u>  Every economy $\mathcal{E}$ which satisfies the following conditions (a),b), c), d), e), and f)) has an equilibrium.

For each agent $i = 1,2,\ldots,n$

a 1)  $X_i$ is closed and convex,

a 2)  $P_i$ has open graph in $R^\ell \times X \times Y \times \dot{\Omega}$, and

a 3)  $P_i(x,y,p)$ is convex and $x_i \in \mathrm{Bdry}\{P_i(x,y,p) \cap X_i\}$ for each $(x,y,p) \in X \times Y \times \dot{\Omega}$.

For each firm $j = 1,2,\ldots,m$

b 1)  $Y_j$ is closed and convex,

b 2)  $\mathcal{Y}_j$ is a continuous correspondence, and

b 3)  $\mathcal{Y}_j(x,y)$ is a closed convex set containing $0$ for each $(x,y) \in X \times Y$.

c) The attainable set $A = \{(x,y) \in X \times Y : y_j \in \mathcal{Y}_j(x,y)$ for all $j$ and

$$\sum_i x_i \leqq \sum_j y_j + \sum_i \omega_i\}$$ is nonempty and bounded.

d) The maps $\varphi_i$, $\mu_i$, and $\Psi_j$ are continuous.

For each $i = 1,2,\ldots,n$ and each $(x,y,p) \in X \times Y \times \dot{\Omega}$,

e 1) there exists a $z_i \in pr_i(A)$ such that

$$pz_i < p\omega_i + \mu_i(x,y,p) - (\varphi_i(x,y,p)-p) (x_i - \omega_i).$$

(The symbol $pr_i$ denotes projection on the $i^{th}$ coordinate.)

e 2) Let $B$ be an $\ell \times \ell$ orthonormal matrix representing a rotation of $R^\ell$

which sends $\dfrac{\varphi_i(x,y,p)}{\|\varphi_i(x,y,p)\|}$ to $\dfrac{p}{\|p\|}$. ($B$ represents the direction and size of

the price distortion caused by the commodity taxes.) Let $T : R^\ell \to R^\ell$ be

this rotation with the origin translated to $x_i$; i.e., $T(z) = B(z-x_i) + x_i$.

Then we require:

e 2.1) $T(P_i(x,y,p)) \cap X_i \neq \emptyset$, and e 2.2) if $\varphi_i(x,y,p)x_i = \varphi_i(x,y,p)\omega_i + \mu_i(x,y,p)$

and $P_i(x,y,p)$ contains points $z_i \in X_i$ for which $\varphi_i(x,y,p)z_i \leqq \varphi_i(x,y,p)x_i$.

then at least one such $z_i$ must satisfy $T(z_i) \in X_i$.

f) For each $(x,y,p) \in X \times Y \times \dot{\Omega}$ for which $\Psi_j(x,y,p)y_j \geqq \Psi_j(x,y,p)z_j$ holds

for each $z_j \in \mathcal{Y}_j(x,y) \cap pr_j(A)$, $j=1,2,\ldots,m$, and

$$\varphi_i(x,y,p) (x_i - \omega_i) \leqq \mu_i(x,y,p) \quad i=1,2,\ldots,n, \quad \text{we must have}$$

$$\sum_i \mu_i(x,y,p) = \sum_i (\varphi_i(x,y,p) - p) (x_i - \omega_i) + \sum_j py_j.$$

Conditions a) through d) are either self-explanatory or standard

assumptions; condition f) simply requires that the tax authority balance

tax revenues with expenditures. It states that the aggregate lump sum

transfers $\sum_i \mu_i(x,y,p)$, which also include after tax profits by assumption,

must be equal to aggregate commodity tax revenues from consumers,

$\sum_i (\varphi_i(x,y,p) - p) (x_i - \omega_i)$, plus the sum of tax revenues from firms and after

tax profits, which is $\sum\limits_{j} py_j$. When this holds, and $\varphi_i(x,y,p) (x_i - \omega_i) = \mu_i$

holds for each $i$, then we will get $p(\sum\limits_{i} x_i - \sum\limits_{j} y_j - \sum\limits_{i} \omega_i) = 0$, which

is Walras' Law.

The condition e 1) is analagous to the minimum wealth requirement

which is used to guarantee the continuity of budget correspondences. Since

$\varphi_i(x,y,p) (x_i - \omega_i) = \mu_i$ is equivalent to $p(x_i - \omega_i) = \mu_i - (\varphi_i(x,y,p) - p)(x_i - \omega_i)$.

the term $(\varphi_i(x,y,p) - p) (x_i - \omega_i)$ measures the change of income due to the

price change from $p$ to $\varphi_i(x,y,p)$. Condition e 1) thus requires the existence

of affordable consumption vectors when the loss in income due to the commodity

taxes is considered as a lump sum tax.

Our reason for using e 1) as a minimum wealth requirement rather than,

for example, the more direct requirement that

(*)               there exists $z \in X_i$ such that $\varphi_i(x,y,p) (z - \omega_i) < \mu_i(x,y,p)$

is that this latter requirement may rule out reasonable taxation mechanisms.

Note that if condition * is satisfied, then the budget correspondence defined

by $C_i(x,y,p) = \{z_i \in X_i : \varphi_i(x,y,p) (z_i - \omega_i) \leqq \mu_i(x,y,p)\}$ will be nonempty

valued and continuous. We will give an example of why (*) is too strong after

discussing condition e 2). This condition places bounds on the size of the

price distortions caused by the commodity taxation. The first part     (e 2.1)

requires that if the preferred set $P_i(x,y,p)$ is "twisted" about $x_i$

in the same direction and angle as the move from $\dfrac{\varphi_i(x, y, p)}{\|\varphi_i(x,y,p)\|}$ to

$\dfrac{p}{\|p\|}$, then this twist cannot be so large as to completely move $P_i(x,y,p)$

out of the consumption set. We remark that this will be satisfied whenever

$x_i \in \text{int} X_i$, since $x_i \in \text{Bdry} \{P_i(x,y,p) \cap X_i\}$. Thus the problem arises only at boundary

points of $X_i$. The condition will also be satisfied at any point $x_i$ for which $P_i(x,y,p) \cap X_i$ contains all $z \in R^\ell$ for which $z \geqq x_i$ and $z \neq x_i$, since the rotation is never more than $90°$. Thus e 2.1) is not viewed as unreasonably strong. The second part of e 2) requires that the twist should never be so large as to move out of $X_i$ all consumptions preferred to $x_i$ which cost no more than $x_i$. This condition will also be satisfied at any $x_i \in \text{int} X_i$; however it is a strong restriction on the boundary. A condition which guarantees e 2.1) is that $X_i = R_+^\ell$ and $\text{Bdry} X_i$ is an indifference curve.[4]/ Sontheimer [13] has counterexamples to show that the theorem is false with all the assumptions except e 2.1). Sontheimer's model, which is our model without externalities, with $\varphi_i$ depending only on p, and with $\mu_i$ a constant plus $(\varphi_i - p)(x_i - \omega_i)$, uses conditions different from e 2 ) to overcome this problem. However, neither his conditions nor our's are more general.

We now give examples of why (*) above may be too strong for some tax mechanisms. In our first example it is shown that keeping the budget correspondence $C_i$ nonempty is a significant problem. In our second example we show that even if $C_i$ can be made nonempty valued (*) may fail and $C_i$ may in fact not be continuous.

<u>Example 1</u>  Consider an exchange economy with two goods $x_1$ and $x_2$, and denote the prices of these commodities by $p_1$ and $p_2$ respectively. Suppose there is a single consumer with consumption set $R_+^2$ and initial endowment $\omega = (2,2)$. Let the prices faced by the consumer be $\varphi(p) = (3p_1/5, 7p_2/5)$; thus, good 2 is taxed to finance a subsidy on good 1. Condition f) requires that the lump sum transfer $\mu(x,p)$ satisfy $\mu(x,p) = (\varphi(p) - p)(x - \omega)$ whenever

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[4]/We observe that if there is no twist; e.g., if all commodities are taxed at the same rate, then e 2 ) is satisfied.

$(x,p)$ satisfies $\varphi(p)x = \varphi(p)\omega + \mu(x,p)$.

If we define $\mu$ by $\mu(x,p) = (\varphi(p) - p)(x - \omega)$ for all $(x,p)$, which is the most natural way, then the budget set will be empty for some choices of $(x,p)$. The reader may verify that if $p_1 = p_2 = 1$ and $x = (12,1)$, then $\varphi(p)\cdot\omega + \mu(x,p) = 4 - 22/5 < 0$ so no point $x' \in R_+^2$ is affordable. This problem arises because $\mu$ must be defined and continuous even at $(x,p)$ at which $x$ is not affordable.

It is natural to ask whether a proof technique exists which requires only that the $C_i$'s be nonempty at $(x,p)$ at which $x_i$ is affordable for each i. Even if this can be done, the next example indicates that (*) is too strong. The budget correspondence can easily fail to be continuous even when restricted to $x_i$ which are affordable. Example 2 provides an illustration of this phenomenon. It can be embedded in a two agent model in which each consumer is returned the tax he pays and which has its only equilibrium at a point at which the budget correspondence is not continuous.

Example 2   Consider a consumer in a one person pure exchange economy with two goods. In Figure 1, the agent's initial endowment vector is $\omega$, and his consumption set $X$ is the set of all points lying on or above the curve consisting of the straight line segment $[a,x_1]$ and the curve starting at $x_1$ and passing through $A$. The tax mechanism consists of subsidizing good 2 by a tax on good 1, and we must have $\mu(x,p) = (\varphi(p) - p))(x - \omega)$ at any $x$ for which $\varphi(p)x = \varphi(p)\omega+\mu$ (condition f). The latter statement is equivalent to requiring that $p(x - \omega) = 0$ whenever $\varphi(p)x = \varphi(p)w + \mu$. In Figure 1, the lines labeled $p_i, i = 1,2$, are the hyperplanes $\{z: p_i z = p_i \omega\}$, and the lines labeled $\varphi_i$, $i = 1, 2$, are the hyperplanes $\{z: \varphi(p_i)z = \varphi(p_i)x_i = \varphi(p_i)\omega + \mu(x_i,p_i)\}$. Consistent with our assumptions about the taxes, $\varphi_i$ is always steeper than $p_i$.

Good 1

Good 2

$\phi_1$

$\phi_2$

$P_1$

$P_2$

$X_1$

$X_2$

$X$

$A$

$a$

$b$
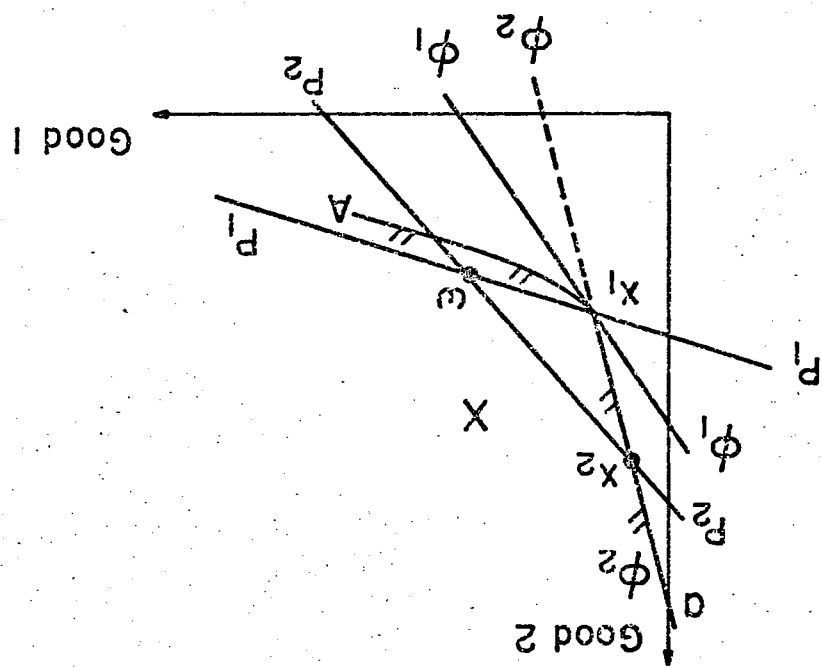
As drawn, the minimum wealth condition 1) fails to be satisfied at each $(p_i, x_i)$. Such a situation arises simply because the tax mechanism may prevent $\omega$ from being an affordable consumption, so that at points $x$ on the boundary of $X$ the budget set $C(x,p)$ may contain only boundary points. As this example is drawn, the constraint correspondence $C$ actually fails to be lower-hemi-continuous at $(p_2, x_2)$. Consider a sequence $(p^n, x^n)$ such that $p^n \to p_2$, $x^n \in [x_1, x_2)$ and $p^n(x^n - \omega) = 0$. Then the hyperplanes determined by $\varphi(p^n)$ will be less steep than $\varphi_2$ and will never contain $x_2$. Therefore, one cannot extract a sequence of points from the $C(p^n, x^n)$ which converges to $a \in C(p_2, x_2)$.

In this example, however, e 1) is satisfied, and e 2) will be satisfied if, for example, we take $P$ to be defined by $P(x) = \{z \in \text{int } X : pr_1(z) > pr_1(x)\}$ (for a different specification of $P$, e 2) may fail). The minimum wealth condition * may frequently fail simply because, with commodity taxes, the initial endowment $\omega$ may not be affordable (even when the proceeds of the tax are returned by lump sump sum transfer!). Figure 1 makes it apparent that our theorem will include cases when the budget correspondence is not continuous.

## III.  PROOF OF THE THEOREM

Our main tool for proving existence of equilibrium will be the following

extension of a theorem of G. Debreu on the existence of equilibrium in a

generalized  N-person game  [ 3 ].

Lemma 1   (Shafer and Sonnenschein [11])    Let  $\Gamma = (X_i, P_i, \mathcal{A}_i)_{i=1}^{i=n}$   satisfy

α)   each  $X_i$  is a nonempty compact convex subset of  $R^\ell$  $(X = \prod_{i=1}^{n} X_i)$,

β)   each  $P_i$  is a preference correspondence:  $P_i: X \twoheadrightarrow X_i$   such that,

βl)  $P_i$  has open graph in  $X \times X_i$,

β2)  for each  $x \in X$,  $x_i$  is not in the convex hull of  $P_i(x)$,   and

γ)   each  $\mathcal{A}_i$  is a (constraint) correspondence:  $\mathcal{A}_i: X \twoheadrightarrow X_i$   such that

γl)  $\mathcal{A}_i$  is a continuous correspondence,

γ2)  for each  $x \in X$,  $\mathcal{A}_i(x)$  is nonempty, compact, and convex.

Then there exists an equilibrium for  $\Gamma$,  i.e., there exists an  $\bar{x} \in X$

such that, for each  i,

δl)  $\bar{x}_i \in \mathcal{A}_i(\bar{x})$

δ2)  $P_i(\bar{x}) \cap \mathcal{A}_i(\bar{x}) = \emptyset$.

First we prove a special case of the theorem.

Lemma 2  If  $\delta$  satisfies the conditions of the theorem and in addition

$\varphi_i(x,y,p) \equiv p$  for each  i = 1,2,...,n  and each  $(x,y,p) \in X \times Y \times \dot{\Omega}$, then

$\delta$  has an equilibrium.

Proof of Lemma 2   We convert  $\delta$  into an  n + m + 1  person game.  By the

standard technique of intersecting each  $X_i$  and  $Y_j$  with a sufficently large

compact set (see Debreu [ 4 ]), we may assume each $X_i$ and $Y_j$ is compact.

Also, in place of the local nonsatiation assumption that $x_i \in Bdry\{P_i(x,y,p) \cap X_i\}$

we may assume that $x_i \in BdryP_i(x,y,p)$ holds whenever $(x,y) \in A$. (A is the

set of attainable states.)

Let $\Delta = \{p \in \Omega : \sum_i p_i = 1\}$.[5] The first $n$ agents are described as follows.

Agent $i$ has choice set $X_i$, constraint correspondence $C_i : X \times Y \times \Delta \longrightarrow X_i$

defined by

$$C_i(x,y,p) = \{z_i \in X_i : pz_i \leqq p\omega_i + \mu_i(x,y,p)\},$$

and preference correspondence $\hat{P}_i$ defined by $\hat{P}_i(x,y,p) = P_i(x,y,p) \cap X_i$.

The agents $j = 1,2,\ldots,m$ are described as follows. Agent $j$ has choice

set $Y_j$, constraint correspondence $\mathcal{Y}_j$, and preference correspondence

$\hat{P}_j : X \times Y \times \Delta \longrightarrow Y_j$ defined by

$$\hat{P}_j(x,y,p) = \{z_j \in Y_j : \psi_j(x,y,p)z_j > \psi_j(x,y,p)y_j\}.$$

The last agent, the "market player", has choice set $\Delta$, constraint corres-

pondence $\mathcal{A} : X \times Y \times \Delta \longrightarrow \Delta$ defined by $\mathcal{A}(x,y,p) \equiv \Delta$, and preference correspon-

dence $\hat{P} : X \times Y \times \Delta \longrightarrow \Delta$ defined by

$$\hat{P}(x,y,p) = \{q \in \Delta : q(\sum_i x_i - \sum_j y_j - \sum_i \omega_i) > p(\sum_i x_i - \sum_j y_j - \sum_i \omega_i)\}.$$

Aside from the continuity of the $C_i$, it is evident that $\Gamma$ satisfies

the conditions of Lemma 1. The fact that e 1) is simply the usual minimum

wealth requirement in this case guarantees the continuity of the $C_i$ functions

(Debreu [ 4 ].)

Let $(x^*,y^*,p^*)$ be an equilibrium of $\Gamma$. We will show that it is an

equilibrium of $\mathcal{E}$. We have $p^*x_i^* \leqq p^*\omega_i + \mu_i(x^*,y^*,p^*)$ $i = 1,2,\ldots,n$.

and each $y_j^*$ is a profit maximizing vector. Thus by condition f),

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[5] In this model excess demand is not necessarily homogeneous of degree zero in prices: thus, the equilibrium price set may depend on the normalization which is chosen. For example, this will be the case for specific taxes.

$\sum_i p^*(x_i^* - \omega_i) \leqq \sum_i \mu_i(x^*, y^*, p^*) = p^* \sum_j y_j^*$, so $p^*(\sum_i x_i^* - \sum_j y_j^* - \sum_i \omega_i) \leqq 0$.

This latter condition, together with the fact that $p^*$ maximizes the value

of excess demand (by equilibrium for the market player), implies that

$\sum_i x_i^* - \sum_j y_j^* - \sum_i \omega_i \leqq 0$, so that $(x^*, y^*) \in A$. Then local        nonsatiation

implies $p^* x_i^* = p^* \omega_i + \mu_i(x^*, y^*, p^*)$, so that E 1) and E 3) are satisfied,

and with f) we now get E 2). Thus $(x^*, y^*, p^*)$ is an equilibrium for $\mathcal{E}$.

Before proving the main theorem, we establish a technical lemma which allows

us to choose the rotations described in e 2) continuously as functions of

$p$ and $\varphi_i$.

<u>Lemma 3</u>   There exists a continuous map $B : \dot{\Omega} \times \dot{\Omega} \to R^{\ell^2}$ such that for each

$(p, q) \in \dot{\Omega} \times \dot{\Omega}$, $B(p, q)$ is an $\ell \times \ell$ orthonormal matrix with positive determi-

nant such   that   $B(p, q) \frac{q}{\|q\|} = \frac{p}{\|p\|}$.

<u>Proof of Lemma 3</u>   For $k = 2, 3, \ldots, \ell$, let $y^k \in R^\ell$ be the vector with

$(k-1)^{st}$ coordinate 1,   whose k'th coordinate is $-1$, and which has all other

coordinates 0. Let $S_+^{\ell-1} = \{q \in R_+^\ell : \|q\| = 1\}$. Then it is easy to verify

that for any $q \in S_+^{\ell-1}$, the set $\{q, y^2, \ldots y^\ell\}$ is linearly independent. We

now apply an orthogonalization process to the sequence $q, y^2, \ldots, y^\ell$ to obtain

an equivalent orthonormal sequence $q, a_2(q), \ldots, a_\ell(q)$. This process, as

described in Gantmacher ([7], pp. 256-258, especially formulas 35-37), yields

the $a_i(q)$ as continuous functions of $q$. Define $A(q)$ to be the matrix

whose first row is $q$ and whose $i^{th}$ row is $a_i(q)$ for $i > 1$. Then $A(q)$

is an orthonormal matrix which varies continuously with $q$, has positive

determinant, and satisfies $e^1 A(q) = q$, where $e^1 = (1, 0, \ldots, 0) \in R^\ell$. Let

$B(p, q) = A'(p) A(q)$, and extend $B$ to $\dot{\Omega} \times \dot{\Omega}$ in the obvious way.

<u>Proof of the Theorem.</u> We convert $\mathcal{E}$ to an economy $\tilde{\mathcal{E}}$ satisfying the

conditions of Lemma 2. Choose B as in Lemma 3, and for each $i=1,2,\ldots,n$

define $T_i: X \times Y \times \dot{\Omega} \times X_i \to R^\ell$ by $T_i(x,y,p,z_i) = B(p,\varphi_i(x,y,p))(z_i - x_i) + x_i$.

Define preference correspondences $\tilde{P}_i: X \times Y \times \dot{\Omega} \to\!\!\!> R^\ell$ by

$$\tilde{P}_i(x,y,p) = T_i(x,y,p,P_i(x,y,p)).$$

Condition a 2), the continuity of $T_i$, and the fact $T_i(x,y,p,\cdot)$

is bijective means $\tilde{P}_i$ will have open graph in $X \times Y \times \dot{\Omega} \times R^\ell$. The linearity

of $T_i(x,y,p,\cdot)$ implies $\tilde{P}_i(x,y,p)$ is convex, and the fact that $T_i(x,y,p,.)$

is bijective and condition e 2.1) yield

$x_i \in \text{Bdry}\{\tilde{P}_i(x,y,p) \cap X_i\}$.

Let $\tilde{\varphi}: X \times Y \times \dot{\Omega} \to \dot{\Omega}$ be such that $\tilde{\varphi}_i(x,y,p) \equiv p$ for all $(x,y,p)$, and

$\tilde{\mu}_i: X \times Y \times \dot{\Omega} \to R$ be defined by

$$\tilde{\mu}_i(x,y,p) = \mu_i(x,y,p) - (\omega_i(x,y,p)-p)(x_i - \omega_i).$$

Then the economy.

$\tilde{\mathcal{E}} = (X_i, \omega_i, \tilde{P}_i, Y_j, \mathcal{Y}_j, \tilde{\varphi}_i, \tilde{\mu}_i, \Psi_j)$ satisfies the conditions of Lemma 2, so

it has an equilibrium $(x^*, y^*, p^*)$. We now verify that this is an equilibrium

for $\mathcal{E}$. Clearly only E1 and E3 need to be verified.

We have $p^* x_i^* = p^* \omega_i + \tilde{\mu}_i(x^*, y^*, p^*)$ so

$p^*(x_i^* - \omega_i) = \mu_i(x^*, y^*, p^*) - (\varphi_i(x^*, y^*, p^*) - p^*)(x_i^* - \omega_i)$, and thus

$\varphi_i(x^*, y^*, p^*)(x_i^* - \omega_i) = \mu_i(x^*, y^*, p^*)$. Therefore E 1) is satisfied. Choose

$z \in P_i(x^*, y^*, p^*) \cap X_i$. It must be shown that $\varphi_i(x^*, y^*, p^*)(z_i - \omega_i) > \mu_i(x^*, y^*, p^*)$.

Suppose that $\varphi_i(x^*, y^*, p^*)(z_i - \omega_i) \leq \mu_i(x^*, y^*, p^*)$. Then by e 2.2) there must

be a $z' \in P_i(x^*, y^*, p^*)$ such that $\varphi_i(x^*, y^*, p^*)(z' - \omega_i) \leq \mu_i(x^*, y^*, p^*)$ and

$T_i(x^*, y^*, p^*, z') \in X_i$. Write $z'' = T_i(x^*, y^*, p^*, z') = G(z' - x_i^*) + x_i^* \in X_i$,

where $G$ is the matrix $B(p^*, \varphi_i(x^*, y^*, p^*))$. By construction,

$$\frac{G\varphi_i(x^*, y^*, p^*)}{\|\varphi_i(x^*, y^*, p^*)\|} = \frac{p^*}{\|p^*\|}, \quad \text{and since } G \text{ is orthonormal} \quad \frac{\varphi_i(x^*, y^*, p^*)}{\|\varphi_i(x^*, y^*, p^*)\|} = \frac{p^*}{\|p^*\|} G.$$

Thus $\varphi_i(x^*, y^*, p^*)z' \leqq \varphi_i(x^*, y^*, p^*)x_i^*$ is equivalent to $p^* G z' \leqq p^* G x_i^*$ is

equivalent to $p^*[G(z' - x_i^*) + x_i^*] \leqq p^* x_i^*$, which in turn is equivalent to

$p^* z'' \leqq p^* x_i^*$. But since $z'' \in \tilde{P}_i(x^*, y^*, p^*) \cap X_i$, this contradicts the optimality

of $x_i^*$. Thus E3 is satisfied and the proof is completed.

IV.  NOTES

We conclude with two notes.  The first of these explains an efficient method for proving the existence of equilibrium under the assumption of continuity of consumers' budget correspondences.  It provides an alternative method for achieving the result of Mantel [8] and Shoven [12].  The second note shows how to interpret the model to include the provision of public goods and services.

1.  Assume that preferences can be represented by continuous utility functions and that consumers' budget correspondences are continuous and non-empty valued.  The latter assumptions are strong and as we have pointed out, they are not implied in our framework by the condition that initial endowments lie interior to consumption sets even if the commodity taxes each person pays are returned as a lump sum transfer.  In this case, one can immediately assoc-iate with each economy a generalized  n  person game which satisfies the condi-tions of the Debreu Lemma [3], and such that an equilibrium of the generalized game is a competitive equilibrium for the economy.  (If in fact preferences are not representable  by utility functions, then the lemma communicated in [11] can be applied.)  The substantial point which most distinguishes our result from the treatments of taxation equilibrium provided by Mantel [8] and Shoven [12] and makes it more of a descendant of the work of Sontheimer [13], is that we do not require (or obtain) continuity of the budget correspondence. Furthermore, the natural budget correspondences of our model are frequently empty (see example 2).  The conditions  e 1), e 2),  and "the twist" (see the Theorem) are all directed to proving equilibrium in the absence of contin-uous budget correspondences.

2. To interpret the model to include the provision of public goods and services by the government requires some minor adjustment. Corresponding to each $(x,y) \in X \times Y$ we may suppose that the government has an idea of what public goods it would like to provide at that state. Let $G(x,y) \in - R_+^\ell$ be the set of input vectors which can produce the desired goods and services. If the map $G: X \times Y \longrightarrow - R_+^\ell$ is continuous, convex and nonempty valued, and if there exists a $u \in R^\ell$ such that $u \in G(x,y)$ for all $(x,y)$, then we can treat the government as if it is a firm which has production correspondence $\mathcal{Y}_{m+1} X \times Y \longrightarrow R^\ell = Y_{m+1}$ given by $\mathcal{Y}_{m+1}(x,y) = G(x,y) - \{u\}$. The government will, for each $(x,y,p)$ maximize $pz$ subject to $z \in \mathcal{Y}_{m+1}(x,y)$ which is equivalent to finding that $g \in G(x,y)$ with lowest cost. In this way the government can be considered behaviorally to act as a firm. The definition of the set of attainable sets should then read:

$$A = \{(x,y) \in X \times Y \mid x_i \in X_i, \ i = 1,2,\ldots,n,$$

$$y_j \in \mathcal{Y}_j(x,y) \ j = 1,2,\ldots,m+1$$

$$\text{and} \ \sum_{i=1}^{n} x_i - \sum_{j=1}^{m+1} y_j \leq \sum_i \omega_i + u\}$$

and condition f) of the theorem should read:

$$\sum_i \mu_i(x,y,p) = \sum_i (\varphi_i(x,y,p) - p)(x_i - \omega_i) + \sum_{j=1}^{m+1} p y_j + p u .$$

With public goods introduced in this way, each $P_i(x,y,p) \cap X_i$ can be interpreted as the set of commodity vectors preferred to $x_i$ by consumer i, given the public goods quantities the government wants to produce at state $(x,y)$. Similarly, each $\mathcal{Y}_j(x,y)$ is the set of feasible productions of firm j at state $(x,y)$ and at the corresponding public goods provision the government would provide at state $(x,y)$.

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