Discussion Paper No. 856

# BARGAINING AND THE RIGHT TO REMAIN SILENT

by

Lawrence M. Ausubel*

and

Raymond J. Deneckere**

September 1989

*Department of Managerial Economics and Decision Sciences, J.L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, Illinois 60208.

**This research was supported in part by National Science Foundation Grant SES-86-19012. Financial Assistance was also provided, for Lawrence Ausubel, by the Lynde and Harry Bradley Foundation, and for Raymond Deneckere, by the Kellogg School of Management's Beatrice/Esmark Research Chair. The idea for this paper arose three years ago during a visit of both authors to the Institute for Mathematical Studies in the Social Sciences at Stanford University in the summer of 1986. We would like to thank the IMSSS for its hospitality, and Faruk Gul and Robert Wilson for an especially valuable conversation we had at that time. Any errors in the paper are wholly our responsibility.

*At the outset, if a person in custody is to be subjected to interrogation, he must first be informed in clear and unequivocal terms that he has the right to remain silent. ... The warning of the right to remain silent must be accompanied by the explanation that anything said can and will be used against the individual in court. This warning is needed in order to make him aware not only of the privilege, but also of the consequences of forgoing it. It is only through an awareness of these consequences that there can be any assurance of real understanding and intelligent exercise of the privilege. Moreover, this warning may serve to make the individual more acutely aware that he is faced with a phase of the adversary system -- that he is not in the presence of persons acting solely in his interest.*

--Chief Justice Earl Warren, United States Supreme Court, June 13, 1966, in *Miranda v. Arizona*, 384 U.S. 436, 467-469.

## 1. Introduction

The negotiation process transmits information in at least two ways. First, any time that an informed party responds (positively or negatively) to an existing offer on the bargaining table, he may reveal some of his private information to his partners in the negotiations. Second, whenever that party places his own new counteroffer on the table (or refrains from doing so), the form of the proposal potentially conveys some information. Together, these two vehicles for information transmission may result in the rapid disclosure of the informed party's information.

Consider a bilateral bargaining situation where one of the parties possesses private information which the other party wishes to learn. It is reasonable to think that the first channel ("passive revelation") can be more readily exploited to expose the informed agent's information than can the second channel ("active revelation"). The uninformed agent obtains

# BARGAINING AND THE RIGHT TO REMAIN SILENT

by

Lawrence M. Ausubel*

and

Raymond J. Deneckere*

This paper analyzes a class of alternating-offer bargaining games with one-sided incomplete information. If sequential equilibria are required to satisfy the additional restrictions of stationarity, monotonicity, pure strategies and no free screening, the Silence Theorem is shown to hold: When the time interval between successive periods is sufficiently short, the informed party never makes any serious offers in the play of alternating-offer bargaining games. A class of parametric examples suggests that the time interval required to assure silence is not especially brief.

KEYWORDS: Noncooperative bargaining; alternating offers; stationarity; Coase Conjecture; incomplete information.

---

*Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, Illinois 60208.

information via passive revelation by making an offer which the informed agent finds either attractive or unattractive, depending on his private information, and merely waiting for the informed agent's response. In contrast, active revelation relies on the informed party's willingness to voluntarily choose to frame a proposal which reveals his information. The uninformed party can utilize the first device to force the informed party to disclose; however, the informed party has the option of refraining from making counteroffers, and thus can avoid the second means of information transmission.

In this article, we formally derive a result of this type. Consider the $(k,\ell)$-alternating-offer bargaining game[1] of one-sided incomplete information. Restrict attention to the set of sequential equilibria which satisfy the additional restrictions of stationarity, monotonicity, pure strategies and no free screening.[2] Our main result then is the Silence Theorem: there exists a sufficiently short (but still positive) time interval between successive offers such that ***the informed party never makes any serious counteroffers*** in any of these equilibria. All information revelation then occurs only through passive responses by the informed party to offers of the uninformed party.

Our result thus provides a justification for studying the bargaining game of one-sided incomplete information in which only the uninformed party is permitted to make offers. This game, while extensively and successfully studied in earlier papers,[3] has also been criticized for artificially

---

[1]We introduced this terminology in Ausubel and Deneckere (1989b): k offers by the uninformed agent are followed by $\ell$ counteroffers by the informed agent, whereupon the game repeats until agreement is reached. The (1,1) extensive form is the standard alternating-offer game introduced by Rubinstein (1982).

[2]The assumptions of stationarity, monotonicity, pure strategies and no free screening were introduced by Gul and Sonnenschein (1988).

[3]Papers on the bargaining game with one-sided incomplete information where the uninformed party makes all the offers, and the related problem of durable goods monopoly, include: Bulow (1982); Stokey (1981); Fudenberg and Tirole (1983); Sobel and Takahashi (1983); Fudenberg, Levine and Tirole (1985); Gul, Sonnenschein and Wilson (1986); and Ausubel and Deneckere (1989a,b).

restricting the actions of the informed party.[4] In contrast, our current result establishes that, for an interesting class of equilibria, the outcome of an alternating-offer game is ***as if*** the extensive form permitted offers only by the uninformed party. Exogenously, both traders are permitted to make offers; endogenously, equilibrium counteroffers by the informed party degenerate to null moves.

The intuition for the Silence Theorem is at once simple and compelling. Our restrictions on sequential equilibrium mandate that, at each of his moves in the game, the informed agent partitions the interval of remaining possible valuations into two subintervals (one possibly degenerate).[5] In particular, at times when it is the informed agent's turn to make an offer, the remaining valuations partition into a high subinterval (who ***speak*** by making a serious offer) and a low subinterval (who effectively ***remain silent*** by making a nonserious offer). Now suppose you are the informed party: you have the options of speaking or remaining silent. Choosing to speak reveals a high valuation, which is information that bargaining partners can exploit; remaining silent indicates a low valuation. In short, you recognize that "anything you say can and will be used against you." Therefore, regardless of valuation, you decline to speak, since "you have the right to remain silent."[6]

The existing article most closely related to the present paper, and on which we significantly rely, is that of Gul and Sonnenschein (1988). Gul and Sonnenschein examined the standard (1,1)-alternating-offer game under

[4]For example, see footnote 2 of Grossman and Perry (1986).

[5]The structure in which informed agents' valuations are partitioned into exactly two subintervals at each informed agent move was introduced by Grossman and Perry (1986); it is necessitated by their notion of perfect sequential equilibrium.

[6]The two phrases quoted in this paragraph were taken from the standard "rights card" used by the San Francisco Police Department in the aftermath of the Miranda decision. See American Jurisprudence Proof of Facts, Bancroft-Whitney Co., San Francisco, 1967, Vol. 19, p. 80.

one-sided incomplete information, for the case of a "gap" between the uninformed party's valuation and the (lowest possible) informed party's valuation. They formulated the four restrictions on sequential equilibrium and demonstrated that these imply the "no delay" result: for any $\epsilon > 0$, there exists a sufficiently short (but still positive) time interval between offers such that the probability of trade within time $\epsilon$ exceeds $1 - \epsilon$. Our departure from Gul and Sonnenschein is two-fold. First, we prove a ***uniform version*** of the Coase Conjecture[7] for ***(k,ℓ)-alternating-offer games*** in the case of ***no gap***.[8] Second, we use the uniform Coase Conjecture merely as a lemma in proving our main result, that the informed party chooses never to speak.

Grossman and Perry (1986) examine alternating-offer bargaining in the case of a gap and prove that there exists at most one "perfect sequential equilibrium." Ausubel and Deneckere (1989b) characterize the entire set of sequential equilibria for the (k,ℓ)-alternating-offer game in the case of no gap.[9] Rubinstein (1985) considers alternating-offer bargaining where the uncertainty concerns the rate of time preference of the informed party. In a model with two types, he shows that there is a continuum of sequential equilibria but generally a unique "bargaining sequential equilibrium."

---

[7]The no delay result is closely connected to the Coase Conjecture for durable goods monopoly and bargaining where the uninformed seller makes all the offers. The Conjecture states that, for any $\epsilon > 0$, there exists a sufficiently short (but still positive) time interval between offers such that the initial offer is always within $\epsilon$ of the lowest buyer valuation.

Coase (1972) introduced the intuition for the Conjecture. Gul, Sonnenschein and Wilson (1986) proved the Coase Conjecture to hold for the case of "the gap," and to be true for "no gap" under an assumption of stationarity. Ausubel and Deneckere (1989a) showed the Conjecture to be false without this additional assumption.

In the alternating-offer game, a Coase Conjecture type result (such as Lemma 3.2 below) implies no delay — since the initial offer is very low, buyer acceptance occurs very quickly. For the case of "no gap", it can conversely be shown that the no delay result implies the Coase Conjecture.

[8]The Silence Theorem does not generally hold for the case of a "gap", so it cannot be proved using the no delay result of Gul and Sonnenschein (who explicitly assume a "gap"). Moreover, a uniform version of the Coase Conjecture (i.e., that prices are uniformly low, relative to the state, as the game evolves) is required for proving the Silence Theorem.

[9]We also proved that, for both the gap and no gap cases, the game where only the informed party makes offers has a unique sequential equilibrium.

Admati and Perry (1987) examine a different alternating-offer, extensive-form game which circumvents the no delay result.

The structure of our paper is as follows. In the next section, we describe the model and the equilibrium concept. In Section 3, we prove the Silence Theorem. Section 4 concludes. Proofs of the lemmas are relegated to an Appendix.

## 2. The Model

Consider a situation where two parties are bargaining over the price at which a single item is to be sold. The seller's valuation for the object is common knowledge, for convenience normalized to equal zero. However, the buyer's type, $q \in I = [0,1]$, is private information. Let $f(q)$ be a left-continuous, weakly-decreasing function describing the buyer's valuation. We assume that $f(q) > 0$ for $q < 1$, with $f(0) = 1$ and $f(1) = 0$. The assumption $f(0) = 1$ amounts to a simple normalization. Note, however, that unlike Gul and Sonnenschein (1988), we assume that there is ***no gap*** between the seller's and the lowest buyer's valuation, i.e., $\lim_{q\uparrow 1} f(q) = 0$.[10]

We also assume that q is uniformly distributed on I. This distribution, as well as the valuation function $f(\bullet)$, are common knowledge. Observe that the implied distribution of buyer valuations is given by $F(v) = 1 - \inf\{q: f(q) \le v\}$, and that the support of F is a subset of [0,1] containing the seller's valuation $s = 0$.

The seller and buyer both exhibit impatience; in fact, we assume a common discount rate of r. Thus, if trade occurs at a price p at time t the

---

[10] Any buyer types with valuations below the seller's are not effective players in the game and hence are deleted.

seller derives a net surplus of $pe^{-rt}$, and the buyer (of type q) earns $[f(q) - p]e^{-rt}$.

Players alternate in making offers at discrete moments in time, spaced z apart. The seller proposes in even periods (by convention, the initial period is taken to be zero) and the buyer proposes in odd periods. Immediately after an offer has been made, the other party can either accept or reject the offer. Acceptance terminates the game; rejection yields the opportunity to make a counteroffer in the next period. Let $h_n$ denote an n-period history of prices and rejections, and let $H_n$ denote the set of all $h_n$. Let $h_n'$ denote $h_n$ followed by a price offer in period n, and let $H_n'$ denote the set of all $h_n'$. The strategy of the seller is a sequence of functions $\sigma^s = \{\sigma_n^s\}_{n=0}^{\infty}$, where $\sigma_n^s$: $H_n \rightarrow \Re$ for n even and $\sigma_n^s$: $H_n' \rightarrow \{Y,N\}$ for n odd. Similarly, the strategy for the buyer is a sequence of functions $\sigma^b = \{\sigma_n^b\}_{n=0}^{\infty}$, where $\sigma_n^b$: $H_n' \times I \rightarrow \{Y,N\}$ for n even and $\sigma_n^b$: $H_n \times I \rightarrow \Re$ for n odd. We assume that the buyer's strategy is measurable in the second argument (his type). A strategy profile is denoted by $\bar{\sigma} = \{\sigma^s,\sigma^b\}$.

Let W denote the set of probability distributions on I and let $Z \subset W$ denote the set of uniform distributions on intervals [a,b], where $0 \leq a \leq b \leq 1$. Distributions in Z will be denoted by the endpoints of their supports (e.g., $(a,b) \in Z$). Seller beliefs are defined for each history of the game by functions $g_n$: $H_n \rightarrow W$ and $g_n'$: $H_n' \rightarrow W$. Specifically, $g_n$ denotes the seller's beliefs at the start of period n, and $g_n'$ denotes her beliefs following the offer of the period n. We require that these beliefs do not change after the seller's own move, that is, $g_{n-1}' = g_n = g_n'$ for n even. Finally, let $\bar{g}_n = \{g_n,g_n'\}$ and let $\bar{g} = \{\bar{g}_n\}_{n=1}^{\infty}$

We will say that the pair $\{\bar{\sigma},\bar{g}\}$ is a sequential equilibrium if beliefs are derived through Bayesian updating whenever this is possible, and if strategies are "sequentially rational" in the sense that at every information set a player's strategy maximizes his expected payoff given his beliefs and the strategies of his opponents.

It is well known that in the seller-offer game, where the buyer has no opportunities to make a counteroffer, the seller successively skims through the buyer's possible valuations. A somewhat analogous proposition remains true in even periods of the alternating-offer game.[11]

<u>Lemma 2.1</u>: For any sequential equilibrium and any even number n there exists a function Q: $H_n' \to I$ such that for all $h_n' \in H_n'$, $\sigma_n^b(h_n', q) = Y$ if and only if $q \le Q(h_n')$.

While buyer acceptances thus lead to a truncation of the seller's beliefs concerning the buyer's type, the same need not be true about buyer offers. Following Gul and Sonnenschein (1988), we will henceforth make two assumptions which guarantee that buyer offers also truncate the seller's beliefs.

<u>A.1</u> **(Pure Strategies)**: Along the equilibrium path, the seller's offer and acceptance behavior is deterministic.

<u>A.2</u> **(No Free Screening)**: For all odd n, and for all $h_n \in H_n$, let $\psi(h_n) = \{p: \sigma_n^b(h_n, q) = p \text{ for some } q \in I\}$. Then if $p, p' \in \psi(h_n)$ and $g_n'(h_n, p) \neq g_n'(h_n, p')$, either $\sigma_n^s(h_n, p) = Y$ or $\sigma_n^s(h_n, p') = Y$.

---

[11]The proof of this lemma is standard; see, e.g., Fudenberg, Levine and Tirole (1985, Lemma 1), and Ausubel and Deneckere (1989a, Lemma 2.1).

Assumption 1 guarantees that at any stage of the game both the seller and the buyer make at most one ***serious*** offer (an offer which has positive probability of acceptance).[12] Assumption 2 rules out cheap talk, that is, non-payoff-relevant moves which reveal information. More precisely, the seller is required to form the same update following different ***nonserious*** offers (offers which have zero probability of acceptance). Without loss of generality, we will henceforth assume that there is a unique nonserious offer in each period. The two assumptions taken together imply that the equilibrium paths of our equilibria display a simple and intuitive structure.[13]

Lemma 2.2: For every sequential equilibrium $(\bar{\sigma},\bar{g})$ satisfying A.1 – A.2, there exists a unique nondecreasing sequence $q_0,q_1,q_2,\ldots$ called the ***states*** generated by $\bar{\sigma}$, and for each i there exists a unique $h_i \in H_i$ (and if i is even a unique $h_i' \in H_i'$) that occur with positive probability under $\bar{\sigma}$, such that:

(i) if i is even, then $\sigma_i^b(h_i',q) = Y$ if $q \in (q_i,q_{i+1}]$ and $\sigma_i^b(h_i',q) = N$ if $q \in (q_{i+1},1]$;

(ii) if i is odd, then $\sigma_i^b(h_i,q) = p_i$ if $q \in (q_i,q_{i+1}]$ and $p_i$ is the unique serious offer at i, and $\sigma_i^b(h_i,q) = \bar{p}_i$ if $q \in (q_{i+1},1]$ and $\bar{p}_i$ is the unique nonserious offer at i.

Furthermore, if i is even, then $g_i(h_i) = g_i'(h_i') = (q_i,1)$. If i is odd, then $g_i(h_i) = (q_i,1)$; in addition, $g_i'(h_i,p_i) = (q_i,q_{i+1})$ and $g_i'(h_i,\bar{p}_i) = (q_{i+1},1)$.

---

[12]Indeed, for the seller, A.1 guarantees directly that no randomization occurs, and hence that there is at most one serious offer. The fact that the seller never accepts offers with a probability in (0,1) makes it suboptimal for the buyer to make more than one serious offer in any given period (the lowest offer always dominates).

[13]Actually, to obtain the conclusions of Lemma 2.2, the full strength of A.2 is not needed. It would suffice that A.2 hold along the equilibrium path.

Proof: See Gul and Sonnenschein (1988).

Equilibria satisfying A.1 – A.2 can still be quite complex in the sense that the buyer's offer and acceptance behavior can depend on the entire history of offers and counteroffers. Several authors (Gul, Sonnenschein and Wilson (1986); Gul and Sonnenschein (1988)) have advocated refinements in which the buyer's strategy is Markovian, i.e., where the buyer's behavior is allowed to depend on the previous history only insofar as it affects the current state. While we do not necessarily endorse this notion of stationarity, we believe it is interesting to explore the consequences. We will therefore follow Gul and Sonnenschein (1988) in making the additional assumptions:

A.3 **(Stationarity of the Buyer's Offer Behavior)**: For every n and m odd, $q \in I$, and every $h_n \in H_n$ and $h_m \in H_m$, $g_n(h_n) = g_m(h_m)$ implies $\sigma_n^b(h_n, q) = \sigma_m^b(h_m, q)$.

A.4 **(Monotonicity of the Buyer's Acceptance Behavior)**: For every n and m even, for every $h_n \in H_n$, $h_m \in H_m$, and $q_m \le q_n$ such that $g_n(h_n) = (q_n, 1)$ and $g_m(h_m) = (q_m, 1)$, and for every $p \in \mathfrak{R}$, there exists $p^* \ge p$ such that $\sigma_m^b((h_m, p^*), q) = \sigma_n^b((h_n, p), q)$ for every $q \in I$.

Assumption A.4 is actually a hybrid assumption, requiring not only stationarity of the buyer's acceptance behavior, but also a certain type of monotonicity in this acceptance behavior as a function of the current state.[14] To understand the precise meaning of A.4, let us define, for each n even and each $h_n \in H_n$ with $g_n(h_n) = (q,1)$, the acceptance function $P(h_n,x) = \sup\{p: Q(h_n,p) \geq x\}$. Thus, $P(h_n,x)$ is the highest price a buyer of type x will accept after history $h_n$.

It is straightforward to verify that if m is even, and if $h_m \in H_m$ is such that $g_m(h_m) = (q,1) = g_n(h_n)$, then $P(h_n,\bullet) = P(h_m,\bullet)$. Thus, A.4 implies that the buyer's acceptance is only a function of the current state. Furthermore, it is straightforward to verify that if m is even and, instead, $g_m(h_m) = (q',1)$ for some $q' < q$, then $P(h_m,\bullet) \geq P(h_n,\bullet)$. Thus, the buyer's acceptance behavior is monotone in the sense that the presence of additional high-valuation buyer types (those in the interval $(q',q]$) does not lead a particular buyer type to lower his acceptance price.

One final remark: the reader should observe that, in any stationary sequential equilibrium and in any "cycle" of offer and counteroffer, there must be a positive probability of trade. Indeed, suppose that there were two consecutive periods in which only nonserious offers were made. By stationarity, the buyer would continue to make nonserious offers. The seller then must eventually make a serious offer, since there always exists a positive price which has a positive probability of acceptance. Note that she could have accelerated this offer by two periods, and stationarity would have assured her the same continuation profits. This contradicts the optimality of the seller's strategy.

---

[14] As is evident from our proofs, we really need assume A.4 only for q such that $q = q_n$, that is, for equilibrium states. Furthermore, we only need the existence of $p^* \geq p$ when $Q(h_n,p) = q_{n+1}$.

## 3. The Silence Theorem

In this section, we establish the main result of the paper. It is useful to begin by stating two lemmas, which are proved in the Appendix. The first lemma is closely related to Lemma 3.1(iii) of Grossman and Perry (1986)[15]:

Lemma 3.1: For any valuation function f(•), consider any sequential equilibrium of the alternating-offer bargaining game. Suppose, after any history in which the buyer has not previously deviated, the seller maintains beliefs that the buyer's type is at most $\bar{q}$ (and so the buyer's valuation is at least $f(\bar{q})$). Then the seller will reject any counteroffer less than $(\frac{\delta}{1+\delta})f(\bar{q})$ and will not offer any price less than $(\frac{1}{1+\delta})f(\bar{q})$.

For any distribution of types implied by f(•), any real interest rate r, and any time interval between periods z, let $\Sigma(f,r,z)$ denote the set of sequential equilibria of the alternating-offer game which satisfy Assumptions A.1 – A.4. For $0 < M \leq 1 \leq L < \infty$ and $0 < \alpha < \infty$, let $F_{L,M,\alpha}$ denote the set of all functions f(•) such that $M(1-q)^\alpha \leq f(q) \leq L(1-q)^\alpha$ for all $q \in [0,1]$. This notation permits us to state a second lemma, which is closely related to the main theorem of Gul and Sonnenschein (1988), but instead treats the case of "no gap" and establishes the same type of uniformity as in Theorem 5.4 of Ausubel and Deneckere (1989a):[16]

---

[15] Grossman and Perry's Lemma 3.1 concerns only sequential equilibria which satisfy the support restriction, whereas our Lemma 3.1 treats all sequential equilibria.

[16] The proof of the lemma draws heavily on Gul, Sonnenschein and Wilson (1986), and Gul and Sonnenschein (1988). We learned a lot from these authors, and are glad to be able to acknowledge our intellectual debt here.

<u>Lemma 3.2</u> **(The Alternating-Offer, Uniform Coase Conjecture):** For every $0 < M \leq 1 \leq L < \infty$, $0 < \alpha < \infty$, and $\epsilon > 0$, there exists $\bar{z}(L,M,\alpha,\epsilon) > 0$ such that for every $f \in F_{L,M,\alpha}$, for every z satisfying $0 < z < \bar{z}(L,M,\alpha,\epsilon)$ and for every equilibrium belonging to $\Sigma(f,r,z)$, the initial serious (seller or buyer) offer is less than or equal to $\epsilon$.

An important observation should be made concerning Lemma 3.2. Kreps and Wilson's (1982) definition of sequential equilibrium for finite games implies that whenever (after any history) the seller revises her beliefs about the buyer, she posits a new distribution function that has its support entirely contained in the support of the initial distribution of buyer types. While it may be desirable to also impose this restriction on equilibria of infinite games, the proof of Lemma 3.2 does not depend on such a restriction and, therefore, we will not make such a restriction in the current paper. Rather, we will allow the seller's beliefs to wander outside the initial support of buyer valuations, after histories which have zero probability of occurrence.[17] In the proof of Theorem 3.3, below, this will enable us to restate a version of Lemma 3.2 which holds at the start of *any* (as opposed to just the initial) period following an equilibrium history (where the seller may have narrowed her beliefs so that high-valuation buyer types have zero posteriors).

With our intermediate results in hand, we may now prove the main theorem:

---

[17]For finite games (similar to the infinite game we are considering here), Fudenberg and Tirole (1988) have formulated a notion of equilibrium which is in the spirit of sequential equilibrium but relaxes its consistency requirements. Their notion, termed "perfect Bayesian equilibrium," permits a type with zero prior probability to attain a positive posterior following a zero-probability history. In fact, were it not for the fact that our game is infinite, our requirement of sequentiality would coincide with perfect Bayesian equilibrium.

<u>Theorem 3.3</u> **(The Silence Theorem)**: Let f belong to $\mathcal{F}_{L,M,\alpha}$ and let r be any positive interest rate. Then there exists $\bar{z} > 0$ such that, whenever the time interval between offers satisfies $0 < z < \bar{z}$ and for every equilibrium belonging to $\Sigma(f,r,z)$, the informed party ***never makes any serious offers*** in the play of the alternating-offer bargaining game.

<u>Proof</u>. We begin by demonstrating that a version of Lemma 3.2 also holds along equilibrium histories of the game: there exists $\bar{z} > 0$ such that for every z $(0 < z < \bar{z})$ and for every $(\bar{\sigma},\bar{g}) \in \Sigma(f,r,z)$, the next serious offer ***after a state of q*** is at most $\epsilon f(q)$, where q is any state entering an even-numbered period along the equilibrium path of $(\bar{\sigma},\bar{g})$. The proof is as follows. Let $f_q(\bullet)$ denote the ***rescaled*** residual valuation function from $f(\bullet)$ when the state is $q \in [0,1)$, i.e., $f_q(x) = f[q + (1-q)x]/f(q)$, for all $x \in [q,1]$. As in Lemma 5.3 of Ausubel and Deneckere (1989a), if $f \in \mathcal{F}_{L,M,\alpha}$, then $f_q \in \mathcal{F}_{L',M',\alpha}$, where $L' \equiv L/M$ and $M' \equiv M/L$. At the same time, let $(\bar{\sigma}_q,\bar{g}_q)$ denote the continuation of $(\bar{\sigma},\bar{g})$ from the time that the state reaches q (without prior deviations). Importantly, observe that when all prices and valuations in $(\bar{\sigma}_q,\bar{g}_q)$ are appropriately rescaled upward (via multiplication by $1/f(q)$), $(\bar{\sigma}_q,\bar{g}_q)$ becomes a sequential equilibrium for valuation function $f_q$ (i.e., $(\bar{\sigma}_q,\bar{g}_q) \in \Sigma(f_q,r,z)$).[18] Thus, for positive $z < \bar{z}(L',M',\alpha,\epsilon)$, Lemma 3.2 implies that the initial serious offer in $(\bar{\sigma}_q,\bar{g}_q)$ is at most $\epsilon$. By rescaling, the next serious offer in $(\bar{\sigma},\bar{g})$ after q is at most $\epsilon f(q)$.

---

[18]Observe that this is the sentence of the proof which necessitated our discussion, earlier in this section, of <u>not</u> restricting sequential equilibria to have the property that the seller's revised beliefs be entirely contained in the support of the initial distribution of buyer types. If we had made that restriction in our definition of "sequential equilibrium," then it would not necessarily be the case that $(\bar{\sigma}_q,\bar{g}_q)$ is a sequential equilibrium for valuation function $f_q$. The reason is that $(\bar{\sigma}_q,\bar{g}_q)$ was derived from a sequential equilbrium $(\bar{\sigma},\bar{g})$ for valuation function f. In $(\bar{\sigma},\bar{g})$, after a history in which beliefs are entirely contained in (q,1], it is still possible (off the equilibrium path) for beliefs to subsequently be revised outside (q,1]. This translates to beliefs $\bar{g}_q$ possibly wandering outside the initial support from $f_q$.

We may now easily establish the Silence Theorem. Suppose the theorem did not hold. Then for any $\bar{z} > 0$, there would exist $f \in \mathcal{F}_{L,M,\alpha}$, positive time interval $z < \bar{z}$, sequential equilibrium $(\bar{\sigma},\bar{g}) \in \Sigma(f,r,z)$, and buyer types q and q′ $(0 \leq q < q' < 1)$ with the property that, at some point in the equilibrium, the interval $(q,q']$ of buyers makes a serious counteroffer. To be more precise, there is an odd-numbered period j such that, along the equilibrium path of $(\bar{\sigma},\bar{g})$, the set of buyers remaining at the start of period j is $(q,1]$. In period j, the buyers partition into two nondegenerate subintervals as follows: buyers in $(q,q']$ make a serious counteroffer p; whereas buyers in $(q',1]$ make a nonserious counteroffer.

We now will show that buyer q′ can profitably deviate by mimicking $(q',1]$ in making a nonserious counteroffer.[19] Suppose that q′ follows the prescribed equilibrium. Since q′ reveals himself to be contained in $(q,q']$ when he offers p, the seller immediately comes to maintain beliefs that the valuation of q′ is at least $f(q')$. Since p is defined to be a serious counteroffer, it must be accepted by the seller; by Lemma 3.1, $p \geq (\frac{\delta}{1+\delta})f(q')$. Hence, the payoff (evaluated in period j) to q′ from equilibrium play equals $f(q') - p$, which is bounded above by $(\frac{1}{1+\delta})f(q')$. Alternatively, q′ may deviate by making a nonserious counteroffer. This deviation is undetectable and, hence, the state entering period j + 1 equals q′. By stationarity, the next serious offer must occur in either period j + 1 or j + 2. Let $\bar{z}$ be any positive time interval less than $\bar{z}(L',M',\alpha,\frac{1}{4})$. By the version of Lemma 3.2 proven two paragraphs above, the next serious offer will be at most $\frac{1}{4}f(q')$. Hence, the payoff from deviating is bounded below by $\delta^2[f(q') - \frac{1}{4}f(q')] = \frac{3}{4}\delta^2 f(q')$. Let $\bar{z}$ also be chosen sufficiently small that $\delta$

[19] Since q′ will strictly prefer to deviate and f(•) is left-continuous, it is in fact the case that a <u>positive measure</u> of buyer types in $(q,q']$ can profitably deviate by mimicking types $(q',1]$.

$\equiv e^{-rz}$ satisfies $\frac{3}{4}\delta^2 > (\frac{1}{1+\delta})$, whenever $0 < z < \bar{z}$. Then the payoff from deviating exceeds the equilibrium payoff, providing a contradiction. Q.E.D.

The Silence Theorem holds not only for the (1,1)-alternating-offer game but, in fact, for all alternating-offer games in which k (≥ 1) seller offers are followed by $\ell$ (≥ 0) buyer counteroffers. In $(k,\ell)$-alternating-offer games, the definition of stationarity is appropriately modified by requiring the buyer's offer behavior to depend only on the current state ***and the period modulo $(k + \ell)$***, and the definition of monotonicity is modified similarly.[20] Let $\Sigma^{k,\ell}(f,r,z)$ denote the set of sequential equilibria of the $(k,\ell)$-alternating-offer game which satisfy stationarity, monotonicity, pure strategies and no free screening. Then Lemma 3.2 continues to hold for $\Sigma^{k,\ell}(f,r,z)$. Meanwhile, recall that Lemma 3.1 required that, for $\delta \approx 1$, the seller reject counteroffers less than $\approx \frac{1}{2} f(\bar{q})$. Similarly, for the $(k,\ell)$-alternating-offer game, an analogue to Lemma 3.1 requires the seller to reject counteroffers less than $\approx (\frac{k}{k+\ell})f(\bar{q})$.[21] Hence, the logic behind the proof of Theorem 3.3 carries through for general $(k,\ell)$; we have:

<u>Theorem 3.4</u>: Let f belong to $F_{L,M,\alpha}$ and let r be any positive interest rate. Let $k \geq 1$ and $\ell \geq 0$. Then there exists $\bar{z} > 0$ such that, whenever the time interval between offers satisfies $0 < z < \bar{z}$ and for every equilibrium belonging to $\Sigma^{k,\ell}(f,r,z)$, the informed party ***never makes any serious offers*** in the play of the ***$(k,\ell)$-alternating-offer bargaining game***.

---

[20] To be more precise, A.3 is modified to read: "For every n and m which are periods in which the buyer makes offers and such that $n \equiv m \pmod{k+\ell}$ ..." Similarly, A.4 is modified to read: "For every n and m which are periods in which the seller makes offers and such that $n \equiv m \pmod{k+\ell}$ ..."

[21] To be more precise, in period $n \equiv j \pmod{k+\ell}$, where $k \leq j \leq k+\ell-1$, the seller will reject any counteroffer less than $[\delta^{k+\ell-j}(1-\delta^k)/(1-\delta^{k+\ell})]f(\bar{q})$. In period $n \equiv i \pmod{k+\ell}$, where $0 \leq i \leq k-1$, the seller will not offer any price less than $[1-\delta^{k-i}(1-\delta^{\ell})/(1-\delta^{k+\ell})]f(\bar{q})$. See Ausubel and Deneckere (1989b, Theorem 5).

If k = 0, then we actually find ourselves in the game where the buyer makes all the offers. In Ausubel and Deneckere (1989b, Theorem 4), we proved that this game has a unique sequential equilibrium. All buyer types pool by making an offer demanding all the surplus; this offer is immediately accepted. The Silence Theorem no longer literally holds in this case; nevertheless, the informed party never reveals any information via his own offers.

## 4. Conclusion

Theorem 3.3 tells us that, for discount factors sufficiently close to one, the informed party fully exercises his right to remain silent. However, it provides us no guidance as to whether a particular discount factor is "sufficiently close" to one. For example, at an interest rate of 10% per year and a time interval of a few days between offers, is silence mandatory? To shed some light on this question, we will consider the family of parametric examples which are ***invariant under rescaling***. Let $f(q) = (1 - q)^{1/\alpha}$, for any $\alpha > 0$. Using the formula $F(v) = 1 - f^{-1}(v)$, we see that the valuation function $f(\bullet)$ corresponds to the distribution function $F(v) = v^{\alpha}$. Observe that the rescaled residual valuation function of $f(\bullet)$ is given by $f_q(x) = (1 - x)^{1/\alpha}$; conditional distributions formed by truncation are merely rescaled versions of the initial distribution. For this family of examples, it is natural to examine sequential equilibria which not only satisfy A1 – A4 but also are themselves ***invariant under rescaling***. Additionally, it is sensible to restrict attention to equilibria in which the seller's beliefs are not permitted to wander outside the support [0,1] of the prior distribution $F(\bullet)$.

For the purposes of this Conclusion, let us redefine "state" to now denote the highest remaining buyer ***valuation***. Invariance under rescaling requires that the continuation strategies, starting from any equilibrium state, look the same. Therefore, all offers, counteroffers, acceptance functions and value functions must be linear in the state. Moreover, either there exists a serious buyer counteroffer in every (odd-numbered) period or else the buyer never makes serious counteroffers in any period.

Let us assume for the moment that there does exist a serious buyer counteroffer in every odd-numbered period. (This will be true when $\delta$ is sufficiently small.) The offer-counteroffer structure along the equilibrium path can be described as follows, using constants $0 < \phi, \gamma, \eta, \theta, \mu \leq 1$. When it is the seller's turn to offer and the support of remaining buyer valuations equals the interval $[0,x)$, the seller proposes a price of $\phi x$. Buyers in the subinterval $[\gamma x, x)$ accept, whereas buyers in $[0,\gamma x)$ reject. Of the rejecting buyer types, an upper subinterval $[\theta\gamma x,\gamma x)$ proposes a serious counteroffer of $\eta\theta\gamma x$ in the next period, whereas the lower subinterval $[0,\theta\gamma x)$ makes a nonserious counteroffer and awaits the seller's next offer. Finally, let $V([a,b))$ denote the seller's expected present value of continuing the game when it is her turn to move and the set of remaining buyer valuations is the interval $[a,b)$. Then $V([0,x))$ is given by $\mu x$, where $\mu \equiv V([0,1))$.

The following conditions on the parameters $\{\phi,\eta,\theta,\mu\}$ can easily be derived. First, a buyer of valuation $\theta\gamma x$ must be indifferent between proposing the counteroffer $\eta\theta\gamma x$ and awaiting the offer $\phi\theta\gamma x$ one period later, i.e.,

$$(4.1) \qquad 1 - \eta = \delta\,(1 - \phi)\ .$$

Second, the counteroffer $\eta\theta\gamma x$ must make the seller indifferent between acceptance and continuing the game with beliefs $[\theta\gamma x, \gamma x)$:

$$(4.2) \qquad \eta\theta \;=\; \delta V([\theta,1)) \;.$$

Third, the seller must be optimizing when choosing $p = \phi x$:

$$(4.3) \qquad \mu \;\equiv\; V([0,1)) \;=\; \max_{p} \Big\{ p\Phi([v(p),1)) + \delta\eta\theta v(p)\Phi([\theta v(p), v(p))) + \delta^2\Phi([0,\theta v(p)))V([0,\theta v(p))) \Big\},$$

where $v(p)$ denotes the solution to $v(p) - p = \delta[v(p) - \eta\theta v(p)]$ and where $\Phi([a,b))$ denotes the probability that the buyer's valuation is contained in $[a,b)$ (i.e., $\Phi([a,b)) = b^\alpha - a^\alpha$ whenever $0 \le a \le b \le 1$).

Let $\underline{\delta}(\alpha)$ denote the critical value at which the buyer stops speaking in the class of equilibria we consider when the parameter is equal to $\alpha$. Then for any $\delta < \underline{\delta}(\alpha)$, it must be the case that the subinterval $[\theta\gamma x, \gamma x)$ is nondegenerate, while for $\delta > \underline{\delta}(\alpha)$, the subinterval $[\theta\gamma x, \gamma x)$ is degenerate. Consequently, when $\delta = \underline{\delta}(\alpha)$, the above system of equations must yield a solution with $\theta = 1$. Note, then, that (4.2) reduces to $\eta = \delta V([1,1]) = (\frac{\delta}{1 + \delta})$, Rubinstein's (1982) complete information solution.[22]

Also observe that $\phi$ is the maximizer of (4.3). Maximizing (4.3) and substituting $\theta = 1$ and $\eta = (\frac{\delta}{1 + \delta})$ yields:

---

[22]In any sequential equilibrium, after any history in which the seller's beliefs are concentrated at the upper bound of the support and in which it is the seller's turn to move, she offers a price of $1/(1+\delta)$, which is accepted immediately. This follows from Lemma 3.1, above, and equation (16) of Ausubel and Deneckere (1989b).

$$(4.4) \qquad \phi = (1+\delta)^{-1}\left\{(1+\alpha)[1-\alpha\delta^2(1+\delta)\phi/(1+\alpha)]\right\}^{-1/\alpha}.$$

Finally, from (4.1), we see that $\phi = 1 - [\delta(1+\delta)]^{-1}$. Substituting this into (4.4) and rearranging terms yields:

$$\omega(\delta) \equiv \alpha\delta[\delta(1+\delta)-1]^{1+\alpha} - (1+\alpha)[\delta(1+\delta)-1]^{\alpha} + \delta^{\alpha} = 0.$$

Any solution to $\omega(\delta) = 0$ must satisfy $\delta(1+\delta) - 1 > 0$, that is, $\delta > \frac{1}{2}[\sqrt{5}-1]$. Numerical simulations reveal that $\omega(\bullet)$ has a unique zero in $(\frac{1}{2}[\sqrt{5}-1],1)$; this zero is tabulated for various $\alpha$ in Table 1.[23]

Table 1

**Calculation of $\underline{\delta}(\alpha)$, the maximal discount factor, and $\underline{z}(\alpha)$, the minimal time interval between successive periods, such that the informed party ever speaks in the rescale-invariant sequential equilibrium, when $F(v) = v^{\alpha}$ and the real interest rate r = 10% per year.**

| $\alpha$ | $\underline{\delta}(\alpha)$ | $\underline{z}(\alpha)$ |
|---|---|---|
| .10 | .78805 | 28.58 months |
| .25 | .79891 | 26.94 months |
| .50 | .81458 | 24.61 months |
| 1 | .83929 | 21.02 months |
| 2 | .87271 | 16.34 months |
| 4 | .90976 | 11.35 months |
| 10 | .95164 | 5.95 months |

[23]Wilson (1987) tabulates the parameters of the Grossman-Perry (1986) equilibrium at various values of $\delta$, for the case of the uniform distribution and assuming that the serious buyer counteroffer takes the form $\eta\theta\gamma x = [\delta/(1+\delta)]\theta\gamma x$. While there seems to be no justification for this assumption (other than simulations), Wilson found the same critical discount factor of .83929. This should not come as a surprise to the reader. First, the Grossman-Perry equilibrium satisfies rescaling invariance. Second, at the critical $\delta$, the subinterval $[\theta\gamma x,\gamma x)$ collapses to a single point and, then, Grossman-Perry's support restriction justifies the choice $\eta = \delta/(1+\delta)$.

The numbers in Table 1 should be interpreted as follows. Suppose that the distribution function F(•) is linear and that the real interest rate is 10% per year. Then $\underline{\delta}$ = .83929 is equivalent to saying that, in the sequential equilibrium which is invariant under rescaling, the informed party exercises his right to remain silent whenever the time interval between successive periods is less than 1¾ years. Since the extensive form has the parties alternate in making offers, this means that the buyer is silent unless each party has an opportunity to speak less than once every 3½ years! As F(•) becomes arbitrarily concave, the requisite time interval between offers expands to a limiting value of 29.80 months; as F(•) becomes arbitrarily convex, the requisite time shrinks at a very slow rate toward zero. Even with the rather skewed distribution function $F(v) = v^{10}$, the buyer only speaks when the time interval between periods is greater than half a year.

There is a fairly simple intuition why the requisite time interval between periods is made shorter as the distribution function becomes more convex. The force which discourages the informed party from speaking is that making a serious counteroffer would reveal that he has one of the highest remaining possible valuations. However, if the distribution function is very convex, then the uninformed party ***already*** places a high probability on the event that the informed party has one of the highest remaining possible valuations. [For example, if $F(v) = v^{10}$, the buyer's *ex ante* expected valuation is already 0.909.] In other words, even when the distribution function is made extraordinarily convex, the informed party retains the right to remain silent. Unfortunately, against an opponent who holds a sufficiently unfavorable prior distribution, silence becomes almost equally as incriminating as an admission of guilt.

## Appendix

<u>Proof of Lemma 3.1</u>: Define $\bar{p}$ to be the infimum over all prices that the seller accepts or offers in any sequential equilibrium, after any history in which the buyer has not previously deviated and in which the seller maintains beliefs that the buyer's type is at most $\bar{q}$. Since acceptance is individually rational, the seller never accepts an offer (strictly) less than zero. It then follows from the reasoning in Fudenberg, Levine and Tirole (1985, Lemma 2) that the seller never offers less than zero. This establishes that $\bar{p} \geq 0$, a bound which we will now tighten.

Consider any period in which it is the seller's turn to make an offer, after any history in which the buyer has not previously deviated and in which the seller maintains beliefs that the buyer's type is at most $\bar{q}$. Since the seller never accepts or offers a price less than $\bar{p}$, the surplus to buyer q in the continuation game (following rejection) is bounded above by $\delta[f(q) - \bar{p}]$. Knowing this, all buyer types $q \in [0,\bar{q}]$ accept offers p satisfying $f(\bar{q}) - p > \delta[f(\bar{q}) - \bar{p}]$ or, equivalently, $p < (1 - \delta)f(\bar{q}) + \delta\bar{p}$. Consequently, any seller offer p satisfies $p \geq (1 - \delta)f(\bar{q}) + \delta\bar{p}$.

Suppose that $\bar{p} < (\frac{\delta}{1 + \delta})f(\bar{q})$. Then for any $\epsilon > 0$, there exists p satisfying $\bar{p} \leq p < \bar{p} + \epsilon$ such that after some history in which the buyer has not previously deviated and in which the seller maintains beliefs that the buyer's type is at most $\bar{q}$, the seller accepts or offers the price p. Consider $\epsilon = \frac{1}{2}(1 - \delta^2)[(\frac{\delta}{1 + \delta})f(\bar{q}) - \bar{p}]$. A few lines of algebra show that $p < (1 - \delta)f(\bar{q}) + \delta\bar{p}$; the previous paragraph argues that p cannot be a seller offer. Therefore, p is a buyer offer. The seller has the option of rejecting p and counteroffering $p' = (1 - \delta)f(\bar{q}) + \delta\bar{p} - \epsilon/\delta$. Again by the previous paragraph, p′ is accepted with probability one. Another few lines of algebra demonstrate that $\delta p' = \bar{p} + \epsilon$ and, hence, $\delta p' > p$. This establishes that rejecting p and counteroffering p′ is a profitable seller deviation. We conclude that $\bar{p} \geq (\frac{\delta}{1+\delta})f(\bar{q})$. Finally, since any seller offer p satisfies $p \geq (1-\delta)f(\bar{q}) + \delta\bar{p}$, we have $p \geq (\frac{1}{1+\delta})f(\bar{q})$. Q.E.D.

Proof of Lemma 3.2: For any sequential equilibrium $(\bar{\sigma},\bar{g})$, define the ***effective price function*** $\mathcal{P}$: $[0,1] \to [0,1]$ to associate with every buyer $q \in [0,1]$ the price $\mathcal{P}(q)$ he pays to the seller in the equilibrium. (Under the assumption of "no gap," the buyer $q = 1$ never purchases; for convenience, always define $\mathcal{P}(1) = 0$.) To be more precise, if the interval $(q^k, q^{k+1}]$ of buyers purchases at price $p^k$ in equilibrium $(\bar{\sigma},\bar{g})$, we will say $\mathcal{P}(q) = p^k$ for all $q \in (q^k, q^{k+1}]$. Without loss of generality, we will assume that $\mathcal{P}(\bullet)$ is left continuous and nonincreasing (see Ausubel and Deneckere, 1989a, p. 516).

Suppose that Lemma 3.2 does not hold. Then there exists $\epsilon > 0$, a sequence $\{f_n\}_{n=1}^{\infty} \subset F_{L,M,\alpha}$, a sequence of positive time intervals $\{z_n\}_{n=1}^{\infty} \downarrow 0$, and a sequence of stationary equilibria $\{\bar{\sigma}_n, \bar{g}_n\}_{n=1}^{\infty}$ (with effective price functions $\{\mathcal{P}_n\}_{n=1}^{\infty}$) such that $\mathcal{P}_n(0) > \epsilon$ for all $n \geq 1$. Without loss of generality, we may assume that $\{\mathcal{P}_n\}_{n=1}^{\infty}$ converges pointwise for all rationals in $[0,1]$. (This can be assured by taking successive subsequences and applying a diagonal argument.) For every rational $r \in [0,1]$, let $\Phi(r) = \lim_{n\to\infty} \mathcal{P}_n(r)$. Define $\mathcal{P}(0) = \Phi(0)$ and, for every $x \in (0,1]$, define $\mathcal{P}(x) = \lim_{k\to\infty} \Phi(r_k)$, where each $r_k$ is rational and $r_k \uparrow x$. Observe that $\mathcal{P}(\bullet)$ is well defined, left continuous, and nonincreasing. In the second part of the proof, we will demonstrate that the $\mathcal{P}(\bullet)$ so constructed is necessarily continuous. Let us assume this fact for the moment and show that the supposition of $\mathcal{P}_n(0) > \epsilon$ leads to a contradiction.

Assuming the continuity of $\mathcal{P}(\bullet)$, select $\epsilon', \epsilon''$ and rational $x_1, x_2$ satisfying $0 < x_1 < x_2 < 1$ and $\mathcal{P}(0) \geq \epsilon > \epsilon' > \mathcal{P}(x_1) \geq \mathcal{P}(x_2) > \epsilon'' > 0$. By construction, there exists $\hat{n}$ such that $\mathcal{P}_n(0) > \epsilon$, $\mathcal{P}_n(x_1) < \epsilon'$, and $\mathcal{P}_n(x_2) > \epsilon''$ for all $n \geq \hat{n}$.

Define $t > 0$ such that buyer $q = 0$ is indifferent between a price $\epsilon$ at time zero and a price $\epsilon'$ at time t:

$$f_n(0) - \epsilon = e^{-rt}[f_n(0) - \epsilon'], \tag{A.1}$$

where $f_n(0) = 1$.

Observe that, in every equilibrium $(\bar{\sigma}_n, \bar{g}_n)$ $(n \geq \hat{n})$, no offer less than $\epsilon'$ can be accepted until after time t. (Otherwise, buyer $q = 0$, who purchases at time zero or later, would regret his purchase.) Since price has not dropped below $\epsilon'$ at time t, all buyers $q \geq x_1$ remain in the market at time t.

Following Gul and Sonnenschein (1988), we will now specify an "accelerated strategy" for the seller which compresses sales from time interval $[0,t]$ into the shorter interval $[0, t/2]$. For every n, define $N = \lfloor t/4z_n \rfloor$. Restrict attention to $n \geq \bar{n} \geq \hat{n}$, with $\bar{n}$ defined so that $N > 1$. Let $I^j = [1 - (j/N)(1 - \epsilon'), 1 - ((j - 1)/N)(1 - \epsilon'))$, for $j = 1, \ldots, N$. For each such j, select the smallest serious price offered in $\bar{\sigma}_n$ before time t which is contained in $I^j$ (if one exists). Denote the resulting sequence of descending prices $p^1, \ldots, p^m$ $(1 \leq m \leq N)$ and refer to these as the ***good prices***. For each i $(1 \leq i \leq m)$, let $(q_1^i, q_2^i]$ denote the interval of buyers who purchased at $p^i$ in $\bar{\sigma}_n$.

In the play of the accelerated strategy, let k denote the current period and $q^k$ denote the current state, i.e., the set of buyers remaining at the start of period k is $(q^k, 1]$. Our objective is to induce each of the states

$q_2^i$ $(1 \le i \le m)$ in at most the first 2i periods. When $q^k \ge q_2^m$, the seller will have completed the acceleration phase and continues by inducing exactly the same states as in the original strategy from $\bar{\sigma}_n$.

We now describe the accelerated strategy for $q^k < q_2^m$. Define $i(k) = \min\{i \le m: q_2^i > q^k\}$. The following prices, $\hat{p}^k$ and $O^k$, can be defined if $q^k$ is an equilibrium state arising from $(\bar{\sigma}_n, \bar{g}_n)$.[24] If $p^{i(k)}$ was named by the ***seller*** in the original equilibrium, let $\hat{p}^k$ be a seller offer which induces a state of $q_2^{i(k)}$ when the current state is $q^k$. If $p^{i(k)}$ was named by the ***buyer*** in the original equilibrium, let $\hat{p}^k$ be a seller offer which induces a state of $q_1^{i(k)}$ when the current state is $q^k$. (The existence of $\hat{p}^k$ is guaranteed by monotonicity; furthermore, $\hat{p}^k \ge p^{i(k)}$.) Also, let $O^k$ be the serious buyer counteroffer when the state is $q^k$, if a serious counteroffer exists; otherwise, define $O^k = 1$. The seller's strategy when $q^k < q_2^m$ is defined to be:

(A.2) If k is even:

- Offer $\hat{p}^k$.

If k is odd:

- Accept any counteroffer of at least $O^k$.
- Reject any lower counteroffer.

Observe, as in Gul and Sonnenschein (1988), that the following facts hold under the accelerated strategy:

(i) All trades (with any buyer type) which would have occurred in time interval $[0,t)$ under $(\bar{\sigma}_n, \bar{g}_n)$ now occur no later and at prices no more than $(1 - \epsilon')/N$ lower; and

[24] Observe that it is sufficient to specify the accelerated strategy only for states which arise if the buyer does not deviate from his equilibrium strategy.

(ii) All trades (with any buyer type) which would have occurred in time interval $[t,\infty)$ under $(\bar{\sigma}_n,\bar{g}_n)$ now occur at least t/2 sooner and at the same or higher prices.

Statement (i) follows from the fact that the states induced under the accelerated strategy are a subsequence of the states under $(\bar{\sigma}_n,\bar{g}_n)$, and any trade which originally occurred at a price in interval $I^j$ still occurs at a price in the same (or higher) interval. Statement (ii) follows from the fact that the state is brought beyond $q_2^m$ in at most 2m periods and, hence, before a time of $2mz_n \leq 2Nz_n \leq t/2$. Monotonicity guarantees that all sales beyond $q_2^m$ occur at the same or higher prices (but are accelerated by time t/2).

The acceleration strategy thus entails a loss in revenues from buyers $q \in [0,q_2^m]$ but provides a gain due to discounting from buyers $q \in (q_2^m,1]$. Observe that $q_2^m < x_1$ for all $n \geq \bar{n}$. By (i), the monetary loss is less than $(1 - \epsilon')/N$ and the probability of loss is less than $x_1$. Hence, expected losses are bounded above by $x_1(1 - \epsilon')/N \leq [4x_1(1 - \epsilon')/(t - 4z_n)]z_n$.

Let V denote the seller's expected payoff in $(\bar{\sigma}_n,\bar{g}_n)$ starting from when the state is $q_2^m$. V can be bounded below by $e^{-2rz_n}\epsilon''[(x_2 - x_1)/(1 - x_1)]$, as follows. Let $(q',q'']$ be the interval of buyers who purchase with $x_2$ at the price $\mathcal{P}(x_2)$. When the state is $q_2^m$ and it is the seller's move, the seller has the following option: if $\mathcal{P}(x_2)$ was a seller (buyer) offer in $(\bar{\sigma}_n,\bar{g}_n)$, the seller charges a price ($\geq \mathcal{P}(x_2)$) which induces a state of $q''$ ($q'$). In the latter event, buyers in $(q',q'']$ reject the seller's offer and counteroffer $\mathcal{P}(x_2)$, which is accepted. This assures the seller an expected payoff of:

$$e^{-rz_n}\mathcal{P}(x_2)[(q'' - q_2^m)/(1 - q_2^m)] > e^{-rz_n}\epsilon''[(x_2 - x_1)/(1 - x_1)].$$

Meanwhile, when the state is $q_2^m$ but it is the buyer's move, the seller can assure herself the same payoff, only discounted by one period, giving the desired lower bound.

The *ex ante* probability that the state will reach $q_2^m$ is greater than $(1 - x_1)$. The continuation profits upon reaching $q_2^m$ are accelerated from a time not earlier than t to a time not later than t/2, and equal at least V. Hence, the expected gains are bounded below by:

$$(e^{-rt/2} - e^{-rt})(1 - x_1)V > (e^{-rt/2} - e^{-rt})e^{-2rz_n}\epsilon''(x_2 - x_1).$$

Since $\lim_{n\to\infty} z_n = 0$, the expected gains from acceleration exceed the expected losses for sufficiently large n, demonstrating that acceleration is a profitable deviation from $\bar{\sigma}_n$. Thus, our initial assumption that $\mathcal{P}(\bullet)$ was continuous leads to a contradiction.

It remains to be shown that $\mathcal{P}(\bullet)$ is continuous. Suppose otherwise. Since $\mathcal{P}_n(q) \le f_n(q) \le L(1 - q)^\alpha$, for all n and q, it follows that $\mathcal{P}(q) \le L(1 - q)^\alpha$ and hence that $\mathcal{P}(\bullet)$ is continuous at 1. Therefore, there exists x $(0 \le x < 1)$ where $\mathcal{P}(\bullet)$ is discontinuous. Define d = $[\mathcal{P}(x) - \lim_{q\downarrow x} \mathcal{P}(q)]/3$, $\hat{\epsilon} = \mathcal{P}(x) - d$ and $\epsilon' = \mathcal{P}(x) - 2d$. If $x \ne 0$, select $\eta$ $(0 < \eta < x)$ such that $x - \eta$ is rational and $\eta'$ $(0 < \eta' < \min(\eta, 1 - x))$ such that $x + \eta'$ is rational. Also, for any $\gamma \in (0,\eta')$, choose rational $q_\gamma^h \in (x - \gamma/2, x)$ and rational $q_\gamma^\ell \in (x, x + \gamma/2)$. Meanwhile, if x = 0, merely select $\eta'$ $(0 < \eta' < 1)$ such that $\eta'$ is rational. Always choose $q_\gamma^h = 0$ and, for any $\gamma \in (0,\eta')$, choose rational $q_\gamma^\ell \in (0, \gamma/2)$. By construction, for each $\gamma$, there exists $\hat{n}_\gamma$ such that $\mathcal{P}_n(q_\gamma^h) > \hat{\epsilon}$ and $\mathcal{P}_n(q_\gamma^\ell) < \epsilon'$ for all $n \ge \hat{n}_\gamma$. Define t by:

$$\text{(A.3)} \qquad f_n(0) - \hat{\epsilon} = e^{-rt}[f_n(0) - \epsilon'],$$

where $f_n(0) \equiv 1$. Observe that, in every equilibrium $(\bar{\sigma}_n, \bar{g}_n)$ $(n \geq \hat{n}_\gamma)$ a time interval exceeding t must elapse from the moment that $q_\gamma^h$ purchases until the moment that $q_\gamma^\ell$ purchases. ((A.3) implies that $f_n(q_\gamma^h) - \wp_n(q_\gamma^h) < e^{-rt}[f_n(q_\gamma^h) - \wp_n(q_\gamma^\ell)]$.)

We will now specify an accelerated strategy for the seller which compresses all sales from time interval $(t_{n,\gamma}, t_{n,\gamma} + t]$ into the shorter interval $(t_{n,\gamma}, t_{n,\gamma} + t/2]$, where $t_{n,\gamma}$ denotes the time that $q_\gamma^h$ trades in $(\bar{\sigma}_n, \bar{g}_n)$. Similar to the first part of the proof, define $N = \lfloor t/4z_n \rfloor - 1$ and restrict attention to $n \geq \bar{n}_\gamma \geq \hat{n}_\gamma$, with $\bar{n}_\gamma$ defined so that $N > 1$. Define intervals $I^j$ $(j = 1,\ldots,N)$ exactly as before. For each j, select the smallest serious price offered in $\bar{\sigma}_n$ during time interval $(t_{n,\gamma}, t_{n,\gamma} + t]$ which is contained in $I^j$ (if one exists). Denote the resulting sequence of descending prices $p^1,\ldots,p^m$ $(1 \leq m \leq N)$ and define intervals $(q_1^i, q_2^i]$ as before. We will now induce each of the states $q_2^i$ $(1 \leq i \leq m)$ in at most the first $2i + 1$ periods after $t_{n,\gamma}$.

The seller's accelerated strategy is as follows: during time interval $[0, t_{n,\gamma}]$, use the original strategy from $\bar{\sigma}_n$. Beginning at time $t_{n,\gamma} + z_n$, and until the state reaches $q_2^m$, follow the strategy of (A.2). After the state has reached $q_2^m$, the seller continues by inducing the same states as in the original equilibrium $(\bar{\sigma}_n, \bar{g}_n)$.

Fact (i) from above now holds for time interval $(t_{n,\gamma}, t_{n,\gamma} + t]$. Fact (ii) now holds for time interval $(t_{n,\gamma} + t, \infty)$. Additionally, all trades which would have occurred in time interval $[0, t_{n,\gamma}]$ under $\bar{\sigma}_n$ occur identically under the accelerated strategy.

The accelerated strategy may entail a loss in revenues from buyers $q \in (q_\gamma^h, q_2^m]$ but provides a gain due to discounting from buyers $q \in (q_2^m, 1]$. Observe that $q_2^m < q_\gamma^\ell$ and, hence, the probability of loss is less than $(q_2^m - q_\gamma^h) < (q_\gamma^\ell - q_\gamma^h) < \gamma$. Hence, expected losses are bounded above by $\gamma(1 - \epsilon')/N \leq [4\gamma(1 - \epsilon')/(t - 8z_n)]z_n$, discounted from time $t_{n,\gamma}$.

Let V denote the seller's expected payoff in $(\bar{\sigma}_n, \bar{g}_n)$ when the state is $q_2^m$. V can be bounded below by:

$$e^{-rz_n}(1 - e^{-rz_n})(M/2)[(1 - x)/2]^\alpha[(1 - x - \gamma)/(1 - x - \gamma/2)],$$

as follows. When the state is $q_2^m$, the seller has the option of waiting at most one period and offering a price of $(1 - e^{-rz_n})f_n((1 + x)/2) \geq (1 - e^{-rz_n})M[(1 - x)/2)]^\alpha$. Buyers $q \in (q_2^m, (1 + x)/2]$ find this to be an offer they cannot refuse, since the payoff from acceptance dominates obtaining the good for free in the next period. The probability that $q \in (q_2^m, (1 + x)/2]$, conditional on a current state of $q_2^m$, equals $[(1 + x)/2 - q_2^m]/[1 - q_2^m] > (1/2)(1 - x - \gamma)/(1 - x - \gamma/2)$, yielding the desired lower bound.

Since $q_2^m$ is reached with a probability of greater than $(1 - x - \gamma/2)$ and subsequent revenues are at least V and are accelerated by at least $t/2$, the expected gains are bounded below by $(e^{-rt/2} - e^{-rt})e^{-rz_n}(1 - e^{rz_n})(M/2)$ $[(1 - x)/2]^\alpha(1 - x - \gamma)$, discounted from time $t_{n,\gamma}$. For sufficiently large n, note that $1 - e^{-rz_n} > (r/2)z_n$.

We conclude that there exist $k_1, k_2$ $(0 < k_1, k_2 < \infty)$ such that losses are less than $k_1\gamma z_n$ and gains are greater than $k_2 z_n$. Since $\gamma$ can be made arbitrarily small, the expected gains can be made to exceed the expected losses, demonstrating that $\mathcal{P}(\bullet)$ cannot be discontinuous. Q.E.D.

## References


Admati, A. and M. Perry (1987), "Strategic Delay in Bargaining," Review of Economic Studies, 54, 345-364.

Ausubel, L. and R. Deneckere (1989a), "Reputation in Bargaining and Durable Goods Monopoly," Econometrica, 57, 511-531.

Ausubel, L. and R. Deneckere (1989b), "A Direct Mechanism Characterization of Sequential Bargaining with One-Sided Incomplete Information," Journal of Economic Theory, 48, 18-46.

Bulow, J. (1982), "Durable-Goods Monopolists," Journal of Political Economy, 90, 314-332.

Coase, R. (1972), "Durability and Monopoly," Journal of Law and Economics, 15, 143-149.

Fudenberg, D., D. Levine and J. Tirole (1985), "Infinite Horizon Models of Bargaining with One-Sided Incomplete Information," in Game Theoretic Models of Bargaining, A. Roth (ed.), Cambridge University Press, London, 73-98.

Fudenberg, D. and J. Tirole (1983), "Sequential Bargaining with Incomplete Information," Review of Economic Studies, 50 (2), 221-247.

Fudenberg, D. and J. Tirole (1988), "Perfect Bayesian and Sequential Equilibria: A Clarifying Note," mimeo, MIT.

Grossman, S. and M. Perry (1986), "Sequential Bargaining under Asymmetric Information," Journal of Economic Theory, 39, 119-154.

Gul, F. and H. Sonnenschein (1988), "On Delay in Bargaining with One-Sided Uncertainty," Econometrica, 56 (3), 601-612.

Gul, F., H. Sonnenschein and R. Wilson (1986), "Foundations of Dynamic Monopoly and the Coase Conjecture," Journal of Economic Theory, 39, 155-190.

Kreps, D. and R. Wilson (1982), "Sequential Equilibria," Econometrica, 50, 863-894.

Rubinstein, A. (1982), "Perfect Equilibrium in a Bargaining Model," Econometrica, 50, 97-109.

Rubinstein, A. (1985), "A Bargaining Model with Incomplete Information about Time Preferences," Econometrica, 53, 1151-1172.

Sobel, J. and I. Takahashi (1983), "A Multi-Stage Model of Bargaining," Review of Economic Studies, 50, 411-426.

Stokey, N. (1981), "Rational Expectations and Durable Goods Pricing," Bell Journal of Economics, 12, 112-128.

Wilson, R. (1987), "Game-Theoretic Analyses of Trading Processes," in Advances in Economic Theory, T. Bewley (ed.), Cambridge University Press, London, 33-70.