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BRANCHING PROCESSES IN SIMPLE RANDOM WALK

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ABSTRACT. Let $N(a)$ be the number of overcrossings of height a in a simple random walk. For $p < \frac{1}{2}$, the process $N(0)$, $N(1)$, ... is a branching process, which eventually becomes extinct. For $\frac{1}{2} < p$, $N(0)$, $N(1)$, ... is a stationary process which is a branching process with immigration.

1. <u>Introduction</u>. We show that a certain branching process and a branching process with immigration arise in simple random walk when $p \neq \frac{1}{2}$. By simple random walk we mean the sequence of random variables, S_0, S_1, ...,

$$S_0 = 0, \quad S_n = X_1 + \ldots + X_n, \quad n = 1, 2, \ldots$$

where the X_i's are independent and identically distributed,

$$X_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p = q \end{cases}$$

<u>Definition 1</u>. An <u>overcrossing</u> of height a takes place at time n if

$$S_n = a, \quad S_{n-1} = a+1, \quad S_i = a \text{ for some } i < n.$$

<u>Definition 2</u>. $N(a)$ denotes the total number of overcrossings of height a. ($N(a)$ is finite if $p \neq \frac{1}{2}$.)

2. <u>Statement of theorems</u>.

<u>Theorem 1</u>. For $p < \frac{1}{2}$, $N(0)$, $N(1)$, ... evolves as a branching process with

$$E\, t^{N(0)} = (1 - p/q)/(1 - pt/q) \qquad \underline{1/}$$

$$E(\, t^{N(a+1)} \mid N(a) = k) = [q/(1 - pt)]^k, \quad a,k = 0, 1, \ldots$$

$\underline{1/}$ Without further notice, the dummy variables in all generating functions are assumed to be less than 1 in absolute value.

(In other words, each of the elements of the preceding generation independently gives rise to a random number of new elements, according to the progeny generating function q/(1 - pt).)

Theorem 2. For $\frac{1}{2} < p$, $N(0)$, $N(1)$, ... evolves as a branching process, with immigration, with

$$E\ t^{N(0)} = (1 - q/p)/(1 - qt/p)$$

$$E(t^{N(a+1)}\mid N(a) = k) = [p/(1 - qt)]^k[p/(1 - qt)], \quad a,k = 0, 1, \ldots$$

(In other words, each of the elements of the preceding generation independently gives rise to a random number of new elements, according to the progeny generating function p/(1 - qt); in addition, in each generation there is an influx of a random number of new individuals by immigration according to the same generating function p/(1 - qt).)

3. Proofs. The proofs of Theorems 1 and 2 will proceed through a series of steps which follow.

Step 1. Define

T = time at which first overcrossing of 0 takes place. (For $p \neq \frac{1}{2}$, $P(T < \infty) < 1$.)

$V(a)$ = total number of overcrossings of a up to time T.

$R_n(i_1, \ldots, i_n)$ = probability of all paths which start at height 1, overcross height 1 i_1 times, overcross height 2 i_2 times, ... , overcross height n i_n times; do not overcross height 0 enroute, and end up eventually at height 1.

$V_n(s_1, \ldots, s_n) = E(\ s_1^{V(1)} \cdot \ldots \cdot s_n^{V(n)}\mid T < \infty)$

Step 2.

a) $P(T < \infty) = \begin{cases} p/q, & p \le \frac{1}{2} \\ q/p, & \frac{1}{2} \le p \end{cases}$

b) $E(\ t^{V(1)}\mid T < \infty) = \begin{cases} q/(1 - pt), & p \le \frac{1}{2} \\ p/(1 - qt), & \frac{1}{2} \le p \end{cases}$

<u>Proof.</u> a) is a standard fact about simple random walk. Part b) follows from a) by the computation,

$$P(V(1) = k,\ T < \infty) = \begin{cases} (p/q)(p \cdot 1)^k q = p^{k+1}, & p \leq \tfrac{1}{2} \\ 1 \cdot (p \cdot q/p)^k q = q^{k+1}, & \tfrac{1}{2} \leq p \end{cases}$$

<u>Step 3.</u>

$$E(\ s_2^{V(2)} \cdot \ \ldots \ \cdot s_n^{V(n)} \mid V(1) = k,\ T < \infty) = [V_{n-1}(s_2,\ \ldots\ ,\ s_n)]^k$$
$$k = 0,\ 1,\ \ldots$$

<u>Proof.</u> If k=0 then $V(2) = \ldots = V(n) = 0$ and the assertion holds, so assume now that $1 \leq k$. Suppose that $p \leq \tfrac{1}{2}$. From the definition of R_n,

$$P(\ V(1) = i_1,\ \ldots\ ,\ V(n) = i_n,\ T < \infty) = (p/q)R_n(i_1,\ldots,i_n)q$$

For $1 \leq k$, we also have

$$P(\ V(1) = k,\ V(2) = i_2,\ \ldots\ ,\ V(n) = i_n,\ T < \infty) =$$

$$\Sigma(p/q)[pR_{n-1}(j_{12},\ldots,j_{1n})q] \cdot \ldots \cdot [pR_{n-1}(j_{k2},\ldots,j_{kn})q]q$$

where the summation is over all vector sums of k (n-1)-tuples such that

$$(j_{12},\ldots,j_{1n}) + \ldots + (j_{k2},\ldots,j_{kn}) = (i_2,\ldots,i_n)$$

Hence, computing generating functions, we have

$$E(s_1^{V(1)} \cdot \ldots \cdot s_n^{V(n)}\ ;\ V(1)=k,\ T < \infty) =$$

$$(p/q)[qE(s_2^{V(1)} \cdot \ldots \cdot s_n^{V(n-1)};\ T<\infty]^k q$$

from which the result follows by dividing through by $P(V(1)=k, T < \infty) = p^{k+1}$. The proof for $\tfrac{1}{2} \leq p$ is similar and is left to the reader.

<u>Step 4.</u>

$$V_n(s_1,\ \ldots\ ,\ s_n) = V_1(s_1 V_{n-1}(s_2,\ \ldots\ ,\ s_n))$$

<u>Proof.</u> $E(s_1^{V(1)} \cdot \ldots \cdot s_n^{V(n)} \mid T < \infty) =$

$$\sum_k E(s_1^{V(1)} \cdot \ldots \cdot s_n^{V(n)} \mid V(1)=k,\ T < \infty)P(V(1) = k \mid T < \infty) =$$

$$\sum_k s_1^k[V_{n-1}(s_2,\ \ldots\ ,s_n)]^k P(V(1)=k \mid T < \infty) \quad \text{(by Step 3)}$$

This last expression equals the right side of the assertion.

Step 5. Let $Y(0)$, $Y(1)$, $Y(2)$, ... be a branching process with $Y(0) = 1$ and

$$E(t^{Y(n+1)} \mid Y(n) = k) = F^k(t)$$

Define the joint generating function of $Y(1)$, ... , $Y(n)$ to be

$$W_n(s_1, \ldots , s_n) = E(s_1^{Y(1)} \cdot \ldots \cdot s_n^{Y(n)})$$

Then,

$$W_n(s_1, \ldots , s_n) = F(s_1 W_{n-1}(s_2, \ldots, s_n))$$

The joint distribution of $Y(1)$, ... , $Y(n)$ is uniquely determined by this iterative relationship.

Proof.

$$E(s_1^{Y(1)} \cdot \ldots \cdot s_n^{Y(n)}) = \sum_k E(s_1^{Y(1)} \cdot \ldots \cdot s_n^{Y(n)} \mid Y(1)=k) P(Y(1)=k) =$$

$$\sum_k s_1^k [W_{n-1}(s_2, \ldots, s_n)]^k P(Y(1)=k) = F(s_1 W_{n-1}(s_2, \ldots, s_n))$$

Step 6. Under the condition that $T < \infty$, the process $V(1)$, $V(2)$, ... coincides in distribution with a branching process $Y(1)$, $Y(2)$, ... , where $Y(0) = 1$ and

$$E(t^{Y(n+1)} \mid Y(n)=k) = \begin{cases} [q/(1 - pt)]^k , & p \leq \tfrac{1}{2} \\ [p/(1 - pt)]^k , & \tfrac{1}{2} \leq p \end{cases} \quad n,k = 0, 1, \ldots$$

Proof. It follows from Steps 4 and 5 and 2(b) that

$$E(s_1^{V(1)} \cdot \ldots \cdot s_n^{V(n)} \mid T < \infty) = E(s_1^{Y(1)} \cdot \ldots \cdot s_n^{Y(n)})$$

for all $n = 1, 2, \ldots$.

Completion of proof of Theorem 1. Suppose that $p < \tfrac{1}{2}$. If $N(0) = 0$ then automatically $N(1) = N(2) = \ldots = 0$. Suppose that $N(0) = k > 0$. Define

$$(V_i(1), V_i(2), \ldots , V_i(n)) = V_i$$

to be the number of overcrossings of heights 1, 2, ..., n between the time of the i-1 -st and i -th overcrossing of height 0. ($V_1(a)$ is the same as $V(a)$ as defined earlier.) Then,

a) the random vectors V_1, V_2, ... are independent and identically distributed in the sense that

$$P(V_1=v_1, \ldots , V_k=v_k \mid N(0)=k) = P(V_1=v_1 \mid T<\infty)\cdot\ldots\cdot P(V_k=v_k \mid T<\infty)$$

$$P(V_i = v) \text{ does not depend on } i.$$

b) $N(a) = V_1(a) + \ldots + V_{N(0)}$ if $N(0)$ is positive.

c) $P(N(0) = k) = (p/q)^k(1 - p/q), \quad k = 0, 1, \ldots$

By Step 6, the proof of the theorem is now complete.

Completion of proof of Theorem 2. For $\frac{1}{2} < p$, it is no longer true that if $N(0) = 0$ that $N(1)$, $N(2)$, ... are also zero. We must now be concerned with the overcrossings of height $a+1$ <u>after</u> the last overcrossing of height a. The number of such overcrossings plays the role of the immigration into the population at each generation. An easy computation shows that

$$P(N(1) = k \mid N(0)=0, S_0=0) = pq^k, \quad k = 0, 1, \ldots$$

For the rest, the proof is similar to that of Theorem 1 and we leave the details to the reader.

4. **Complements.** Suppose that $p < \frac{1}{2}$. A direct calculation shows that

$$P(N(a)=k) = \begin{cases} 1 - (p/q)^{a+1} \\ (p/q)^a(p/q)^k(1 - p/q), & 0 < k \end{cases}$$

Hence,

$$G_a(t) = E\, t^{N(a)} = 1 - (p/q)^{a+1}(1-t)/(1-pt/q) \tag{4.1}$$

Let $F(t)$ denote the progeny generating function,

$$F(t) = q/(1 - pt)$$

Since $N(a)$ evolves as a branching process, we must have that

$$G_a(t) = G_0(F^{(a)}(t)), \quad a = 1, 2, \ldots \tag{4.2}$$

where $F^{(a)}$ is the a-fold iteration given by

$$F^{(a)}(t) = F(F^{(a-1)}(t)), \quad a = 2, 3, \ldots$$

It is easy to verify directly by induction that (4.1) satisfies (4.2).

For $\frac{1}{2} < p$ we have that

$$E\, t^{N(a)} = (1 - q/p)/(1 - qt/p) = G(t), \quad a = 1, 2, \ldots$$

(Since N(a) is a Markov chain it follows that it is a strictly stationary process.) Since N(0), N(1), ... evolves as a branching process with immigration, we must have that

$$G(t) = G(F(t))p/(1 - qt) \tag{4.3}$$

reflecting the relationship between N(a+1) and N(a) stated in Theorem 2, with

$$F(t) = p/(1 - qt)$$

It is easy to verify that (4.3) holds directly.

For $p < \frac{1}{2}$, N(a) must equal 0 for sufficiently large a. This is consistent with the fact that

$$\frac{d}{dt} q/(1 - pt) \bigg|_{t=1} = p/q < 1$$

In other words, if the expected number of progeny is less than 1, the branching process becomes extinct with probability 1.

5. References. The elementary facts that are needed about random walk and about branching processes can be found in [1].

[1] Feller, W, (1968) An Introduction to Probability Theory and Its Applications, Vol. I, 3rd ed., New York, Wiley.