Discussion Paper No. 309

A 'TRADE OUT OF EQUILIBRIUM' MODEL OF THE STOCK MARKET, I : TRADITIONAL BEHAVIOR

by

Lawrence R. Glosten*

January 1978

* I would like to thank Professors John Roberts, Erhan Çinlar and Theodore Groves for their assistance and advice without implicating them in any of the remaining faults. This research was funded in part by a Richard D. Irwin Foundation Fellowship, and by the National Science Foundation under grant number SOC76-20953.

1. INTRODUCTION

This paper presents a model of the exchange of two assets--one risky, one risk-free--by means of a trading process similar to those found in the New York Stock Exchange. The salient feature is that attention is focused on the actual process, occurring through time or, as Garman (6) terms it, on the "temporal microstructure."

The trading process is used to construct a stochastic process of endowments and prices. This stochastic process is a pure jump Markov process, and hence analysis of the motion of the system can be carried out on the imbedded Markov chain. The analysis is further simplified by the fact that only integral amounts of stock are traded at prices restricted to a finite set, and every allocation is feasible. This implies that the Markov process has a finite state space.

The investor behavior that is assumed in this paper might be termed traditional behavior. Specifically, investors ignore the possibility of speculating on future price changes and ignore the informational content of past transactions. This corresponds to the model traditionally used in the finance literature in which a share of stock is assumed to have some exogenously given random value, or random return. Similarly, in discussions of the stability of the stock market, the distinction between "good" and "bad" speculation has been made where "good" (stable) speculation represents the behavior associated with buying when the price is below the "value of the stock" and selling when it is above (21). This, in fact, corresponds to the behavior assumed here.

Since a very simplistic form of behavior is assumed, the model presented here represents a minimal test of the consistency of the proposed trading process. Under the assumption that speculation (in the usual sense) and updating of information are ruled out, we show that if only unit trades of the stock are allowed, and if the set of allowable prices at which trades take place is finite, then an equilibrium is reached in an almost surely finite number of steps. Furthermore, the equilibrium has certain optimality properties. The essentially static behavior of investors also allows us to compare the trading process outcomes with the competitive outcome. Finally, an example is presented and further properties of the trading process are discussed with reference to this example.

The aim of the research initiated here is to examine the foundations of the efficient markets hypothesis by explicitly considering the organization of the stock market and the behavior of investors faced with this particular institutional arrangement. One criticizm that has been leveled at empirical tests of the efficient markets hypothesis is that the lack of an understanding of the equilibration process in the stock market has led to a lack of a solid foundation on which to base empirical studies ((4), p. 416). In another vein, assuming that markets are strongly efficient (all privately held information is revealed by prices) leads to serious theoretical problems. As Grossman has pointed out, (10) if there is a cost associated with obtaining information, and the market is strongly efficient, then no one will pay to obtain information as long as they can view the equilibrium prices. But then, the market is trivially efficient. This of course, relies on a "no-trading-out-of-equilibrium" assumption. Even though the market may adjust to an information revealing price, the fact that adjustment takes time implies that informed investors may be able to earn a return on their investment in information. Finally, the assumption that stock prices form a geometric Brownean motion is used extensively in the literature on the stock market, particularly with reference to option pricing. The efficient markets hypothesis lies at the heart of this assumption, and hence it would be useful to understand the conditions under which efficiency is an appropriate assumption.

This research is most akin to that body of literature which studies the pricing behavior of a firm facing an unknown demand (see (18) for an excellent survey). As in that research, prices are determined and changed by the actions of market participants, rather than by an uninvolved auctioneer or "invisible hand." The major difference between these studies and the current one essentially involves the differences in institutional framework and the types of markets considered. In contrast to the uncertain demand literature, this research involves a market in which there are not two obviously identifiable and stable goups of buyers and sellers. While the study of firms facing random demand assumes assymetrical behavior for buyers and sellers, such a dichotomy is not appropriate here. Related to this last point is the fact that there is no geographical separation between buyers and sellers (or at least no separation between their brokers). This leads to quite different rules of trading, and much different behavior. Finally, non-competitive behavior arises quite naturally in the study of consumer goods markets, for firms in the process of adjusting prices begin to perceive a demand curve upon which they can base their decisions. It does not appear so far that non-competitive behavior arises quite so naturally in the stock market.

Apart from the interest in determining how markets become efficient, there have been several empirical studies that indicate that the fact that trade takes place out of equilitrium may be of interest in and of itself. Studies by Latané (15) and Latané and Jones (16) have shown that the adjustment to new information may take a matter of months. Thus, if the information aspect of markets is taken into consideration, the disequilitrium aspects of trading may be important.

2. INSTITUTIONAL BACKGROUND

Studies of the stock market have shown that the specialist participates in trades that involve about seventy-five per cent of the volume of trading ((17) p. 146), either as a broker (arranging trades for other investors) or as a dealer (buying and selling on his own account). In the model presented here, the specialist's role is enlarged so that the specialist acts as broker in all trades, playing the neutral role of matching up investors who are willing to trade. That is, when an investor reports a buy price higher than some other investor's ask price, the specialist arranges a trade between the two, according to specified rules.

In the model presented here, the specialist maintains a book which lists the individuals who are currently willing to buy at each price and sell at each price, and the quantities each will supply or demand. Furthermore, within each price, the investors are ordered according to the sequence in which the orders arrived. Now, suppose an individual reports buy and sell prices to the specialist. If the buy price is higher than some selling price recorded in his book, then the specialist engineers a trade between the arriving investor and the investor first in line among those with the lowest selling price. By convention, this trade takes place at the selling price of the individual on the book. After the trade has been consumated, the selling price of the trader registered with the specialist is erased from the book, and the selling prices of the reporting investor is entered. A similar process is followed if the selling price of the arrival is lower than some buying price on the specialist's book. If, on the other hand, the reporting investor's buying price is lower than all selling prices, and his selling price is higher than all buying prices listed with the specialist, then no trade is possible, and the investor's buy and sell prices are recorded in the specialist's book.

Apart from the expanded role of the specialist, the primary differences between the above process and the actual trading process on the New York Stock Exchange involve the range of alternative types of orders submitted to the specialist. In the process described above, only limit buy orders (buy at a specified price or below) and limit sell orders (sell at a specified price or above) are allowed, while on the stock market, a much broader range of orders are allowed. Furthermore, it is assumed that buy and sell orders are submitted at the same time, while on the market this may or may not be the case. Essentially, then, somewhat less flexibility is allowed in the trading process modelled here, but the fundamental structure of trading is quite similar.

In addition to the requirement that all trades take place through the specialist, several additional simplifying assumptions are made. First, it is assumed that at all times individuals have either both buying and selling prices or neither a buy or sell price registered with the specialist. That is, when an individual listed on the book sells or buys, both his buying and selling prices are removed from the book. Similarly, when an investor arrives at the specialist's post, he reports both buying and selling prices. This departure from reality is made for two reasons. First, it simplifies the analysis somewhat by yielding a simpler state space. Secondly, and more importantly, it prevents agents from making disadvantageous trades. For example, suppose an agent is registered with the specialist, and at some point sells a unit of the stock. This changes his endowment, and will ingeneral change his buy and sell prices. Under some circumstances the buying price with the new endowment may be lower than his old buying price, in which case, a trade at the old buying price may lead to a reduction in his well-being.

A further assumption, in addition to requiring that allocations of the stock be in integral amounts is that trades only take place in unit lots. This greatly simplifies the mathematical description of the motion of the system, although by further restricting the nature of trading it may in general eliminate some final allocations that could be reached if orders were merely restricted to integral amounts. Only in the case of certain types of non-convex preferences does this assumption prevent the attainment of "optimal" (in the sense to be defined) states.

Finally, borrowing and short selling are ruled out. That is, each investor must have non-negative holdings of each of the assets at all times. A more realistic assumption would be that individuals could borrow up to some amount determined by a margin requirement. As a first approximation, however, the much simpler assumption is made.

3. MODEL OF THE TRADING PROCESS

In order to model this process, it is first necessary to define the state of the market. Essentially, the state will be defined to be a summary of the specialist's book, including the price at which the last trade was made and a specification of the endowments of all investors. Let $\mathcal{M}$ be the finite set of names of all agents. Buy and sell prices are restricted to some finite set Φ. By taking Φ to be of the form $\{0, \varepsilon, 2\varepsilon, \ldots, Q\varepsilon\}$ it is guarranteed that after any number of trades, the number of dollars of risk free asset held will be in Φ as long as the initial endowment of the risk-free asset is in Φ. In the case of the New York Stock Exchange, ε is one eighth of a cent or in some cases one sixteenth of a cent.

Since the trade of fractional shares is ruled out, and since holdings are required to be nonnegative, there is no loss in generality in the assumption that Φ is bounded. No agent can buy at a price higher than, say, the total amount of money in the world, which is a large, but finite number. If transactions involving fractional shares were allowed, then an investor could conceivably buy an arbitrarily small fraction of a share for an arbitrarily high price per share.

For $B \subset \mathcal{M}$ define P_B to be the set of all permutations of elements of the set B, with $P_\phi = \{ \phi \}$. Define

$S_m = \{(n_o, n_1, \ldots, n_Q) \in N^{Q+1} : \Sigma n_i = m\}$, and

$S'_m = \{(n_o, n_1, \ldots n_Q, n_{Q+1}) \in N^{Q+2} : \Sigma n_i = m\}$ where N^Q is the Q-fold product of the nonnegative integers. An element, n_i, of a vector in S_m is interpreted as giving the number of buying prices on the book which are equal to the ith element of Φ. The interpretation of S'_m

is analogous except that the extra dimension of the set S'_m allows for the selling prices of those investors holding no shares of stock. Since allocations of stock are required to be nonnegative, by convention, the selling price of an individual holding no shares is some number higher than any allowable price. It is assumed that $0 \in \Phi$, so no extra dimension is needed for the set S_m.

Let $A_S = \{(S^1,\ldots,S^I) \in N^I: \Sigma S^i = \Sigma S_0^i = \overline{S}\}$ be the set of feasible allocations of shares, and

$A_W = \{(W^1,\ldots,W^I) \in \Phi^I: \sum_{i=1}^{I} W^i = \sum_{i=1}^{I} W_0^i = \overline{W}\}$, be the set of feasible allocations of the risk free asset. Then we define the state space E by:

$$E = [\bigcup_{B \subset \mathcal{M}} (P_B \times S_{|B|} \times P_B \times S'_{|B|})] \times A_W \times A_S \times \Phi$$

where $|B|$ indicates the number of elements in the set B.

A typical state space vector is of the form:

$$((i_1,\ldots,i_m),(n_1,n_2,\ldots,n_r,0,0,\ldots,0),(i'_1,i'_2,\ldots,i'_m),$$

$$(0,\ldots,0,n_{r+1},n_{r+2},\ldots,n_Q,n_{Q+1}),(S^1,S^2,\ldots,S^I),(W^1,\ldots,W^I),P).$$

The first element of a state space vector gives the agents having listed buy prices, ordered lexicographically by the magnitude of the price (in descending order) and the order of arrival. The second gives the distribution of buying prices. The third and fourth elements have a corresponding interpretation for selling prices, with lower prices listed first. The fifth element gives holdings of the risky asset, the sixth gives holdings of the risk-free asset, and the final element is the price at which the last trade was made.

As an example, suppose $\Phi = \{0,.5,1,1.5,2,2.5,3\}$

$$\mathcal{M} = \{1,2,3,4\} \qquad \overline{S} = 4 \qquad \overline{W} = 20.$$

Then a possible state is:

$$e = ((1,3,4),(0,2,0,1,0,0,0),(3,4,1),$$

$$(0,0,0,0,2,0,0,1),(10,3,5,2),(10,1,2,1),1.5)$$

Thus, investor one has the highest buying price and it is 1.5, both three and four have buying prices of .5, three entered his order before four and two is not registered. Similarly, three and four have the lowest selling price equal to 2 and three is first in line to trade. Investor one has no shares to sell hence his selling price is outside of Φ. Finally, investor one has no shares and ten dollars of risk-free asset, two has one share and three dollars, three has two shares and five dollars, four has one share and two dollars, and the last trade took place at a price of 1.5.

It will be useful later to define the following functions. Let $\pi^i : E \to \Phi \times N$ be the projection of E onto the ith components of A_W and A_S. That is, $\pi(e) = (\pi^i(e))_i$ is the allocation associated with e, and $\pi^i(e)$ is investor i's allocation. Let $\beta(e)$ be the highest buying price associated with e and $\alpha(e)$ be the lowest selling price. Let $\overline{\eta}(e)$ be the name of the individual first in line to buy and $\underline{\eta}(e)$ the individual first in line to sell. Finally, let P(e) be the price at which the last trade took place.

The state described above is only observable by some omniscient observer of the market. Not even the specialist knows the endowments of all investors, and hence he only knows the portion of the state of the market described in his "book". The only portions of the process observable by investor i are the processes $(\pi^i(e_t)$ and $(P(e_t))_t$. Note that unless a trade takes place at time t, $(\pi^i(e_t))_t$ and $(P(e_t))_t$ do not jump at t.

Having specified the state of the market, the motion of the system can be described. Suppose that in the current state of the market, e_n, not all investors have buy and sell prices registered with the specialist. Individuals not registered with the specialist or wishing to alter their buy and sell prices calculate bid and ask prices, report these to their brokers who then inform the specialist. For each individual i, this takes some random amount of time T_n^i. A natural assumption in this case is that given e_n, T_n^i is exponentially distributed with parameter $\lambda^i(e_n)$. That is $P\{T_n^i > t | e_n\} = e^{-\lambda(e)t}$. By putting $\lambda^i(e) = 0$ if i does not wish to report buy and sell prices to the specialist in the state e, we can then define T_n^i for all i where $P\{T_n^i = +\infty | e_n\} = 1$ when $\lambda^i(e_n) = 0$. Define $T_n = \min\{T_n^1, T_n^2, \ldots, T_n^I\}$. Then T_n is the time at which the system will move out of state e_n. Note that assuming T_n^i and T_n^j are independent give e_n for $i \neq j$ implies that:

$$\begin{aligned} P\{T_n > t | e_n\} &= P\{\min(T_n^1, T_n^2, \ldots, T_n^I) > t | e_n\} \\ &= P\{T_n^1 > t,\ T_n^2 > t, \ldots, T_n^I > t | e_n\} \\ &= P\{T_n^1 > t | e_n\}\ P\{T_n^2 > t | e_n\} \ \ldots \ P\{T_n^I > t | e_n\} \\ &= e^{-\sum_{i=1}^{I} \lambda^i(e_n)t} \quad \equiv \quad e^{-\gamma(e_n)t} \end{aligned}$$

That is, given e_n, T_n is exponentially distributed with parameter

$$\gamma(e_n) = \sum_{i=1}^{I} \lambda^i(e_n) .$$

While it may be difficult to argue that T_{n+1}^i is exponentially distributed given e_n, it is possible to justify the fact that T_{n+1} is an exponential random variable. Once it has been argued that T_{n+1} should be exponentially distributed, assuming $(T_{n+1}^i)_i$ is an independent family of exponentially distributed random variables is convenient and does not lead to a result contradicting the assumption that T_{n+1} is an exponential random variable. The argument relies on a result in the theory of the convergence of probability measures, and is presented in an appendix.

Suppose now that $T_n = T_n^i$, that is, person i is the first to reach the specialist. Suppose his bid and ask prices are respectively $b_n^i(e_n)$ and $a_n^i(e_n)$. In general it will be assumed that the asking price is greater than the buying price. A sufficient condition for this to be the case is that investors are risk averse. If $b_n^i(e_n) \geq \alpha(e_n)$ then person i will buy one share from person $\underline{\eta}(e_n)$ at price $\alpha(e_n)$ and investor $\underline{\eta}(e_n)$ will have his bid and ask price erased from the book. If $a_n^i(e_n) \leq \beta(e_n)$ then individual i will sell to investor $\overline{\eta}(e_n)$ at a price $\beta(e_n)$ and investor $\overline{\eta}(e_n)$ will have both buy and sell prices erased from the book. In either of the above cases, person i will trade and hence neither his buy nor sell price will be registered with the specialist. Finally, if $a_n^i(e_n) > \beta(e_n)$ and $b_n^i(e_n) < \alpha(e_n)$, the specialist will enter individual i's buying and selling prices in his book, and no trades take place.

Denote by e_{n+1} the next state reached as a result of this process. Now, again define T_{n+1}^i for all i, to be exponentially and independently distributed with parameters $\lambda^i(e_{n+1})$. Then $T_{n+1} = \min\{T_{n+1}^1, T_{n+1}^2, \ldots, T_{n+1}^I\}$ and the process continues.

Depending on the assumptions concerning individual behavior, this set-up may or may not make sense. If the only determinant of λ^i is i's endowment and whether or not he is represented in the specialist's book, then the "memory-lessness" of the exponential distribution implies that there is no problem with defining T_n^i and T_{n+1}^i. To see this, suppose i is not represented and $T_n^i \neq T_n$, then:

$$P\{T_n^i > T_n + t | e_n, T_n^i > T_n\} = P\{T_n^i > t | e_n\} = \lambda^i(e_n)$$
$$= \lambda^i(e_{n+1}) = P\{T_{n+1}^i > t | e_{n+1}\}.$$

If, on the other hand, λ^i depends, say, on whether or not a trade took place, and further, if a trade took place, at what price it occurred, then the interpretation is changed. In this case, the arrival of a bid and ask price automatically cancels any previous bid and ask prices not yet reported to the specialist. Indeed, if all of the I investors are actually traders on the floor of the exchange, then even this interpretation is not unappealing. Since the specialist is forbidden to show his book to any traders on the floor, floor traders are likely to react immediately to any information that becomes available concerning the state of the market, alterning their buy and sell prices.

Whether or not $(e_t)_t$ is a Markov process depends upon the calculation of buy and sell prices. For example, if buy and sell prices depend both on the current allocation as well as all past prices, then in general $(e_t)_t$ will not be Markov. If, on the other hand, buy and sell prices depend only on the allocation and the last price, and the parameter of the distribution of T_n^i depends only on the current state, then it is clear that $(e_t)_t$ is Markov.

In the case that $(e_t)_t$ is Markov, it is conceptually easy (although possibly computationally difficult) to calculate the transition probabilities, P_t. The derivation of P_t requires only the parameter function governing the jump times, and the transition probability matrix for the discrete time embedded Markov chain. Let the interarrival times $(T_n)_n$ be defined on a probability space (Ω, A, P), and for $w \in \Omega$ define the random variable $N_t: \Omega \to N$ by $N_t(w) = n$ on

$$\{w | \sum_{i=1}^{n} T_i(w) \leq t < \sum_{i=1}^{n+1} T_i(w)\} .$$

Further define $e_{N_t} = e_t$. Then $(e_n)_n$ is a Markov chain and shows each jump of the process e_t. If Q is the transition matrix of $(e_n)_n$ define the matrix A by

$$A(i,j) = \begin{cases} -\gamma(i) & i = j \\ \gamma(i)Q(i,j) & i \neq j \end{cases} \quad \text{for } i,j \in E.$$

Then by Kolmogorov's backward and forward equations (see Cinlar, pp. 253-5)

$$\frac{d}{dt} P_t = AP_t \ , \ \frac{d}{dt} P_t = P_t A$$

Hence, by a well-known result in Markov process theory, $P_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$

There are two ways to incorporate finite memory and preserve some element of the Markov structure of the process. The first method relies on the assumption that investors can remember back T units of time. Define the state at time t, f_t, to be

$$f_t = (e_u)_{t-T \leq u \leq t}$$

In this set up, a state is a function, $(f_t)_t$ is a Markov process, and the state space is enlarged to be the set of functions on [0,T] taking values in E. The state space is no longer finite and is in fact uncountably infinite. Furthermore, there does not appear to be any meaningful way to construct an imbedded discrete time process that is Markov. In any event, this is probably not the correct formulation of the notion of "memory". What is important is not what the investor can actually remember, but rather, what data he has at his disposal. On the market, this data consists of a sequence of prices, or, within the framework of this model, a sequence of past jumps of the process (e_t). This suggests that the appropriate way to incorporate memory is to assume that investors recall the previous n trades. In this case, redefine the state at time m, e_m to be $\hat{e}_m = (e_{m-n}, e_{m-n+1}, \ldots, e_m)$ and expand that state space to E^m. The n, $(\hat{e}_m)_m$ is a Markov chain, with a larger though still finite state space.

To fix the idea of the trading process, consider the state e in the example above, $e_n = (1,3,4),(0,2,0,1,0,0,0),(3,4,1),(0,0,0,0,2,0,0,1),(0,1,2,1),(10,3,5,2),1.5)$.

Suppose investor two's order arrives at the specialist's post and suppose that $a^2(e_n) > 1$. Now, either: $b^2(e_n) \geq 2$ and $a^2(e_n) \leq 1.5$, in which case 2 buys from 3 or sells to 1; or $b^2(e_n) < 2$ and $a^2(e_n) > 1.5$, in which case he does not trade. Suppose $b^2(e_n) = 2.5$. Then the new state is

$$e_{n+1} = ((1,4),(0,1,0,0,0,0,0),(4,1),(0,0,0,0,1,0,0,1),(0,0,3,1),(10,1,7,2),2)$$

If, on the other hand $a^2(e_n) = 1.5$, then

$$e_{n+1} = ((3,4),(0,2,0,0,0,0,0),(3,4),(0,0,0,0,2,0,0),(1,0,2,1),(8.5,4.5,5.2),1.5)$$

Finally, if $b^2(e_n) = 1.5$ and $a^2(e_n) = 2.5$ then

$$e_{n+1} = ((1,2,3,4),(0,2,0,2,0,0,0),(3,4,2,1),(0,0,0,0,2,1,0,1),(0,1,2,1),(10,3,5,2),1.5)$$

In order to specify the state equations for investors' endowments, it is convenient to define the discrete time processes $(W^i_n)_n(S^i_n)_n$ analogously to $(e_n)_n$, namely $W^i_{N_t} = W^i_t$, and $S^i_{N_t} = S^i_t$. In what follows, it is assumed that buy and sell prices depend only on the previous state. Then,

$$\begin{aligned} W^i_n = W^i_{n-1} + I_{\{T_n=T^i_n\}}[\beta(e_{n-1})1_{[0,\beta(e_{n-1})]}(a^i_{n-1}(e_{n-1})) - \alpha(e_{n-1})1_{[\alpha(e_{n-1}),Q\varepsilon]}(b^i_{n-1}(e_{n-} \\ (b^i_{n-1}(e_{n-1})) + \alpha(e_{n-1})1_{\underline{\eta}(e_{n-1})}(i)\sum_{j\neq i} I_{\{T_n=T^j_n\}}1_{[\alpha(e_{n-1}),Q\varepsilon]}(b^j_{n-1}(e_{n-1})) \\ - \beta(e_{n-1})1_{\overline{\eta}(e_{n-1})}(i)\sum_{j\neq i} I_{\{T_n=T^j_n\}}1_{[0,\beta(e_{n-1})]}(a^j_{n-1}(e_{n-1})); \end{aligned}$$

$$\begin{aligned} S^i_n = S^i_{n-1} + I_{\{T_n=T^i_n\}}[1_{[\alpha(e_{n-1}),Q\varepsilon]}(b^i_{n-1}(e_{n-1}) - 1_{[0,\beta(e_{n-1})2}(a^i_{n-1}(e_{n-1})] + \\ + 1_{\eta(e_{n-1})}(i)\sum_{j\neq i} I_{\{T_n=T^j_n\}}1_{[0,\beta(e_{n-1})]}(a^j_{n-1}(e_{n-1})) - \\ - 1_{\underline{\eta}(e_{n-1})}(i)\sum_{j\neq i} I_{\{T_n=T^j_n\}}1_{[\alpha(e_{n-1}),Q\varepsilon]}(b^j_{n-1}(e_{n-1})). \end{aligned}$$

That is, an investor's current endowment is the sum of his previous endowment and the net trade made at the last jump of the market process. Investor i traded at the last jump (jump n) if either 1) his bid and ask price were reported to the specialist at jump n, $I_{\{T_n=T^i_n\}} = 1$, and his bid price was greater than the lowest ask price, $1_{[\alpha(e_{n-1}),Q\varepsilon]}(b^i_{n-1}(e_{n-1})) = 1$, or his ask price was lower than the lowest bid price, $1_{[0,\beta(e_{n-1})]}(a^i_{n-1}(e_{n-1}))=1$; 2) investor i had previously reported to the specialist, and has the lowest asking price, $1_{\underline{\Pi}(e_{n-1})}(i) = 1$, and someone arrived with a higher buying price, or 3) investor i has the highest buying price, $1_{\overline{\Pi}(e_{n-1})}(i) = 1$, and someone arrived with a lower asking price.

4. BEHAVIORAL ASSUMPTIONS

Immediately after a trade, each investor has three decisions to make: what bid and ask prices to report and, through his choice of λ, the expected value of the time until his order reaches the specialist. For most of the modelling to be done, it will be assumed that λ^i will take on one of two values. If investor i has buy and sell prices registered with the specialist, and he does not wish to alter these buy and sell prices, then $\lambda^i(e)$ will be zero. If, on the other hand, investor i is not registered with the specialist, or i wishes to change his buy and sell prices, then $\lambda^i(e) = \lambda$, some predetermined constant.

Though there is reason to believe that, in some models of individual behavior, the timing of orders may be a strategic consideration, this can be handled through the particular buy and sell prices reported. By reporting lower buy prices, and higher sell prices, the investor can (probabalistically) delay the time until he is involved in a trade. Thus, it appears that little is lost by making the above assumption concerning the choice of λ^i. This specification has the added benefit of ruling out the choice of $\lambda^i = +\infty$. If this were allowed, then T^i would be zero, a.s., and the process $(e_t)_t$ would have instantaneous states.

It is difficult to define reasonable rules of behavior for individuals who recognize that the market is not in equilibrium. The fact that investors know something about the way the trading process works, opens the way for a wide range of strategic behavior, and to a large extent, analysis of the stock market and the motion of prices rests on the analysis of the strategies used by investors. As an example, the 1929 crash has been attributed to widespread speculative behavior on the part of investors (5), while calmer times have been attributed to the dominance of more conservative investors interested only in the underlying "value" of the stock (21). Indeed, Zeeman's catastrophe theory of the stock market (22) relies on the hypothesis that chartist speculation is destabilizing while fundamentalist behavior is stabilizing. Thus, it may be unreasonable to expect results concerning the motion of prices to be particularly robust with respect to changes in behavioral assumptions. In fact, the result shown in this paper that trade will stop after an a.s. finite amount of time is probably due as much to the unrealistic and naive behavior that will be ascribed to investors, as it is to the lack of change in the expected payoff.

In this paper, both speculation and the fact that traders may gain some information from the process are ignored. The type of behavior assumed can, at best, be called conservative, and perhaps more accurately, naive; however, it does allow examination of the actual trading process in the simplest of environments.

The investor views the stock as having some underlying value around which the actual price may fluctuate. The value, however, is not known with certainty, and his somehow related to the real, as opposed to the financial operation of the firm that issues the stock. The ith investor has access to some information about the firm, and this, along with an initial allocation of the risk-free asset $W_o^i \in \Phi$ and shares $S_o^i \in N$, is the datum of the problem. Formally, let V be a random variable defined on a probability space, (Ω', A', P'), denoting the value of a share of stock. Each investor i has some information about V which is represented by a σ-algebra on Ω', $H^i \subset A'$. Then, if the ith investor has W^i dollars of risk-free asset, and S^i shares of stock, he views his random wealth to be $W^i + S^iV$. The investor is assumed to have preferences over gambles involving wealth and these preferences satisfy assumptions sufficient to imply the existence of a von Neumann-Morgenstern utility function U^i. That is, if H^i is a σ-algebra representing i's initial information, then, $E[U^i(W^i + S^iV) | H^i]$ represents i's preferences given the information H^i.

Note, that $E[U^i(W^i + S^iV) \mid H^i]$ is a random variable, depending on what information is received. In what follows, the initial information is considered fixed. That is, a particular realization of the above random variable is picked, and to simplify the notation, define:

$$E[U^i(W^i + S^iV) \mid H^i](\omega) = E^i[U^i(W^i + S^iV)] = V^i(W^i, S^i).$$

Admittedly, a somewhat peculiar investor has been defined. Though the trading process occurs over time, the investor ignores any temporal aspect to the problem, and in particular he ignores the possibility of capital gains as a result of trade out of equilibrium. There are several possible vignettes one might draw to justify such behavior. In the examples below, suppose that the risk-free asset is money.

In the first example, suppose that a share of stock represents fractional ownership of some short term operation. During its life, this operation pays no dividends, and at the end of its usefulness it will be liquidated. If $\bar{S}$ shares of stocks are issued and the firm is unlevered, then one share entitles the owner to $1/\bar{S}$ of the book value at the time of liquidation. If the lifetime of the firm is short enough, then the role of discounting could be ignored and the individual's random wealth, if he ignores the possibility of speculation is the sum of the amount of money he has plus the value of his share holdings. This latter amount is the number of shares times the value per share. An important qualification is that the lifetime of this operation is left unspecified. Though it will be shown that the trading process will converge in a finite amount of time, the time to convergence is not bounded. Though the probability that the time of liquidation is before an absorbing state is reached can be made arbitrarily small, it cannot be made zero if the lifetime is finite.

As another example, suppose the investor is concerned only with the dividend income that results from owning a stock. Then, ignoring the possibility of future trades, his current wealth is the discounted stream of dividends per share times the number of shares plus the amount of money he currently holds.

Whether or not these stories are interesting, the reason for this particular formulation is its managability. Assuming the particular form of simplistic behavior facilitates the analysis of the trading process. In future research, behavior will be more realistic and hence the models analyzed will be inherently more interesting

Given the above specification of investor behavior, buy and sell prices are defined so that the investor will be willing to buy at any price less than or equal to his buying price and sell at any price greater than or equal to his selling price. Specifically, the buying price $b^i(W^i,S^i)$ is defined to be the solution to:

$$\min_{q \in \Phi} E^i[U^i(W^i-q+(S^i+1)V)]$$

$$\text{s.t. } 1)\ E^i[U^i(W^i-q+(S^i+1)V)] > E^i[U^i(W^i+S^iV)]$$

$$2)\ W^i-q \geq 0.$$

Note that by 2), if the investor holds none of the risk free asset, his buying price is zero.

If the individual owns no shares of stock, the asking price is some price greater than any element of Φ. If $S^i > 0$ then the selling price $a^i(W^i,S^i)$ is the solution to:

$$\min_{q \in \Phi} E^i[U^i(W^i+q+(S^i-1)V)]$$

$$\text{s.t. } E^i[U^i(W^i+q+(S^i-1)V)] > E^i[U^i(W^i+S^iV)].$$

The strict inequality in the constraint is used to guarantee that trades lead to strict improvement. The fact that Φ is finite guarantees the existence of solutions. Note that if the investor is risk averse, then $a^i(W^i,S^i) > b^i(W^i,S^i)$.

The final choice that the investor makes is the function λ^i. Since the model of investor behavior essentially rules out strategic behavior as far as the timing of orders goes, the definition of buy and sell prices implies that the investor should place an order as soon as possible after a trade, as long as he is not represented on the specialist's book. Furthermore, there is never a reason to change an existing order on the books. Therefore,

$$\lambda^i(e) = \begin{cases} \lambda \text{ if i is not represented on the specialists book in e} \\ 0 \text{ otherwise} \end{cases}$$

for some pre-specified $\lambda > 0$.

We have thus specified all the data of the trading process for the particular behavior under consideration. Furthermore, an equilibrium of this process can be defined. Recall that $\gamma(e) = \Sigma\lambda^i(e)$ is the

parameter governing the distribution of jump times of the process. Following the terminology of the theory of Markov processes, a state e is defined to be absorbing if $\gamma(e) = 0$. The state e is absorbing in the sense that once that state is reached, the time until the next jump is infinite a.s. That is, the system never moves from that state. By the definition of γ, $\gamma(e) = 0$ if and only if $\lambda^i(e) = 0$ for all i; that is , if and only if all investors have orders registered with the specialist, and hence only if $b^i(W^i, S^i) < a^j(W^j, S^j)$ for all i and j. This leads to the following definition:
A state e is called an equilibrium state if $\gamma(e) = 0$.

Before showing the result that equilibrium is reached in a finite amount of time, it is useful to examine the optimality properties of an equilibrium state. Examination of the trade process indicates that trade will continue as long as there are investors willing to trade and that it will stop when there are no longer any mutually advantageous trades given the restrictions to the set of prices Φ and to unit lots. This suggests that in a limited sense, an equilibrium state should enjoy some optimality properties.

With convexity of preferences, the restriction to unit trades causes no problems, for if at a particular allocation an investor is willing to sell or buy n shares at a price of q per share, convexity implies that he is willing to sell or buy one share at a price of q per share. Hence, if a redistribution of n shares is possible at some rate of exchange in Φ, then a redistribution of one share and risk-free asset is possible through the trade process.

The restriction of prices to the set Φ does, however, require modification of the optimality concept. It is easy to construct examples where, limited to prices in Φ, no trade is possible but in a larger set of allowed prices, a mutually advantageous trade is possible. Unfortunately, there is no way to enlarge Φ enough while maintaining finiteness to avoid this problem entirely.

Finally, the usual notion of Pareto optimality is not the appropriate one. Since trading requires strict improvement, there may be the possibility that a redistribution could be accomplished that would make one person better off and leave everyone else indifferent, Since trade only results if two people can make themselves strictly better off by trading together, this reallocation could not be accomplished through the trading process.

The above discussion motivates the following definition.

Definition

An allocation $(W^i,S^i)_i$ is a Pareto optimum relative to Φ if there does not exist another allocation $(\hat{W}^i,\hat{S}^i)_i$ with $\hat{S}^i \in N$ for all i, and $S^i-\hat{S}^i \neq 0$ implies $\left|\frac{\hat{W}^i-W^i}{\hat{S}^i-S^i}\right| \in \Phi$, such that

1) $V^i(\hat{W}^i,\hat{S}^i) \geq V^i(W^i,S^i)$ for all i and $V^i(\hat{W}^i,\hat{S}^i) > V^i(W^i,S^i)$ for at least one i;

and $(\hat{W}^i,\hat{S}^i)$ is feasible. That is,

2) $\Sigma\hat{S}^i = \Sigma S^i$

3) $\Sigma\hat{W}^i = \Sigma W^i$.

if the first inequality in 1) is required to be a strict inequality for all i, then (W^i,S^i) will be called a strict Pareto optimum relative to Φ. Note that due to the indivisibilites, the set of Pareto optima relative to Φ is contained, but not equal to the set of strict Pareto optima relative to Φ.

The fact that trade will continue if there are mutually advantageous trades suggests the following

Theorem 1.

Suppose e is an absorbing state. If U^i is concave and strictly monotonic in wealth for all i, then the allocation $\pi(e)$ is a strict Pareto optimun relative to Φ.

Proof.

Let $\pi(e) = (W^i,S^i)_i$, and suppose $\pi(e)$ is not a strict Pareto optimum relative to Φ. That is, there is another feasible allocation $(W^{i\prime},S^{i\prime})_i$ with $V^i(W^{i\prime},S^{i\prime}) > V^i(W^i,S^i)$ for all i, $S^{i\prime} \in N$, and if $S^{i\prime} \neq S^j$, $\frac{W^{i\prime}-W^i}{S^{i\prime}-S^i} \in \Phi$. For notational simplicity, define

$H = \{i \mid S^{i\prime}= S^i\}$, $F = \{i \mid S^{i\prime} > S^i\}$ and $G = \{i \mid S^{i\prime} < S^i\}$. The strict monotonicity of U^i implies that $W^{i\prime} > W^i$ for $i \in H$. Since U^i is concave, V^i is quasi-concave. Thus, for $\alpha \in [0,1)$

$V^i(\alpha W^i+ (1-\alpha)W^{i\prime},\alpha S^i+ (1-\alpha)S^{i\prime}) > V^i(W^i,S^i)$. In particular, for $i \in F \cup G$,

$\alpha = 1 - 1/|S^{i\prime}-S^i| \in [0,1)$, so

$$V^i(W^i+ \frac{W^{i\prime}-W^i}{|S^{i\prime}-S^i|}, S^i + \frac{S^{i\prime}-S^i}{|S^{i\prime}-S^i|}) > V^i(W^i,S^i).$$

That is, for $i \in F$,
$V^i(W^i + \frac{W^{i'}_i - W^i_i}{S^{i'} - S^i}, S^i + 1) > V^i(W^i, S^i)$. Since $\left|\frac{W^{i'}_i - W^i_i}{S^{i'} - S^i}\right| \in \Phi$, it must be that

$\frac{W^{i'}_i - W^i_i}{|S^{i'} - S^i|} \geq -b^i(W^i, S^i)$. Similarly, for $i \in G$.
$V^i(W^i + \frac{W^{i'}_i - W^i_i}{|S^{i'} - S^i|}, S^i - 1) > V^i(W^i, S^i)$, and hence $\frac{W^{i'}_i - W^i_i}{|S^{i'} - S^i|} \geq a^i(W^i, S^i)$.

$$
\begin{aligned}
\text{Thus, } \Sigma W^{i'} &= \sum_H W^{i'} + \sum_F W^{i'} + \sum_G W^{i'} \\
&\geq \sum_H W^i + \sum_F (W^i - b^i(W^i, S^i)|S^{i'} - S^i|) + \sum_G (W^i + a^i(W^i, S^i)|S^{i'} - S^i|) \\
&= \Sigma W^i + \sum_G a^i(W^i, S^i)|S^{i'} - S^i| - \sum_F b^i(W^i, S^i)|S^{i'} - S^i| \\
&\geq \Sigma W^i + \alpha(e)\sum_G |S^{i'} - S^i| - \beta(e)\sum_F |S^{i'} - S^i| \quad \text{since } e \text{ is an}
\end{aligned}
$$

absorbing state. Now,

$$0 = \Sigma(S^{i'} - S^i) = \sum_G (S^{i'} - S^i) + \sum_F (S^{i'} - S^i) = -\sum_G (S^i - S^{i'}) + \sum_F (S^{i'} - S^i)$$

$$= -\sum_G |S^{i'} - S^i| + \sum_F |S^{i'} - S^i| . \quad \text{Thus, } \sum_G |S^{i'} - S^i| = \sum_F |S^{i'} - S^i| \equiv Z.$$

Then, $\Sigma W^{i'} \geq \Sigma W^i + Z(\alpha(e) - \beta(e)) > \Sigma W^i$.
Hence, $(W^{i'})_i$ is not feasible which is a contradiction.

It is not true that every strict Pareto optimum relative to Φ is an equilibrium allocation. The trading process will continue if two people can be made strictly better off by trading with each other, and hence does not rely on there being an allocation in which everyone can be made better off. The fact that trade will stop if there are no mutually advantageous trades suggests the following

<u>Theorem 2.</u>
If $(W^i, S^i)_i$ is a Pareto optimum relative to Φ, then there exists an absorbing state e such that $\pi(e) = (W^i, S^i)_i$.

<u>Proof.</u>

If $(W^i, S^i)_i$ is a Pareto optimum relative to Φ, then there does not exist a feasible allocation $(W^{i'}, S^{i'})_i$ with $S^{i'} \in N$ for all i, $S^{i'} \neq S^i$ implies $\left|\frac{W^{i'}_i - W^i_i}{S^{i'} - S^i}\right| \in \Phi$ such that $V^i(W^{i'}, S^{i'}) \geq V^i(W^i, S^i)$ for all i with strict inequality for at least one i. In particular, there does not exist i and j such that if $q \in \Phi$, $W^{i'} = W^i - q$, $S^{i'} = S^i + 1$, $W^{j'} = W^j + q$, $S^{j'} = S^j - 1$, $V^i(W^{i'}, S^{i'}) > V^i(W^i, S^i)$, $V^j(W^{j'}, S^{j'}) > V^j(W^j, S^j)$.

Hence, there do not exist two investors who can trade, and thus $(W^i,S^i)_i$ is an equilibrium allocation.

Theorems 1 and 2 are analogues to the two fundamental theorems of welfare economics with the usual condition of optimality replaced by the much weaker "strict optimality relative to Φ." An equilibrium of the trading process will always be a strict Pareto optimum relative to Φ, and any allocation that is Pareto optimal relative to Φ can be reached from some initial endowment (namely the optimum allocation). It is unfortunate that equilibria do not enjoy the much stronger notion of Pareto optimality; however, it can be shown that if the set, Φ, is made finer in an appropriate manner, the set of possible equilibria will converge to a set contained in the set of allocations with integral amounts of stock and that are Pareto optimal.

From a given fixed endowment, let $(W,S)_k$ be the final allocation when trade is restricted to Φ_k. Now, $(W,S)_k$ is a random variable depending upon the arrival process of orders to the specialist. That is, given a particular pattern of order arrivals, the final endowment is determinate. Define P_k to be the set of allocations that are strict Pareto optima relative to Φ_k, and define P^* to be the set of strict Pareto optima with integral shares. That is, $(W^i,S^i)_i$ is in P^* if there does not exist another feasible allocation $(W^{i\prime},S^{i\prime})_i$ with $S^{i\prime} \in N$ for all i such that $V^i(W^{i\prime},S^{i\prime}) > V^i(W^i,S^i)$ for all i. For concreteness, suppose that $\Phi_k = \{i/k!\}_{i=0}^{Qk!}$. Then, Φ_k increases to Φ^* the interval of rationals between 0 and Q. Furthermore, P_k decreases to P^*. Define the probability measure ν_k on the space of allocations by $P\{(W,S)_k \in B\} = \nu_k(B)$. Though it is not in general true that $P\{(W,S)_k \in P^*\}$ increases to 1, if ν_k converges weakly to a probability measure ν ($\nu_k \underset{w}{\rightarrow} \nu$) then $P\{(W,S)_k \in P^*\} \rightarrow 1$. Thus, if Φ is made fine enough, and $\nu_k \underset{w}{\rightarrow} \nu$, the probability that a non-optimal allocation is reached can be made arbitrarily small. The crucial step here is that ν_k converges. Though I have not yet been able to establish this, I conjecture that in fact the weak convergence holds.

The fact that there are only a finite number of states implies that in an a.s. finite amount of time an equilibrium will be reached. To see this, let L(e) be the set of feasible allocations such that $V^i(W^i,S^i) \geq V^i(\pi^i(e))$ for all i.

Let $T(e) = \bigcup_{i,j} \{(\pi(e)\setminus\pi^i(e) + (a^i(\pi^i(e)),-1),\ \pi^j(e) + (-a^i(\pi^i(e)),1)),$

$(\pi(e)\setminus\pi^i(e) + (b^j(\pi^j(e)),-1),\ \pi^j(e) + (-b^j(\pi^j(e)),1))\}$

where $(\pi \setminus \pi^{i\prime},\pi^{j\prime})$ indicates that the ith and jth components of π are replaced by $\pi^{i\prime}$ and $\pi^{j\prime}$ respectively. Then, $T(e)$ is the set of candidates for the allocation reached from e after one trade. Starting from e, if $T(e) \cap L(e) \neq \emptyset$, then in an almost surely finite amount of time a trade will take place, and a new allocation will be reached, $\pi(e_1)$. Now, $L(e)$ has a finite number of elements for all e. Furthermore, $L(e_1) \subset L(e)$ strictly. Thus, in a finite number of steps taking an almost surely finite amount of time, an allocation will be reached, say $\pi(e_N)$, such that $T(e_N) \cap L(e_N) = \emptyset$. Then there will be no further trading, and hence from this time in an almost surely finite amount of time an absorbing state e* will be reached. Thus, we have proven

Theorem 3.

Given the trading process described above, an equilibrium will be reached in an almost surely finite amount of time.

5. EXAMPLES

In this section, a numerical example is shown, and the outcomes of the trading process are analyzed and in particular compared to what would be the competitive outcome.

Suppose there are three investors, each with the same utility function $U^i(w) = 1 - e^{-w}$ and initial endowment of one share and three units of the risk-free asset. Suppose further, that given i's information, V is normally distributed with mean i, (I=1,2,3) and variance 1. If $\Phi = \{j\epsilon\}_{j=o}^{M}$, then the buying and selling prices are respectively:

$$b^i(W^i,S^i) = i - (S^i + \tfrac{1}{2}) - \epsilon \wedge W^i$$
$$a^i(W^i,S^i) = i - (S^i - \tfrac{1}{2}) + \epsilon \text{ if } S^i > 0.$$

Note that in this case only 1 and 3 will trade, and they will either trade at a price of $\frac{1}{2} + \epsilon$ or $1\frac{1}{2} - \epsilon$. In either case, the final allocation of shares will equal the allocation in the competitive outcome. The final allocation of shares and risk-free asset together, however, will never equal the competitive outcome, but the arithmetic mean of the trading outcomes is the competitive allocation.

Given that there is only one possible match up of traders, this result is not particularly surprising. It does indicate, however, that in general there is no hope that the process will result in the competitive outcome, nor even come particularly close to it.

This second example exhibits the effect that random matching of traders has on the outcome. Let the utility functions be as before, and let the common initial endowments be two shares and six units of risk-free asset. Further, suppose that given i's information, V is normally distributed with mean 2i (i=1,2,3) and variance 1. The competitive allocation is then ((10,0),(6,2),(2,4)) and the competitive price is 2. In this case, the buying and selling prices are respectively:

$$b^i(W^i,S^i) = 2i - (S^i + \tfrac{1}{2}) - \varepsilon \wedge W^i$$

$$a^i(W^i,S^i) = 2i - (S^i - \tfrac{1}{2}) + \varepsilon \text{ if } S^i > 0.$$

Then, to start with, 2 and 3, 1 and 3, or 1 and 2 will trade. As might be expected, the number of possible outcomes increases considerably (to 18). In all cases, the final allocation of shares of stock agrees with the competitive allocation of shares; however, in all outcomes of the trading process, investor 2 is at least as well off as in the competitive outcome.

In a sense, investor 2 is a "probabalistic middleman." In terms of shares, investor 2 always ends up with a total net trade of zero. Since there is a certain probability that 1 and 3 cannot meet directly, 2 may buy from 1 and sell to 3 and make a profit on these transactions. This occurs even though investor 2 does not plan to speculate. One might expect that in general there would be a group of investors who would act as middlemen, probabalistically, and hence a group that would on average be better off than in the competitive outcome.

The fact that the outcomes of the trading process diverge from the competitive outcome does not appear in general to be due to indivisibilities. In fact, in both of the above examples, the competitive allocation consists of integral amounts of shares and rational amounts of the risk-free asset. All that one can expect is that trading outcomes lie on a contract curve, modified by the existence of indivisibilities. The second example shows even more clearly why trading outcomes will in general not correspond to the competitive outcome. The fact that trading occurs over time and at different prices seems to imply that not only can one group consistently improve their well-being through trade, but they can actually increase their expected wealth as well through arbitrage.

6. CONCLUSION

We have shown that under essentially static assumptions on investor behavior the trading process described above converges to an equilibrium in an almost surely finite amount of time. Furthermore, an equilibrium is a strict Pareto optimum relative to the indivisibilities and if the process starts from a Pareto optimum relative to the indivisibilities the allocation will not change. Finally, an example shows that in general a trading outcome will not be the competitive outcome, and that (probabalistically) certain investors may serve as middlemen and make arbitrage profits from this activity.

This paper has served primarily to describe the trading process and examine some of its properties under the simplest of behavioral assumptions. As such, it functions as an introduction to further research in which more interesting investor behavior can be examined. As pointed out above, the central purpose of this research is to examine the informational characteristics of a market when trading occurs out of equilibrium. Therefore, the research plan is as follows.

First, it will be assumed that investors ignore the opportunity for speculation, but that they recognize that market signals convey information. Thus, as above, investor i will view his random wealth to be of the form $S^iV + W^i$, but now the possibility of information accumulation is included. Thus, the expectation of his utility is conditioned on his initial information as well as a sequence of past prices. In this case, one would expect the indivisibilities still to lead to finite time convergence, though they might possible prevent complete dissemination of information.

One way to attack the problem of information dissemination is to suppose all preferences and initial endowments are the same, and that investors are only concerned with the mean and variance of the value of a share. Then, by observing buy and sell prices, the investor sees other individuals' subjective means and variances with some random noise. With this model, a possibly useful definition of information dissemination is as follows: if all investors have the same utility function and the same endowment, then information is common to all if no trade takes place. Under certain structural conditions (on utility functions and a priori expectations) one might expect that (with a high probability) information would be disseminated in the above sense.

The next step will be to make the model truly dynamic by assuming that investors recognize the opportunities for speculation. In this case, one would not expect finite time convergence, and hence the analysis will be more concerned with the ergodicity of the process. Another major difference in the analysis will be due to the fact that the stock will no longer have any objectively given random value. Thus, the nature of expectations concerning future prices must be specified and, indeed, one would expect different expectations to have very different impacts on the stochastic process of prices. Since regulation of borrowing and short selling is primarily directed towards maintaining stability, this model will be used to examine these regulations.

Finally, the simple model of two assets will be extended to a model with more than one stock traded. This extension is not necessarily straight forward, for if, as in the stock market, all trades involve money (the risk-free asset) then transferring wealth between markets becomes difficult since one is unable to trade shares of one company for shares of another company directly. This problem has been noted in models of Keynesian unemplyment (9).

7. APPENDIX

The assumption that the interarrival times in the process $(e_t)_t$ are exponentially distributed is justified by the following argument. Consider a fixed state, and let T be the time until the next jump of the process. Consider the following hypothetical situation. Let $Y_{nk}(t)$ be the number of orders investor k enters with the specialist in t units of time after the state e is reached, when there are n total investors. Define $Z_n(t) = \Sigma Y_{nk}(t)$. It is reasonable to assume that individuals make decisions independently, and hence, given e,

a) $(Y_{nk}(t))_k$ is a family of independent random variables taking values in the set of nonnegative integers. As the number of investors gets larger, it should get harder for any one individual to get an order entered due to the limited information processing ability of the brokers and the specialist. That is, $P\{Y_{nk}(t) \geq 1\} \to 0$ as $n \to \infty$ for all k and $t < \infty$. Furthermore, this should hold for the most energetic of investors or b) $\max P\{Y_{nk}(t) \geq 1\} \to 0$ for all $t < \infty$.

As the number of investors becomes large, no single investor should be able to enter more than one order in t units of time. It is assumed that:

c) $\Sigma\, P\{Y_{nk}(t) \geq 2\} \to 0$. Note that

$P\{Y_{nk}(t) \geq 2 \text{ for some } k\} = P(\cup\{Y_{nk}(t) \geq 2\}) \leq \Sigma P\{Y_{nk}(t) \geq 2\} \to 0.$

Finally, as n gets large, the expected total number of orders in t units of time should depend only on the amount of time. It is assumed that

d) $\Sigma P\{Y_{nk}(t) \geq 1\} \to ct$. Note that for n large,

$E[Z_n(t)] = \Sigma E[Y_{nk}(t)] \cong \Sigma\, P\{Y_{nk}(t) = 1\} \to ct.$

By a theorem of B. V. Gnedenko ((7) p. 132-3) under conditions a)-d), $Z_n(t)$ converges in probability to a Poisson random variable Z(t). That is, $P\{Z(t) = j\} = \frac{e^{-ct}(ct)^j}{j!}$. Thus, if the number of investors is large, $Z_n(t)$ is approximately a Poisson process, and T, the time of the first jump of $Z_n(t)$ is approximately exponentially distributed.

BIBLIOGRAPHY

(1) Billingsley, Patrick, Convergence of Probability Measures, John Wiley and Sons, Inc., New York, 1968.

(2) Cinlar, Erhan, Introduction to Stochastic Processes, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1975.

(3) Fama, Eugene F., "The Behavior of Stock Market Prices," The Journal of Business, Vol. 38, #1, (January 1965), pp. 34-105.

(4) ______________, "Efficient Capital Markets: A Review of Theory and Empirical Work," The Journal of Finance, vol. 25, (May 1970) pp. 383-417.

(5) Galbraith, John Kenneth, The Great Crash, Houghton Mifflin Company, Boston, 1961.

(6) Garman, Mark B., "Market Microstructure," Journal of Financial Economics, Vol. 3 #3, (June 1976), pp. 257-275.

(7) Gnedenko, B. V. and A. N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables, Addison-Wesley Publishing Company, Reading, Massachusetts, 1968.

(8) Graham, Daniel A., Edward Jacobsen and E. Roy Weintraub, "Transactions Costs and the Convergence of a 'Trade out of Equilibrium' Adjustment Process," International Economic Review, Vol 13, #1, (February 1972), pp. 123-131.

(9) Grandmont, Jean Michel, "Temporary General Equilibrium Theory," presented at the World Meetings of the Econometrics Society, August, 1975.

(10) Grossman, Sanford, "On the Efficiency of Competitive Stock Markets Where Traders have Diverse Information," The Journal of Finance, Vol. 31, #2, (May 1976), pp. 573-585.

(11) Hurwicz, Leonid, Roy Radner and Stanley Reiter, "A Stochastic Decentralized Resource Allocation Process: Part I," Econometrica, Vol 43, #2, (March 1975) pp. 187-221.

(12) Jennings, Richard W. and Harold Marsh Jr., Securities Regulation Cases and Materials, The Foundation Press, Inc., Mineola, New York, 1972.

(13) Kalai, Ehud, Andrew Postlewaite and John Roberts, "A Game of Barter with Barriers to Trade," Discussion paper no. 302, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston Illinois, 1977.

(14) Kihlstrom, Richard E. and Leonard J Mirman, "Information and Market Equilibrium," The Bell Journal of Economics, Vol. 6, #1, (Spring 1975), pp. 357-376.

(15) Latane, Henry A., "Standardized Unexpected Earnings" unpublished manuscript, University of North Carolina.

(16) ______________ and Charles P. Jones, "Measuring and Using Standardized Unexpected Earnings," unpublished manuscript, University of North Carolina

(17) Leffler, George L., The Stock Market, 3rd Edition, Ronald Press Company, New York, 1967.

(18) Rothschild, Michael, "Models of Market Organization with Imperfect Information: A Survey," Journal of Political Economy, Vol. 81, #6, (November 1973) pp. 1283-1307.

(19) Royden, H. L., Real Analysis, Macmillan Publishing Co., Inc., New York, 1968.

(20) Smidt, Seymour, "A New Look at the Random Walk Hypothesis," Journal of Financial and Quantitative Analysis, September, 1968, pp. 235-261.

(21) Vernon, Raymond, The Regulation of Stock Exchange Members, Columbia University Press, New York, 1941.

(22) Zeeman, E. C., "On the Unstable Behavior of Stock Exchanges," Journal of Mathematical Economics, Vol. 1, #1, (1974), pp. 39-49.