DISCUSSION PAPER NO. 520

THE INDUCTIVE SOLUTION FOR NON-COOPERATIVE GAMES

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March 1982

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ABSTRACT

The inductive solution applies to non-cooperative games in normal form, played once. Each player has a probability distribution over the possible strategies of the other players, and uses this distribution to select an expected-utility-maximizing strategy. The players may use their own choices as inductive evidence about what others' choices will be, by using act-dependent probability distributions.

Various axioms are introduced to constrain the set of possible probability distributions. The axioms are based on a concept of rational probability assessment and on the fact that the players are involved in a game of strategy. It is shown that for symmetric 2-strategy-per-player games with a large number of players, rational players will hold a Beta distribution for the probability that a given proportion of players will choose each strategy. For larger symmetric games they will hold the generalized Dirichlet distribution.

The inductive solution provides an alternative rationale for the minimax zerosum solution and for the theory of Nash equilibria, and justifies these solutions as extensions of standard utility maximization without introducing equilibrium assumptions. It avoids certain logical difficulties inherent in their previous justifications.

According to the accepted theory, players of a non-cooperative game will choose an equilibrium point. This is a stronger assumption than expected utility maximization, in that an equilibrium is to be chosen in a one-shot game even if other strategies could give as high an expected utility, assuming the strategies used by the other players are fixed.

The principle of simplicity prompts us to construct a solution based solely on expected utility maximization, without adding the assumption of an equilibrium choice. This would unify game theory and individual decision theory. It is the basis of the inductive solution described here, which applies to non-cooperative games in normal form that are played only once and thus is an alternative to the von Neumann and Morgenstern minimax solution for the two-person zerosum games (von Neumann and Morgenstern, 1953), and to the theory of Nash equilibria (Nash, 1951). In some cases it coincides with these solutions and provides an alternative justification for their conclusions. For other types of games the inductive value can lead to expectations for the players that are different from the traditional theories.

In Section I the argument for the inductive solution is developed through an example. In Section II axioms are given that state the theory more generally and precisely. Finally, in Section III the inductive solution is compared with the traditional concepts.

I. 2 x 2 Symmetrical Games

The game of Matrix 1 is to be played once non-cooperatively, i.e., without communication or enforceable agreements between the players. It is a symmetrical game of the Apology type according to the classification scheme of Rapoport and Guyer (1966) as modified by Harris (1968).

		Player 2 C_1	C_2
Player 1	R_1	1,1	5,4
	R_2	4,5	0,0

Matrix 1

We will look at the game from the viewpoint of Player 1. He wishes to maximize his expected utility and thus needs a set of utilities for the outcomes with associated probabilities. The utilities are given in Matrix 1, but still lacking are the values of $Pr(C_1|R_1)$, $Pr(C_2|R_1)$, etc., as shown in Matrix 2, (where R_1, R_2 and C_1, C_2 are Player 1's and 2's moves). The corresponding matrix for Player 2 is Matrix 3.

$Pr(C_1\|R_1)$	$Pr(C_2\|R_1)$
$Pr(C_1\|R_2)$	$Pr(C_2\|R_2)$

Matrix 2

$Pr(R_1\|C_1)$	$Pr(R_2\|C_1)$
$Pr(R_1\|C_2)$	$Pr(R_2\|C_2)$

Matrix 3

To fill in Matrix 2, Player 1 needs evidence about what Player 2 will do. He notes that he and Player 2 are in identical positions since the game is symmetrical, so he regards his own move as relevant evidence as to what Player 2's move will be. The symmetries of the game make R_1 and C_1 alike, and R_2 and C_2 are alike, or "corresponding", so the use of one is evidence that the other will be used.

A problem in using his own move as evidence is that he has not as yet decided what move to make. His approach appears to be circular -- how can he use his own move as evidence to estimate what Player 2 will do, as a step in deciding what his own move will be? However, there is a simple way to avoid circularity. Player 1 assigns conditional probabilities according to which each move of Player 2 is more likely given the use of the corresponding move by Player 1. An example is given in

Matrix 4. In this matrix $Pr(C_1|R_1) = .6 > Pr(C_1|R_2) = .5$, etc., so that there is a <u>relationship of evidence</u> between corresponding moves.

.6	.4		.6	.5
.5	.5		.4	.5
<u>Matrix 4</u>.	$Pr(C_i|R_j)$		<u>Matrix 5</u>.	$Pr(R_i|C_j)$

Note that according to Matrix 4 the event of a column choice is probabilistically dependent on the row. This is in contrast to the usual decision-making situation in which the rows of Matrix 4 are identical.

Our discussion so far is summarized by the following assumption:

<u>A1</u>. <u>A player assigns probabilities to the opponent's moves. These probabilities may show dependence on the player's own move.</u>

There are three ways the meaning of probabilities $Pr(C_i|R_j)$ could be misinterpreted, and we will make a short comment on each.

First of all, the fact that $Pr(C_1|R_1) > Pr(C_1|R_2)$ is not intended to suggest a <u>causal</u> relationship between R_1 and C_1. Neither player's move can take a part in causing the other's, since moves are made simultaneously. The relationship is only an <u>evidential</u> one, in the same sense that if a series of specimens of a certain substance is found to conduct electricity, it is more likely that the next one to be examined is also a conductor. No causal relationship is implied from the conductivity of the previously examined specimens to the new one. The probability of the new specimen's conductivity is higher since "probability" is here to be interpreted as "justified degree of belief."

Secondly, we must distinguish between "$Pr(C_1|R_1)$" and "$Pr(C_1|R_1$ <u>and</u> both players know R_1 is chosen)". In the latter interpretation Player 2 would choose C_2 because of the payoff structure of the matrix, so that the conditional probability of C_1 would be zero. But we intend the former interpretation, under which neither player is informed of the other's move. In this case we can consistently state that $Pr(C_1|R_1)$ is greater than $Pr(C_1|R_2)$.

Thirdly, Matrix 4 does not represent a strategic choice on the player's part. Its entries are not chosen by the player on the basis of maximizing his gain. Instead, it represents the player's viewpoint concerning the evidential relevance of the similarity between his role in the game and that of his opponent.

What constraints must conditional probability matrices like Matrix 4 satisfy? We will now show that Player 1's set of possible matrices is tightly constrained-- most matrices, including our example Matrix 4, are inconsistent with the assumption that Player 1 is facing an informed rational opponent.

Since the players are rational in their judgments, we assume that they assess the strength of evidence in a similar way. Neither player has any privileged information about the strength of evidence of one move for another. This leads to:

<u>A2</u>. <u>The conditional probability matrices assigned by the players have the same symmetry as the game</u>.

This means that if Player 1 assigns Matrix 4, for example, Player 2 will assign the transpose, Matrix 5.

Player 1 knows this and thus is able to calculate Player 2's choice of move: according to Matrix 1 and Matrix 5, player 2 will choose C_1 since its utility is 2·6 and is greater than C_2's utility of 2·0. But this conclusion contradicts the original assumption that Player 1 holds Matrix 4. By Matrix 4, whatever Player 1 does, Player 2 will choose C_2 with probability of at least .4. Therefore, Player 1 must believe that unconditionally the probability of C_2 is at least .4. But he has also concluded on the basis of Matrix 5 that the probability of C_2 is 0 since its utility is submaximal. This is a contradiction. Matrix 4 is therefore inadmissible as an assignment of conditional probabilities.

Which conditional probability matrices are admissible? The following principle is a direct consequence of the arguments presented above.

<u>A3</u> <u>A conditional probability matrix is inadmissible if it implies a positive unconditional probability for a move that has submaximal expected utility</u>.

Thus, for example, in the 2 x 2 game of Matrix 1, if the probabilities of C_1 and C_2 are positive, their expected utilities must be equal. The set of matrices for Player 1 consistent with this principle is given by Matrices 6 and 7.

$q + \frac{1}{4}$	$\frac{3}{4} - q$
$1 - q$	q

with $\frac{3}{8} \leq q \leq \frac{3}{4}$

<u>Matrix 6</u>

1	0
$1 - q$	q

with $\frac{3}{4} < q \leq 1$

<u>Matrix 7</u>

These matrices are not difficult to derive. Using the notation of Matrix 2, we use the fact that $Pr(C_1|R_1)$ lies in the interval $[0, 1]$ and equals $1 - Pr(C_2|R_1)$ and the analogous fact for $Pr(C_1|R_2)$ is used. Matrix 6 represents a situation in which both moves for a player offer equal expected utility. The form of the matrix is found by letting $p = Pr(C_1|R_1)$ and $q = Pr(C_2|R_2)$, equating the expectations of the two rows, and then eliminating p. The restriction $\frac{3}{8} \leq q$ follows from the premise that the evidential relation between corresponding moves is non-negative and therefore $Pr(C_1|R_1) \geq Pr(C_1|R_2)$.

Matrix 7 reflects the alternative situation to Matrix 6, one in which the expectations of the rows are not equal. It is derived as follows. If the rows' expectations are unequal, then the columns' will be unequal, too. The conditional probability matrix must allow that the unconditional probability of one of the columns be zero. The eight possible matrices of this type are Matrices 8 through 15.

$$8.\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad 9.\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad 10.\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad 11.\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$12.\ \begin{bmatrix} p & 1-p \\ 1 & 0 \end{bmatrix} \qquad 13\ \begin{bmatrix} 1 & 0 \\ 1-q & q \end{bmatrix} \qquad 14.\ \begin{bmatrix} 0 & 1 \\ 1-q & q \end{bmatrix} \qquad 15.\ \begin{bmatrix} p & 1-p \\ 0 & 1 \end{bmatrix}$$

<u>Matrices 8 - 15</u>. $p,q \in (0,1)$

Several of these are inadmissible because they violate the principle of non-negative evidence, that $Pr(C_1|R_1) \geq Pr(C_1|R_2)$ and $Pr(C_2|R_1) \leq Pr(C_2|R_2)$. This eliminates Matrices 10, 12 and 14. Other matrices, namely 8, 11, 15 and some values of Matrix 13, allow a conditional probability of zero for one column, but it is the column that would have greater rather than lesser expected utility, so these matrices must be eliminated.[1] Matrix 9 remains, and Matrix 13 remains with the stipulation that $q > \frac{3}{4}$. These possibilities are summarized by Matrix 7, so Matrices 6 and 7 give all admissible probabilities.

Note that Matrices 6 and 7 form a family indexed by one parameter q.

This solution can be compared with traditional Nash equilibrium theory. There are three Nash equilibria for the game of Matrix 1. Two of these are the pure strategy solutions (R_2, C_1) and (R_1, C_2) yielding payoffs

(4,5) and (5,4) respectively. These violate A2, since one player's conditional probability matrix is not the transpose of the other's. The remaining Nash equilibrium has the mixed strategy pair $[(5/8\ R_1,\ 3/8\ R_2),\ (5/8\ C_1,\ 3/8\ C_2)]$. Each player has an expectation of 2.5.

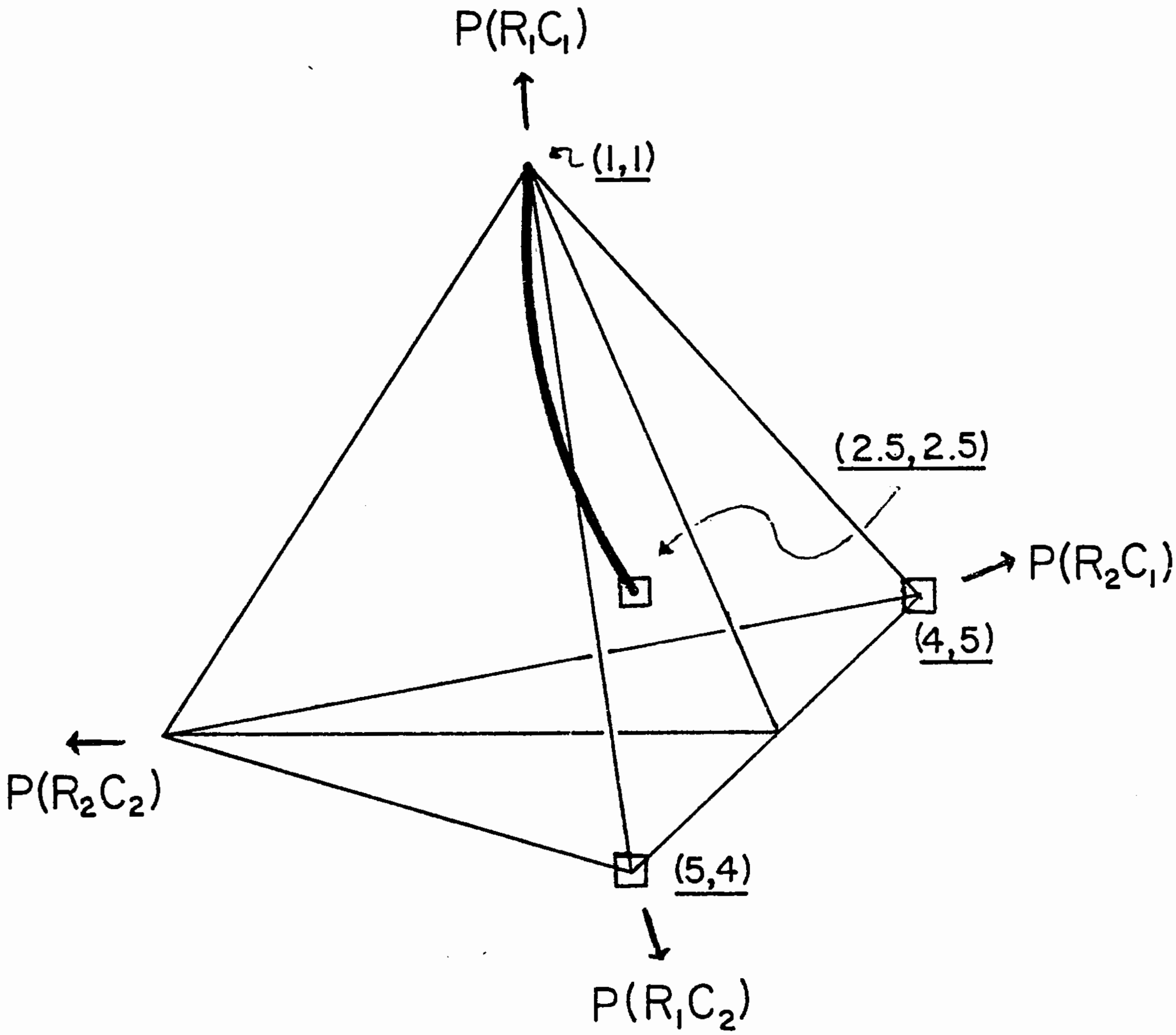
This Nash equilibrium corresponds in our theory to Matrix 6 with $q = 3/8$. This is the unique conditional probability matrix with identical rows that is allowed by our theory, i.e., it is the only admissible matrix having probabilistic independence of moves.

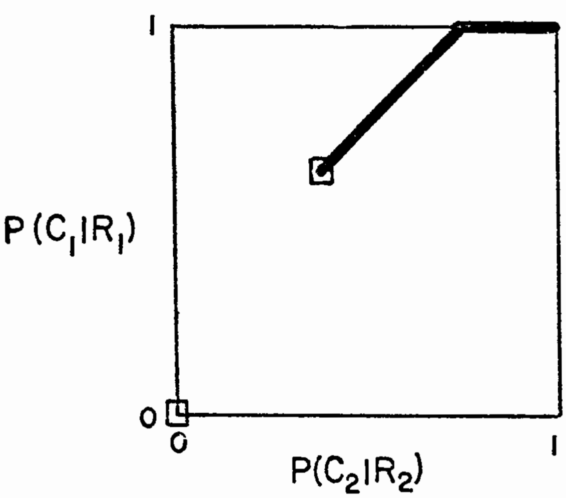
The locations of the Nash equilibrium and the inductive solutions are shown in Figure 1. The tetrahedron shows the joint probability of every pair of moves.[2]

The value $q = 3/8$, like the symmetrical Nash solution, gives each player an expectation of 2.5. The other admissible matrices have lower expectations, down to a value of 1 for $q = 1$.

The physical realization of this game that suggested the name "Apology" involves two friends who wish to make up after a fight. They can go and apologize (R_1 and C_1) or stay home and wait for the other to visit (R_2 and C_2). Each wants an apology to be made but would rather the other do it. If they both visit they will miss each other. Our theory is more pessimistic than Nash's since the more the evidential dependence of the moves the more likely they will do the same thing and receive low payoffs.

Probability matrices satisfying A1 - A3 for a given game will be called strategic for the game. If a certain expectation can be achieved by a strategic probability matrix, it will be called an inductive value of the game. The set of strategic probability functions and inductive values is the inductive solution. These terms are chosen to stress the concept of inductive evidence among moves that forms the basis of the solution.





	C_1	C_2
R_1	1,1	5,4
R_2	4,5	0,0

Figure 1. The inductive solutions (solid line) and Nash equilibria (squares) for the Apology game of Matrix 1, shown in the spaces of unconditional and conditional probabilities. Payoff vectors associated with certain solutions are indicated.

The Inductive Solution of n-person Games -- Summary of Results in Section II

The example outlined above, a 2 x 2 game, results in a family of strategic probability matrices with only one parameter. What happens if a game has more than two players or more than two moves per player?

Suppose there are n players, player i having m_i moves. To be able to select a move, player i must assign a probability to each pattern of play by the n-1 opponents, conditional on each strategy of the player himself. It might seem that the number of parameters required to determine these probabilities would greatly increase compared to the 2 x 2 case. For general n-person games there is indeed a large increase, but for symmetrical n-person games the number of parameters is small, if an additional postulate is introduced. (This assumption will be stated formally as Axiom IV in Section II.) For symmetrical m x m ... x m games, the players' strategic probability values are determined by m parameters or fewer. For example, for 2 x 2 ... x 2 games the set of values of the probabilities described involves only one parameter, just as in the 2 x 2 case.

Details will be given in Section II, but for completeness of this summary we will describe the further assumption now. It deals with the manner in which players evaluate one move as evidence to determine the probability of another move, and is a version of what has been termed Johnson's sufficiency postulate (Goode, 1971), or the λ-principle (Carnap, 1981).

Suppose we have an n-person game that is symmetrical in players and in which each player's move has a unique move alike by symmetry in each of the other players' strategy sets. Such moves are said to be corresponding or of the same type. The players will hold a probability function that specifies the probability of a strategy type α by player i given certain specified moves by the other players. The axiom of sufficiency states that the value of this probability depends only on a specification for each of the other players, whether he chooses type α or not. If certain of the other players do not choose type α, the value of probability is independent of their particular choices. The number of players choosing α is "sufficient" information.

The results of Section II will be summarized now. It is shown that a single parameter λ can be defined that measures the degree of evidential independence of any two moves. On the basis of the axioms λ can be shown to be independent of the pair of moves chosen for its measurement (Theorem 1).

In choosing an optimal move, a player needs to know the probabilities of all (n-1)-tuples of strategies of the other players given each of his own choices. General formulae for these probabilities are given in Theorem 2.

The well-known two-person Prisoners' Dilemma games have been generalized to n people in a simple way by defining Commons Dilemma games. These have been used to model inflation, pollution, energy use (Dawes, 1980) and animal behavior (Treisman, 1977). Theorems 3 and 4 give the inductive value of these games, the latter stating that as the number of players becomes large, either each player's probability density function for the proportion of players choosing a given move follows the Beta function, or else all players are certain to choose the same move.

Finally, Theorem 5 of Section II states that a strategic probability function always exists.

II. Formal Exposition

A game G is defined as a triple $\langle N,\mathscr{S},\overline{U}\rangle$. The set N is the finite set of the players, $\{1,2,\ldots,n\}$. Vector $\mathscr{S}$ consists of strategy sets $(S_1,S_2,\ldots,S_n)$, where S_i is the set of strategies under the control of player i. Payoff function $\overline{U}$ is a vector-valued function from $S_1 \times S_2 \times \ldots \times S_n$ to $\mathbb{R}^n$. A strategy n-tuple $\bar{s}_n$ is defined as a vector of n strategies $(s_{1\alpha_1},\ldots,s_{n\alpha_n})$ such that $s_{i\alpha_i} \in S_i$ for all i. The set of all games of this type is labeled $\mathscr{G}$.

The event set of G, $\mathscr{E}_G$, is the set of statements that partially or completely describe a play of G. Examples are "Player 1 uses s_{11} and Player 2 uses s_{23}" and "Either Player 1 uses s_{11} or Player 4 uses s_{42}."

Formally $\mathscr{E}_G$ can be defined as the power set of $\times_{i=1}^n S_i$. Any $e \in \mathscr{E}_G$ will then be a set of strategy n-tuples. Each member of $\times_{i=1}^n S_i$ is a complete description of a play of G. The "occurrence" of event e or the statement that e is "true" means that the play actually used is a member of this set. If there is more than one n-tuple in e, then the event e will not specify the play completely, but will mean only that the strategy n-tuple used is one of the n-tuples in e.

In this notation "$s_{i\alpha}$" is to be interpreted sometimes as the name of a strategy, and sometimes as the event that it is used, but it should be clear from the context which meaning is intended. The two events stated above would then be written "$s_{11} \cap s_{23}$" and "$s_{11} \cup s_{42}$". All impossible events, (e.g., $s_{11} \cap s_{12}$) are regarded as identical and written φ. All necessary events (e.g., $s_{11} \cup \overline{s_{11}}$) are also regarded as identical and written U. The set $\mathscr{E}_G \times (\mathscr{E}_G - \varphi)$ is the set of conditional events of G. If e' is true, then conditional event $e|e'$ has the same truth value as e. If e' is false then the truth or falsity of $e|e'$ is undefined.

The probability function Pr_G is from $\mathcal{S}_G \times (\mathcal{S}_G - \varphi)$ to the interval [0,1]. The number $Pr_G(e|e')$ is the players' rational degree of belief in the conditional event $e|e'$, $e \in \mathcal{S}_G - \varphi$.

Note that Pr_G is not specific to each player but the same for all, following the approach that all are rational and so evaluate evidence in the same way.

In the development of a probability function an unconditional probability measure would normally be a primitive, and a conditional probability function would then be defined as a quotient of unconditional probabilities. However, in the context of games of strategy situations may arise in which the denominator would be zero. For example, a player may know that he will not use strategy $s_{i\alpha}$ because he knows it has a suboptimal expected utility, yet he will require values for the probabilities of the others' various strategies given $s_{i\alpha}$. These probabilities are needed for the purpose of knowing the expected utility of $s_{i\alpha}$, even though it is suboptimal. Thus Pr_G must allow the assessment of probabilities of events conditioned on an event of measure zero.

This means we cannot define conditional probability as a quotient of unconditional probabilities. Conditional probability must be axiomatized directly, and we use the following set of axioms given by Renyi (1955).

Suppose X is a non-empty set, E is an algebra of sets on X, and P is a function from $\mathcal{S} \times (\mathcal{S} - \varphi)$ to the real numbers. The triple $\langle X, \mathcal{S}, P \rangle$ is a <u>finite conditional probability space</u> if for any $A \in \mathcal{S}$ and $B, C \in \mathcal{S} - \varphi$

1) $P(A|B) \geq 0$
2) $P(B|B) = 1$
3) $P(A|B) + P(X-A|B) = 1$
4) if $B \cap C \neq \emptyset$, $P(A \cap B|C) = P(A|B \cap C) \cdot P(B|C)$

Applying this to the context of games, the first axiom restricting Pr_G is X

<u>AI</u> (probability). For $G \in \mathcal{B}$, $\langle X_{i=1}^{m} S_i, \mathcal{S}_G, Pr_G \rangle$ is a finite conditional probability space.

A justification for AI can be given based on our interpretation of Pr_G as conditional degree of belief, based on the use of the "Dutch book" introduced by Ramsey (1931) and de Finetti (1931, 1964). A person's degree of belief in the conditional event $e|e'$ is manifest in his decision-making behaviour, and reflects the stakes the person is willing to accept for a bet on event $e|e'$. (A bet on a conditional event $e|e'$ is won if e and e' are true, lost if e is false and e' is true, and called off if e' is false.) Suppose the decision-maker's disposition to accept bets is based on a probability function that violates AI. Carnap and Jeffrey (1971) have shown that the decision-maker will be willing to accept every bet in a certain set of bets, among the events of which there are logical interconnections such that the set is logically certain to result in an overall loss to the bettor. Accepting the set of bets is clearly irrational, so it follows that Axiom I is a requirement for any probability function. The next axiom, Axiom II, requires that Pr_G be independent of the labeling of the players and strategies.

First we introduce the notion of a permutation of a game G. Permutation σ is a permutation of the set of all players' strategies $S_1 \cup \ldots \cup S_n$ with the special property that two strategies belonging to a single player always go into two strategies belonging to a single player. If σ has this property it leads to a <u>permutation of the players</u>, labeled π, in that player $\pi(i)$ in G' takes on the "role" of player i in G. Permutation σ also defines a permutation of the strategy n-tuples, which we shall label ν. Then G', the <u>permutation of the game</u> G <u>under</u> σ is defined as the game having player set $N' = N$, strategy set $\mathscr{S}' = \{S_1', \ldots, S_n'\}$ with $S'_{\pi(i)} = \{\sigma(s_{i1}), \sigma(s_{i2}), \ldots\}$, and payoff function $\bar{U}'$ such that the $\pi(i)$'th element of $\bar{U}'(\nu(\bar{s}_n))$ is equal to the i'th element of $\bar{U}(\bar{s}_n)$.

We also define e', the <u>permutation of an event</u> e <u>under</u> σ. If $e \in \mathscr{E}_G$, e' is defined as the event in which for each i and each α, any occurrence of $s_{i\alpha}$ in e is replaced by $\sigma(s_{i\alpha})$.

Axiom II can now be stated. It requires that Pr_G be independent of the labelling of players or strategies, and also affine transformations of a player's utilities.

AII (symmetry). For any game G, any $e_1 \in \mathcal{E}_G$ and $e_2 \in \mathcal{E}_G - \varphi$, let G', e_1' and e_2' be permutations under σ, and let G'' be derived from G' by an affine transformation of a player's utilities. Then $Pr_G(e_1|e_2) = Pr_{G''}(e_1'|e_2')$.

The next axiom states the possible relationships of evidence among the strategies. First we define the class of games among whose strategies we expect to find relationships of positive evidence. Suppose G' is the game permutation of G under σ, and that the payoff function $\overline{U}'$ is identical to $\overline{U}$. Permutation σ is called a symmetry of G, and G is said to be symmetrical under σ.

Every game has at least one symmetry, the identity permutation I, and some games have additional symmetries. Matrix 1, for example, has I and also the permutation $(R_1C_1)(R_2C_2)$.

For certain games the set of symmetries uniquely defines a correspondence between any two players' strategy sets. Two strategies, $s_{i\alpha}$ and $s_{j\beta}$, are regarded as corresponding if some symmetry of G takes $s_{i\alpha}$ into $s_{j\beta}$. The class of games $\mathcal{S}_{UC}$ ("UC" for "unique correspondence") is those games such that for all $i,j \in N$ and all $s_{i\alpha} \in S_i$ there is exactly one β such that a symmetry of the game takes $s_{i\alpha}$ into $s_{j\beta}$.

In Matrix 1 for example, R_1 and C_1 are corresponding strategies, as are R_2 and C_2. Since there is no symmetry taking R_1 into C_2, or R_2 into C_1, the correspondence is unique and Matrix 1 is in $\mathcal{S}_{UC}$.

For any game in $\mathcal{S}_{UC}$, the players obviously will have an equal number of strategies. The set $\mathcal{S}_{UC}$ may be partitioned into $\mathcal{S}_{UC}^m$, for $m = 1,2,\ldots$, where $\mathcal{S}_{UC}^m$ is the set of all games in $\mathcal{S}_{UC}$ with exactly m strategies for each player.

From now on we will assume that for any game in $\mathcal{S}_{UC}$, the players' strategies are labelled in agreement with the unique correspondence, i.e., $s_{i\alpha}$ and $s_{j\beta}$ are corresponding strategies if and only if $\alpha = \beta$.

Corresponding strategies are "alike." Axiom III states that players regard the use of a strategy by one player as non-negative evidence for the use of its corresponding strategy by another player.

AIII (evidence). Let $\bar{s}_k$ be a strategy k-tuple for a set of k players not including player i, and let $\bar{s}_k'$ be like $\bar{s}_k$ except that player j uses $s_{j\beta}$ rather than $s_{j\alpha}$. If $s_{j\alpha}$ is a corresponding strategy of $s_{i\alpha}$ and $s_{j\beta}$ is not, then

$$Pr_G(s_{i\alpha} | \bar{s}_k) \geq Pr_G(s_{i\alpha} | \bar{s}_k').$$

The next axiom, the sufficiency postulate, restricts the manner in which strategies can be evidence for each other. It states that if we wish to determine the probability of player i's strategy $s_{i\alpha}$ given the choices of a set of players not including player i, then it is sufficient to know for each player whether that player chose strategy of type α or not. If player j did not choose $s_{j\alpha}$ which particular strategy he did in fact choose is irrelevant.

Since the axiom bases relevance entirely on the relation of correspondence, and since correspondence is determined by the symmetry properties of the game, then the axiom is in effect stating that the symmetry properties of the game are the only basis for claiming evidential links among players' strategies.[3]

AIV (sufficiency). Let $\bar{s}_k$ and $\bar{s}_k'$ be defined as in AIII. If neither $s_{j\alpha}$ nor $s_{j\beta}'$ are corresponding strategies of s_{ij} then

$$Pr_G(s_{ij} | \bar{s}_k) = Pr_G(s_{ij} | \bar{s}_k').$$

Next we take account of the fact that the players are engaged in a game of strategy. Axiom V states that each player is certain that the others will act to maximize their expected payoff, using the probability function Pr_G to calculate their expectations.

Player i's expected utility for $s_{i\alpha}$ is defined as

$$E(s_{i\alpha}) = \sum Pr_G (\bar{s}_n | s_{i\alpha}) \, u_i(\bar{s}_n)$$

where the sum ranges over all n-tuples $\bar{s}_n$ containing $s_{i\alpha}$, and where $u_i(\bar{s}_n)$ is the i'th element of vector $\bar{U}(\bar{s}_n)$.

AV (strategic rationality). If $E(s_{i\alpha}) \neq \max_{s \in S_i} E(s)$,

then $Pr_G(s_{i\alpha}|U) = 0$, where U is the universal event.

Axiom IV (sufficiency) places restrictions on the players' probability function only if the game has three or more strategies per player. The case m = 2 is a "loose end" which is taken care of by the following axiom. Without this axiom the theorems to follow would be limited to the case $m \geq 3$.

The axiom states that the players of a two-strategy-per-player game must have a probability function identical to one for the three-strategy game formed by augmenting each player's strategy set with a "dummy" strategy, i.e., one which all players are sure will not be used.

AVI (extension). Let $G_3 \in \mathscr{G}^3_{UC}$, and let G be like G_3 except that s_{i3} is eliminated from each player's strategy set. Then Pr_G has an extension on $\mathscr{S}_{G_3} \times (\mathscr{S}_{G_3} - \varphi)$ satisfying AI to AV.

It is clear that any extension, Pr_{G_3} will be such that $Pr_{G_3}(s_{i3}|U) = 0$.

This completes the list of axioms. If Pr_G satisfies AI-AVI, it will be called strategic for the same G.

Theorem 1 states that if a function Pr_G is strategic it can be assigned a number $\lambda(Pr_G)$ which measures the degree to which one player's strategy is evidentially independent of the others, i.e., the "weakness of evidence" among the strategies. If $\lambda(Pr_G) = 0$ a player's move is conclusive evidence that the other player will use the corresponding move. If $\lambda(Pr_G) = \infty$ the players' moves are evidentially irrelevant to each other. The exact definition of $\lambda(Pr_G)$ is given in Theorem 1.

The fact that a single number will do all this is not trivial or obvious, and it will not be true unless Pr_G satisfies AIV, the axiom of sufficiency, since the strategies of the game G will in general be distinct and the probability of a given event will change if the strategies named in it are changed. One might think that a long series of indices would be necessary, but it turns out that only one, $\lambda(Pr_G)$ is required.[4]

Theorem 1. Let $G \in \mathscr{G}^m_{UC}$ and let Pr_G satisfy AI-AIV and AVI. Let $i,j \in I_m$ with $i \neq j$. It is possible to choose α,β such that $\alpha \neq \beta$ and P_α, $P_{\alpha|\beta}$ are not both 0.

$$\text{Define} \quad \lambda(Pr_G) = P_{\alpha|\beta}/(P_\alpha - P_{\alpha|\beta}) \quad \text{if } P_\alpha \neq P_{\alpha|\beta}$$

$$= \infty \quad \text{if } P_\alpha = P_{\alpha|\beta} \neq 0$$

where $P_\alpha = Pr_G(s_{i\alpha}|U)$ and $P_{\alpha|\beta} = Pr_G(s_{i\alpha}|s_{j\beta})$. Then $\lambda(Pr_G)$ is independent of the choice of α, β, i and j.

Hereafter $\lambda(Pr_G)$ will be abbreviated λ.

To choose an optimal strategy a player must know the likelihood of the other players' strategies conditional on each strategy of his own. That is, he must know the values of $Pr_G(\bar{s}_{-i}|s_{i\alpha})$ for all α and all $\bar{s}_{-i}$, the latter defined as the (n-1)-tuples of strategies for the other players.

Theorem 2 gives the possible values of $Pr_G(\bar{s}_{-i}|s_{i\alpha})$, stating that they are determined by at most m parameters, where m is the size of each player's strategy set.

Of the total set of parameters, m-1 are determined by choosing values of the a priori probabilities of the m strategies, i.e., $Pr_G(s_{i\alpha}|U)$ for $\alpha = 1$ to m. These values are required to sum to 1, resulting in m-1 independent parameters. The final parameter is $\lambda(Pr_G)$, the measure of evidential independence of Pr_G.

Theorem 2. Let $G \in \mathscr{S}^m_{UC}$ and let $\bar{s}_{-i}$ be an (n-1)-tuple of strategies for the players other than i, in which n_α players choose strategies of type α. If Pr_G satisfies AI to AIV and AVI, then

either 1) $\lambda \in [0,\infty)$ and for all $\bar{s}_{-i}$, all $i \in N$ and all $s_{i\alpha} \in S_i$,

$$Pr_G(\bar{s}_{-i}|s_{i\alpha}) = \frac{\prod_{i=1}^{n_\alpha}(i+P_\alpha\lambda)}{(n-1+\lambda)\cdots(2+\lambda)(1+\lambda)} \prod_{\beta\neq\alpha} \prod_{i=1}^{n_\beta} (i-1+P_\beta\lambda)$$

where P_α abbreviates $Pr_G(s_{i\alpha}|U)$ and where the product is taken to be 1 if the index set is empty.

or 2) $\lambda = \infty$ and $Pr_G(\bar{s}_{-i}|s_{i\alpha}) = \prod_{\beta=1}^{m} P_\beta n_\beta$.

Theorem 2 is deficient in two respects. First, it is supposed to state a condition on the function Pr_G but the nature of the condition is somewhat clouded by the use of λ. The parameter λ is a function of Pr_G so that conditions on λ are in fact conditions on Pr_G, but it would be clearer to eliminate λ and deal with values of Pr_G directly. This seems to be impossible in general, but possible for a specific class of games as shown in Theorem 3.

The second deficiency is that Theorem 2 does not complete characterize strategic probability functions, since AV, the axiom of rationality a component of the definition of "strategic" is absent in Theorem 2's list of conditions. However AV is introduced in the next theorem, and strategic probability functions are characterized by a single formula.

Theorem 2 deals with Commons Dilemma games which have generated a great deal of interest because of their elegent properties and their implications for sociology and ecology. Values of the strategic probability functions for these games can be given by a simple expression as stated in Theorem 3.

Dawes (1980) has defined the Commons Dilemma games as those in which each player has two strategies, a cooperation strategy C and a defection strategy D, and the payoffs functions are linear in the number of cooperators and have equal slopes. These are written

$$U_C(n_C) = \frac{n_C}{n}(a+1) - a \qquad (2)$$

$$U_D(n_C) = \frac{n_C}{n}(a+1)$$

where $U_C(n_C)$ and $U_D(n_C)$ are the payoffs for cooperation and defection respectively if n_C players cooperate. (The functions given above are a normalized form such that U_D has a minimum of 0, and U_C has a maximum of 1.) In order that the games be strategically non-trivial it is required that $a > 1/(n-1)$. Commons Dilemma games are labelled $\mathcal{G}_{CD}$. Clearly $\mathcal{G}_{CD}$ is a subset of $\mathcal{G}^2_{UC}$. In the two-player case a Commons Dilemma game becomes the familiar Prisoners' Dilemma.

<u>Theorem 3</u>. Let $G \in \mathcal{G}_{CD}$ with payoff functions as given above. Let P_C and P_D be the players' unconditional probabilities of cooperation and defection, i.e., $P_C = Pr_G$ (i uses C|U), $P_D = Pr_G$ (i uses D|U). If Pr_G is strategic for G one of the following holds:

1) $P_C = 1,\ P_D = 0,\ P_{C|D} \in \left[0, \frac{n}{(n-1)(a+1)}\right)$

$P_{D|C} = 0$, so that all players cooperate and have expected payoff 1.

or 2) $P_C \in [0,1],\ P_D = 1 - P_C$

$P_{C|D} = \frac{n}{(n-1)(a+1)} P_C,\ P_{D|C} = \frac{n}{(n-1)(a+1)} - P_{C|D}$

the probability that i players cooperate,

$$Pr_G(n_C = i) = \frac{n!}{i!(n-i)!} \frac{\prod_{j=1}^{i}\left(j-1+P_C\frac{n}{(n-1)a-1}\right)\prod_{j=1}^{n-1}\left(j-1+P_D\frac{n}{(n-1)a-1}\right)}{\prod_{j=0}^{n-1}\left(j+\frac{n}{(n-1)a-1}\right)}$$

and all players have expected payoff P_C.

or 3) $P_C = 0$, $P_D = 1$, $P_{D|C} \in \left(\frac{n}{(n-1)(a+1)}, \infty\right]$

and $P_{C|D} = 0$ so that no players cooperate and all have expected payoff 0.

Example 1. Suppose an 8-player Commons Dilemma game has payoff functions

$$U_C(n_C) = n_C - 8$$
$$U_D(n_C) = n_C$$

where n_C is the number of cooperators.

Normalizing so as to follow the format of (2) gives the equivalent game with payoff functions

$$U_C(n_C) = .25\, n_C - 1$$
$$U_D(n_C) = .25\, n_C$$

so that a = 1.

Theorem 3 gives a set of possible values of P_C, P_D, $P_{C|D}$ as shown in Figure 2. To choose some point in the set of inductive solutions, players might evaluate $P_{C|D}$ of $P_{D|C}$ to measure the strength of evidence between their own and their opponents' moves.

The next theorem, Theorem 4, gives limiting values of Pr_G for a large number of players.

Theorem 4. Let $G \in \mathscr{G}_{CD}$ with payoffs as given above. If Pr_G is strategic for G, one of the following holds in the limit as $n \to \infty$

1) $P_C = 1$, $P_D = 0$, $P_{C|D} \in \left[0, 1/(a+1)\right)$,
$P_{D|C} = 0$ so that all players cooperate and have expected payoff 1.

or 2) $P_C \in [0,1]$, $P_D = 1 - P_C$,
$P_{C|D} = 1/(a+1)\, P_C$, $P_{D|C} = 1/(a+1) - P_{C|D}$, and
the probability density that proportion x of the players cooperate is

$$f(x) = \frac{\Gamma(1/a)}{\Gamma(P_{C}/a)\Gamma(P_{D}/a)} \; x^{P_C/a - 1} (1-x)^{P_D/a - 1}$$

i.e., x has the Beta distribution with parameters P_C/a and P_D/a. All players have expected payoff P_C.

or 3) $P_C = 0$, $P_D = 1$, $P_{D|C} \in \left(1/(a+1), \infty\right]$, $P_{C|D} = 0$ so that all players defect and have expected payoff 0.

The Beta distribution strictly defined is the one given in part 2) of Theorem 4. The discrete distributions of parts 1) and 3) are limiting values of the Beta as one or the other of its parameters goes to zero. If we generalize the definition of the distribution to include these extremes, we can make the simple statement that in Commons Dilemma games the proportion of cooperators will follow the Beta distribution in the limit.

One part of the theory we would like to describe but cannot at this point, is a method of determining unique appropriate values of Pr_G. However, we will speculate about some possible directions the theory might take. Probabilities might be specified by axioms that are a part of the general theory of inductive logic (Carnap and Jeffrey, 1971) or on the other hand the further axioms might be like AV (strategic rationality) and deal specifically with game situations. Another possibility is that further axioms are based not on the abstract game as the axioms here are, but on the particular physical realization of the game. For example, the evidential relations among the moves might depend on the actions used to make the moves or on the similarities of the events yielding the outcomes and not solely on the payoff values.

Theorems 3 and 4 can be generalized in two ways. First the limiting Beta distribution for large n is not a property only of Commons Dilemma games, but holds true for any game in $\mathscr{G}_{UC}$.

Secondly if a game in $\mathcal{G}_{UC}^{m}$ has more than two strategies per player, m > 2, then the limiting distribution is the generalized Dirichlet distribution (Ferguson, 1973), an m-parameter generalization of the Beta distribution.

We will not prove these two statements since this can be done with methods almost identical to the proofs of Theorems 3 and 4.

Finally, Theorem 5 guarantees that strategic probability functions exist for any finite game. This is so since the set of inductive solutions includes every Nash equilibrium that reflects the symmetry of the game, and such an equilibrium always exists.

<u>Theorem 5</u>. For all $G \in \mathcal{G}$, there exists a Pr_G that is strategic for G.

III. Discussion

Act-Dependent Probabilities and Newcomb's Paradox

The most controversial aspect of the theory, in our view, is that the conditional probabilities of Matrix 2 may show dependence on the decider's action. In the terminology of Gibbard and Harper (1978) we allow act-dependent probabilities. The question of whether such probabilities are appropriate has usually not been dealt with explicitly by formal treatments of decision-making, (one exception is Jeffrey (1965)), but usually there has been an implicit attitude that actions and states of nature must be probabilistically independent, at least in the context of decisions involving an inanimate nature.

Recently the issue of act-dependence has arisen in the discussion of a decision-making puzzle known as Newcomb's Paradox. The Paradox has a close relationship to our inductive solution, so we will state one version of it here. A decision-maker (DM) can choose either 1) the contents of boxes A and B, or 2) the contents of box A only. Box B contains $1000, while A contains either $1,000,000 or nothing. The contents of box A depend on a predictor (PR) who attempted to predict DM's choice. If PR predicted DM would choose action 1), he put $0 in box A. If PR predicted action 2), he put $1,000,000 in box A. PR's action was taken before DM's decision. DM knows the above rules but does not know PR's action. DM is reliably informed that PR's prediction, be it 1) or 2), is correct with probability .9. The "paradox" arises in the following manner. Expected utility principles lead to a choice of box A alone (its expected utility is $900,000 versus A & B's expected utility of $101,000) while the Sure-thing Principle leads to a choice of boxes A and B, since the choice of Box A is a dominated strategy. The payoff and probability matrices are shown as Matrix 16 and 17.

		States	
		$1,000,000 in Box A	$0 in Box A
DM's	Box A & B	$1,001,000	$1,000
	Box A	$1,000,000	$0

Matrix 16

Payoffs to DM

.1	.9
.9	.1

Matrix 17

Conditional Probabilities

A number of authors have advocated the choice of boxes A and B (Gibbard and Harper, 1976) and others the choice of box A alone (Bar-Hillel and Margolit, 1972).

The dilemma facing DM is closely related to that of our game players. A DM who chooses Box A alone is, like our players, assigning act-dependent probabilities as is clear from Matrix 17. His subjective probability of receiving a large reward in Box A varies depending on his own choice. If he is permitted to use his own move as evidence about PR's choice, then box A alone will be the rational choice.

Those who regard the choice of one box by DM as rational may accept the logic of the inductive solution, while those who favour the choice of two boxes will tend not to since it involves act-dependent probabilities.

Comparison with minimax and equilibrium solutions.

It should be clear that our theory involves a different approach than the traditional solutions. We will make some comparisons now and point out some of the advantages of the inductive solution.

1) The inductive solution avoids the necessity of introducing mixed strategies.

The traditional solution concepts require that a player be able to construct any probabilistic combination of pure strategies, that is, be able to produce a sample from his pure strategies, each one to be selected with a specified probability. The sampling must be available at no further cost above the straightforward choice of a pure strategy. Without this assumption, Nash equilibria and minimax strategies would not exist for a large class of games.

How is this assumption to be interpreted in terms of the real world? Clearly the randomizing device must be either external or internal. People are rarely equipped with an external device of this type, at least for the conflicts typical of everyday life. If the device is to be regarded as internal, difficulties arise. A number of psychological experiments have shown that people are poor generators of random numbers. When asked to produce a simple uniform distribution, they have shown systematic biases (Baddeley, 1966; Chapanis, 1953). We believe that the requirement that a player be able to generate mixed strategies limits the traditional theories' applicability to real behaviour.

Even if players were usually able to randomize, it would be possible in principle to specify games in which they cannot. What would constitute rational behaviour in such games? The traditional theories would be silent, and so as general guides to rational behaviour they are incomplete from a theoretical point of view.

The inductive solution does not require randomization since players choose one move or another directly. In those situations where two or more moves have maximal expected utility, the player will be uncertain what he will do before he makes a decision and will hold a probability distribution over his own choices. This will not be an objective probability distribution in the sense of long-term relative frequencies generated by some device, but will be a statement of the player's degrees of belief. Having arrived at an appropriate probability distribution consistent with the arguments of the inductive solution, the player will have to make a final choice if more than one strategy has maximal utility, but there is no requirement of randomization involved in the final choice.

2) The inductive value admits the possibility of mutual cooperation in the Prisoners' Dilemma game.

According to the traditional theory, in the example Prisoners' Dilemma game of Matrix 18 both players would choose outcome (0,0), an equilibrium point, rather than (1,1), which is Pareto-optimal.

Some writers have regarded this as unsatisfactory on the logical grounds that rational players should not engage in "mutual punishment", and also on the empirical grounds that real players usually do not act this way.

	C_1	C_2
R_1	1, 1	-3, 5
R_2	5,-3	0, 0

Matrix 18

The inductive solution of the Prisoners' Dilemma game includes mutual cooperation as a possible outcome. The game of Matrix 18 for example has strategic probability matrices for Player 1 given by Matrices 19, 20 and 21.

1	0
1-q	q

with $.8 \leq q \leq 1$

Matrix 19

$2 - \frac{5}{4}q$	$\frac{5}{4}q - 1$
$1 - q$	q

with $.8 \leq q \leq 1$

Matrix 20

p	1 - p
0	1

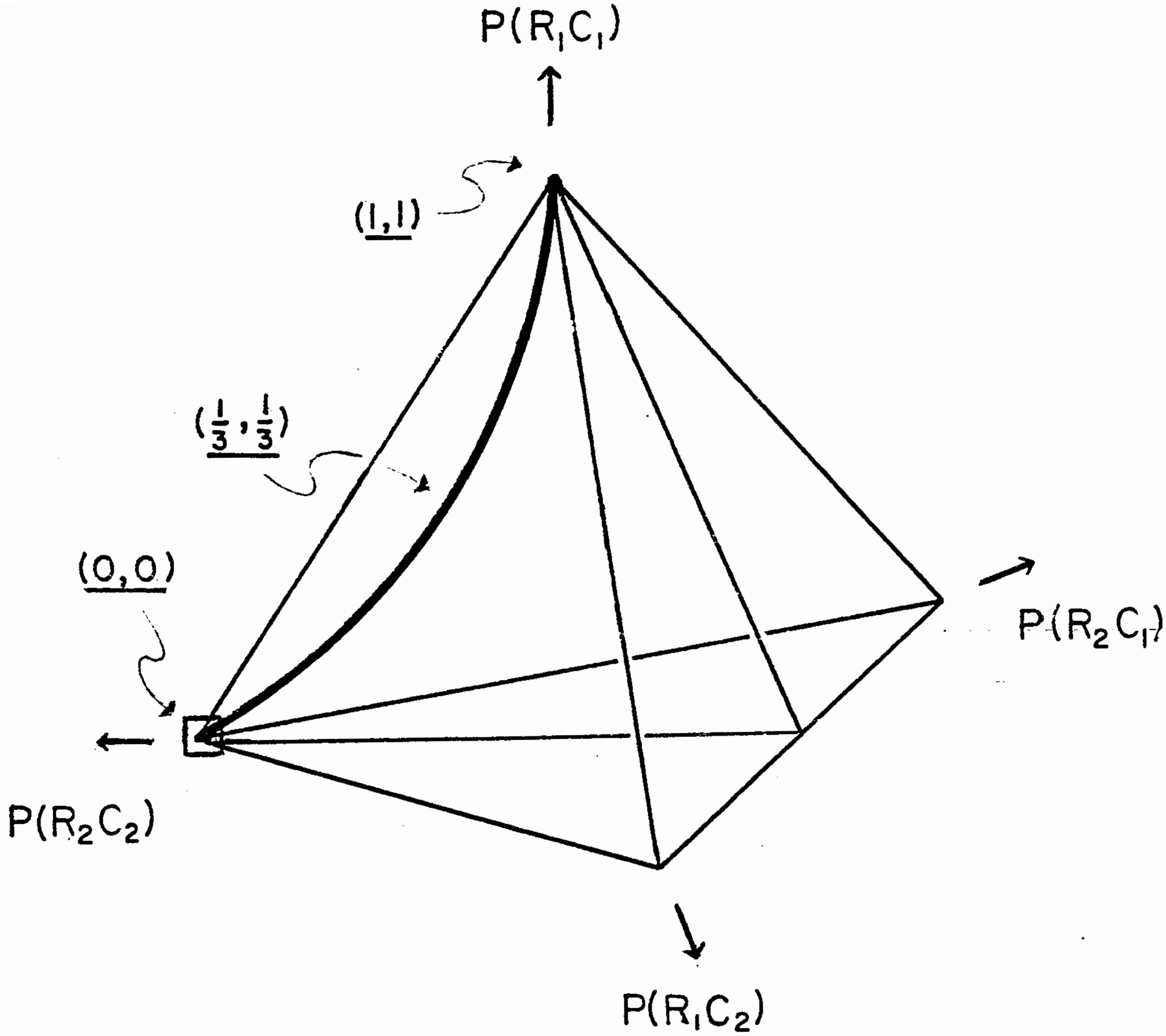
with $0 \leq p < .75$

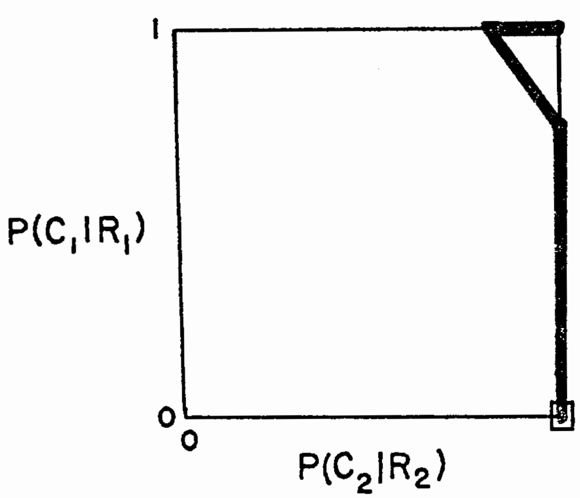
Matrix 21

In the event of Matrix 19, both players are certain of a mutually cooperative choice since the utilities of R_1 and C_1 are higher than those of R_2 and C_2. For Matrix 21, R_2 and C_2 are certain. If the players hold a probability matrix of the form of Matrix 20, any of the four outcomes is possible since the utilities of both strategies are equal. The unconditional probabilities associated with Matrix 20 may be calculated. They are $Pr(R_1 \text{ and } C_1) = (8 - 5q)(1 - q)/q$ and $Pr(R_2 \text{ and } C_2) = (5q - 4)$ and thus the probability of mutual cooperation can be positive, as shown in Figure 3.

Note that the inductive solution sometimes rationalizes cooperation and sometimes non-cooperation, depending on the players' views of the evidential

Figure 3 About Here





	C_1	C_2
R_1	1,1	-3,5
R_2	5,-3	0,0

Figure 3. The inductive solutions (solid line) and Nash equilibrium (square) for the Prisoners' Dilemma game of Matrix 18.

relationships among the moves. The theory is not an argument for the cooperative choice, but it explains only the possibility of this choice.

At least two other justifications of mutual cooperation have appeared. One is the theory of metagames (Howard, 1971) and the other is based on the theory of infinitely repeated games (Selten, 1978).

Comparing these with the present approach, each stresses a different aspect of how the Prisoners' Dilemma game is imbedded in real-world situations. The theory of metagames may be interpreted to suggest that players typically do not choose strategies in an isolated game, but adopt dispositions to play such games in a certain way (Robinson, 1975). The theory of infinitely repeated games suggests that conflict situations may occur again and again involving the same opponents. Our own solution emphasizes relationships of evidence. The three approaches are complementary in explaining people's behaviour.

As well as the empirical question of how to predict behaviour there is the theoretical question of what constitutes rational behaviour in an abstract situation. Here we believe the inductive solution has an advantage. The other concepts do not (and for some authors were not intended to) address themselves to a one-shot game with the two specified moves. They introduce either mixed strategies and extra games in the case of the theory of infinitely repeated games, or extra strategies in the case of the metagames theory. Rational behaviour is delineated for a situation different than the one facing the prisoners. The inductive solution, in our view, answers the question as it is given.

3) The inductive solution avoids certain logical difficulties in the rationale for the minimax theory.

The most careful justification of the minimax theory that we know of is the treatment given by von Neumann and Morgenstern (1953). Except for the novel approach of Harsanyi (1973), all subsequent justifications seem to be variations on the original.

They offer two lines of argument, one based on the idea of "security levels", (1953, their section 14.5 for games with saddlepoints, and section 17.8 for games without saddlepoints), and the other known as the "indirect proof" (section 17.3). The two arguments complement one another in von Neumann and Morgenstern's view, the security argument showing that the minimax solution is satisfactory, and the indirect proof that it is uniquely so (p. 148, footnote 5). We will evaluate each argument now.

Regarding security levels, they point out two advantages of the minimax strategy for Player 1:

A) it guarantees 1 a value v, no matter what 2 does, v being the highest such lower bound (1953, 14:C:d, 17:C:d).

B) it limits Player 2's gain to -v, no matter what 2 does, -v being the lowest such upper bound (1953, 14:C:e, 17:C:e)

On this basis they assert that the "good way for Player 1 to play" is to use a minimax strategy (1953, 14:C:a, 17:C:a). They regard this as a <u>definition</u> of "good way" in the context of games (1953, 14:5:2, second paragraph).

First of all it is not clear what the innate advantage is of consideration B), limiting the opponent's maximum. Players are not malicious. Any of the attractiveness of B) to Player 1 is due to the game being zerosum, so that B) indirectly results in a higher minimum payoff to player 1 himself, but this has already been stated as consideration A), and thus B) is superfluous.

Focusing on consideration A), why should Player 1 be concerned with his minimum possible payoff as opposed to, say, his maximum? He is not assumed to be risk-averse. Clearly the concern with security levels is a version of the well-known maximin rule of decision-making under uncertainty.

The maximin rule of decision-making strikes us as arbitrary in general, and especially inappropriate in the context of games. Note that by using the maximin decision rule (i.e., the minimax strategy ($\frac{1}{2}$,$\frac{1}{2}$)) in a game such as Matrix 22, Player 1 is also, unfortunately, placing an upper bound on his own payoffs,

1	-1
-1	1

Matrix 22

the worst possible upper bound. Player 1 can never achieve more than 0, because of his own strategy choice. The minimax strategy is the unique one for which this is so and thus is uniquely bad from this viewpoint. To be concerned with the lower bound and ignore the upper bound is to be arbitrary or pessimistic.

It seems specially inconsistent for these authors to advocate ($\frac{1}{2}$,$\frac{1}{2}$) on the grounds of consideration B), that Player 1 is placing the lowest roof on the opponent's gain, yet ignore the fact he is at the same time limiting his own gain in a similar way (Ellsberg, 1956; McLennen, 1977)

This critique applies as well to games with saddlepoints. In the case of Matrix 23, each player uses his first pure strategy according to minimax, but each is thereby decreasing his own maximum possible gain.

3	4	5
2	5	0
1	0	6

Matrix 23

The maximin rule of decision-making generated substantial interest at the time of von Neumann and Morgenstern's writing, but since then its shortcomings have become well-known. It is odd that the security level argument which we have argued to be a version of the maximin rule has continued to be accepted in the theory of games.

A proponent of the argument might respond that a player by deviating from $(\frac{1}{2},\frac{1}{2})$ opens himself up to exploitation. But the game is to be played only once and there is no provision for Player 2 spying on Player 1 or engaging in any form of precognition about his strategy choice. Player 1 can therefore ignore the exploitation argument.

Von Neumann and Morgenstern themselves explicitly reject dynamic arguments that assume sequential play (1953, 17:3:1). The "indirect proof", to which we now turn, was intended to justify such dynamic considerations, including possibilities of being "found out", as heuristic guides even though the game is played only once.

They state the indirect proof as follows:

> "[The indirect proof] consists in imagining that we have satisfactory theory of a certain type, trying to picture the consequences of the imaginary intellectual situation, and then drawing conclusions from this as to what the hypothetical theory must be like in detail . . . Let us now imagine that there exists a complete theory of the zerosum two-person game which tells a player what to do and which is absolutely convincing. If the players knew such a theory, then each player would have to assume that his strategy had been 'found out' by his opponent. The opponent knows the theory and he knows that a player would be unwise not to follow it. Thus the hypothesis

of the existence of a satisfactory theory legitimizes our investigation of the situation when a player's strategy is found out by an opponent."

(1953, pp. 147-148)

Only the minimax strategy remains sensible after it has been discovered and thus, it is argued, it must be the rational strategy. Von Neumann and Morgenstern's view is that the indirect argument justifies the application of the spyproof argument. Each player should play _as if_ his strategy were to be found out by the opponent, but he were not to find out the opponent's.

What have von Neumann and Morgenstern really demonstrated? They have not shown that minimax strategies must be the unique solution. They can conclude only that _if a unique solution exists_, it must be the minimax. This is a weaker statement since it is conditional on the solution being unique.

A theory of rational action can fall into one of three categories:

A) the theory can state that no strategies are rational in the situation,

B) the theory can specify a unique strategy as rational,

C) the theory can specify a set of two or more strategies as rational.

The indirect argument shows that the premise B) leads to the minimax, but to derive the minimax without qualification A) and C) must be ruled out. Von Neumann and Morgenstern accept the view that the indirect argument must be complemented by some further proof to show that a solution does exist

(1953, p. 148). They state that the security level properties of the minimax and its guaranteed existence for all finite two-person zerosum games, provide this proof. They have thus attempted to rule out the possibility of A) but have not attended to C).

If the possibility is admitted that two or more strategies are rational, the argument that an opponent can deduce one's strategy and therefore a player must act as if his strategy had been found out, cannot be made.

It seems odd to us that von Neumann and Morgenstern use an argument that involves B) as a premise, since for many games (e.g., Morra) the minimax strategy itself is not unique, and their own n-person solution is not unique.

A further clue that something is amiss is shown by Matrix 24, published by Harsanyi (1964).

3, 1	0, 2
1, 4	2, 0

Matrix 24

An argument based on security levels leads to the maximin strategies (1/4,3/4), (2/5,3/5). An argument based on the indirect proof leads to a choice of the equilibrium point (4/5,1/5), (1/2,1/2). The two arguments lead to divergent strategy choices and thus contradict each other.

Of course Matrix 24 is not a zerosum game, whereas the sections of von Neumann and Morgenstern's works quoted above are intended to apply only to zerosum games. However there is no reason why the arguments <u>if valid</u> should not apply to Matrix 24 even though they were not intended to. The contradiction that arises shows that there is a logical fault in the arguments.

The inductive solution avoids the concept of security levels, and modifies the indirect proof to a form which we believe to be logically acceptable.

For zerosum games it leads to expectations for the players identical to the minimax value. In games such as Matrices 22 and 23, it is appropriate to regard the moves as providing no evidence for one another. In this case the inductive value of the games is identical to the von Neumann and Morgenstern value of the mixed strategy extensions of these games. The players' degrees of belief in the occurrence of the various outcomes are numerically identical to the minimax solution's relative frequency probabilities of these outcomes. In a sense the inductive solution provides an alternative reason for coming to the conclusions reached by the traditional theory. This particular application of the inductive solution does not involve the use of act-dependent probabilities which some authors have found philosophically objectionable.

4) The inductive solution gives a unified basis for individual decisions and games

The most significant advantage of the inductive solution in our view is a philosophical one: the special rationale of an equilibrium and its use in the context of static games are eliminated, and replaced by consideration of the probabilities the players can hold given they are in a game situation. This brings a non-cooperative solution under the domain of regular decision analysis.

Appendix - Proofs

We will abbreviate $Pr_G(s_{i\alpha}|U)$ by P_α, and abbreviate $Pr_G(s_{i_1\alpha_1} \cap \ldots s_{i_k\alpha_k}|s_{j_1\beta_1} \cap \ldots s_{j_\ell\beta_\ell})$ where all players $\{i_1,\ldots,i_k, j_1,\ldots,j_\ell\}$ are distinct, by $P_{\alpha_1,\ldots,\alpha_k|\beta_1,\ldots,\beta_\ell}$. Thus, for example, $Pr_G(s_{i\alpha}|s_{j\beta})$ is written $P_{\alpha|\beta}$. These are well-defined since by AII (symmetry) the probabilities are independent of the particular players using each strategy. The integers $\{1,\ldots,k\}$ will be denoted I_k.

Five lemmas will now be stated for use in the proofs of the theorems. The first three follow easily from the axioms of probability and the proofs will be omitted.

Lemma 1. Let e and e_i for $i \in I_k$ be events such that $e_i \cap e_j = \varphi$ for $i \neq j$ and $\bigcup_{i \in I_k} e_i = e$. If Pr_G satisfies AI (probability), then

$$\sum_{i \in I_k} Pr_G(e_i|e) = 1.$$

Lemma 2. If Pr_G satisfies AI (probability) then

$$P_{\alpha_1\ldots\alpha_{n-1}|\alpha_n} = P_{\alpha_1|\alpha_2\ldots\alpha_n} \times P_{\alpha_2|\alpha_3\ldots\alpha_n} \cdots \times P_{\alpha_{n-1}|\alpha_n}.$$

Lemma 3. If Pr_G satisfies AI (probability), if $e \subseteq e' \in \mathcal{S}_G$ and $e'' \in \mathcal{S}_G - \varphi$ then $Pr_G(e|e'') \leq Pr_G(e'|e'')$.

Lemma 4. Let $G \in \mathcal{L}^m_{UC}$. Suppose Pr_G satisfies AI (probability), AII (symmetry), AIII (positivity) and AIV (sufficiency). If m = 1, then $P_{\alpha|\alpha} = P_\alpha = 1$. If $m \geq 2$ one of the following, L1), L2),...,or L6), is satisfied:

L1) for all $\alpha,\beta \in I_m$, $\alpha \neq \beta$,

either $1 > P_{\alpha|\alpha} > P_\alpha > P_{\alpha|\beta} > 0$

or $1 > P_{\alpha|\alpha} > P_\alpha = P_{\alpha|\beta} = 0$

L2) for all $\alpha,\beta \in I_m$, $\alpha \neq \beta$,

$$1 = P_{\alpha|\alpha} > P_\alpha \geq P_{\alpha|\beta} = 0,$$

L3) for all $\alpha,\beta \in I_m$, $\alpha \neq \beta$

$$1 > P_{\alpha|\alpha} = P_\alpha = P_{\alpha|\beta} \geq 0,$$

L4) for all $\alpha,\beta \in I_m$, $\alpha \neq \beta$

$$P_{\alpha|\alpha} = P_\alpha = P_{\alpha|\beta} = 0 \text{ or } 1,$$

L5) for all $\alpha,\beta \in I_m$, $\alpha \neq \beta$

$$\text{either } 1 = P_{\alpha|\alpha} = P_\alpha > P_{\alpha|\beta} > 0$$

$$\text{or } \quad 1 > P_{\alpha|\alpha} > P_\alpha = P_{\alpha|\beta} = 0,$$

L6) for all $\alpha,\beta \in I_m$, $\alpha \neq \beta$

$$P_{\alpha|\alpha} = 1, \; P_\alpha = 0 \text{ or } 1, \text{ and } P_{\alpha|\beta} = 0.$$

Proof of Lemma 4:

If $m = 1$, then Lemma 4 is satisfied as a consequence of Lemma 1.

From now on we will assume that $m \geq 2$. By AI (probability) for any α,

$$P_\alpha = \sum_{\delta \in I_m - \{\alpha\}} P_{\alpha|\delta} P_\delta + P_{\alpha|\alpha} P_\alpha .$$

Choosing a fixed $\beta \neq \alpha$ it follows from AIV (sufficiency) that $P_{\alpha|\delta} = P_{\alpha|\beta}$ so that

$$P_\alpha = P_{\alpha|\beta} \sum_{\delta \in I_m - \{\alpha\}} P_\delta + P_{\alpha|\alpha} P_\alpha$$

therefore $$P_\alpha = P_{\alpha|\beta} (1 - P_\alpha) + P_{\alpha|\alpha} P_\alpha$$

and $$P_\alpha (1 - P_{\alpha|\alpha}) = P_{\alpha|\beta} (1 - P_\alpha), \text{ for } \beta \neq \alpha. \qquad (3)$$

It will now be shown that

$$P_\alpha \geq P_{\alpha|\beta} \quad \text{for } \alpha,\beta \in I_m, \; \alpha \neq \beta. \qquad (4)$$

If $P_\alpha = 1$ the assertion (4) follows from AI (probability). If $P_\alpha < 1$ then $P_{\alpha|\alpha} \geq P_\alpha$ by AIII (positivity) and the assertion (4) follows from combining this inequality with (3).

Since $P_\delta > 0$ for some $\delta \in I_m$ by AI (probability), and since $1 \geq P_{\delta|\delta} \geq P_\delta$ by AI (probability) and AIII (positivity), the following four conditions are exhaustive: (It follows from the present proof that they are also mutually exclusive, although we will not need to use this fact now.)

C1) $\exists\delta\ (1 > P_{\delta|\delta} > P_\delta > 0)$

C2) $\exists\delta\ (1 = P_{\delta|\delta} > P_\delta > 0)$

C3) $\exists\delta\ (1 > P_{\delta|\delta} = P_\delta > 0)$

C4) $\exists\delta\ (1 = P_{\delta|\delta} = P_\delta > 0)$.

The rest of the proof of Lemma 4 will be structured as follows. Under the headings "C1" to "C4" we will assume in turn each condition C1 to C4. It will be shown that C1 , C2 and C3 lead to statements L1, L2, and L3, respectively of the lemma, and that C4 leads to either L4, L5 or L6 of the lemma. In each case α and β will be two arbitrarily chosen strategy types with $\alpha \neq \beta$, and δ will be a strategy type satisfying the appropriate condition, C1, C2, C3 or C4.

<u>C1</u>). First assume $\delta = \alpha$. Then by C1 and by (4) we have for all $\beta \neq \alpha$, $1 > P_{\alpha|\alpha} > P_\alpha \geq P_{\alpha|\beta}$ and $P_\alpha > 0$.

Applying (3) shows that $P_{\alpha|\beta}$ must be greater than zero, so that

$1 > P_{\alpha|\alpha} > P_\alpha > P_{\alpha|\beta} > 0$ and case L1 holds.

Alternatively assume $\delta \neq \alpha$ but $P_\alpha > 0$. Then $P_\delta < 1$. By this inequality, by C1 and by (3) with δ and α substituted for α and β respectively, we have $P_\delta(1 - P_{\delta|\delta}) = P_{\delta|\alpha}(1 - P_\delta) > 0$, and $1 - P_\delta > 1 - P_{\delta|\delta}$, and therefore

$$P_\delta > P_{\delta|\alpha} > 0. \qquad (5)$$

Multiplying each term of (5) by the positive quantity $P_\alpha|P_\delta$ and expressing $P_\delta P_{\alpha|\delta}$ as $P_\alpha P_{\alpha|\delta}$ gives

$P_\alpha > P_{\alpha|\delta} > 0.$ (6)

By AIV (sufficiency), and (6)

$P_\alpha > P_{\alpha|\beta} > 0.$ (7)

Since $P_\delta > 0$, then $P_\alpha < 1$. This and (7) implies that both sides of (3) are positive, so $P_{\alpha|\alpha} > P_\alpha$. Then $1 > P_{\alpha|\alpha} > P_\alpha > P_{\alpha|\beta} > 0$ so that case L1 holds.

The final possibility under C1 is that $P_\alpha = 0$ and $\delta \neq \alpha$. By (4) and AI (probability) for all β, $1 \geq P_{\alpha|\alpha} \geq P_\alpha = P_{\alpha|\beta} = 0$. But since $P_{\delta|\alpha} > 0$ which may be shown similarly to the derivation of (5), and since $P_{\delta|\alpha} + P_{\alpha|\alpha} \leq 1$, then $P_{\alpha|\alpha} < 1$.

To show $P_{\alpha|\alpha} > 0$, by AI (probability)

$$\sum_{\gamma \in I_m, P_\gamma > 0} P_\gamma = 1.$$

But if $P_\gamma > 0$, then $P_\gamma > P_{\gamma|\alpha}$, shown in a manner similar to (7).

Thus

$$\sum_{\gamma \in I_m, P_\gamma > 0} P_{\gamma|\alpha} < 1.$$

But since $P_\alpha = 0$,

$$P_{\alpha|\alpha} = 1 - \sum_{\gamma \in I_m, P_\gamma > 0} P_{\gamma|\alpha}$$

so that $P_{\alpha|\alpha} > 0$. Thus $1 > P_{\alpha|\alpha} > P_\alpha = P_{\alpha|\beta} = 0$ and Lemma 4, case L1 holds.

C2). Suppose $P_\alpha > 0$ and $\alpha = \delta$. Then C2 and (3) imply $P_\alpha > P_{\alpha|\beta} = 0$. By this and C2, $1 = P_{\alpha|\alpha} > P_\alpha > P_{\alpha|\beta} = 0$ and Lemma 4, case L2 holds.

If $P_\alpha > 0$ and $\alpha \neq \delta$, then substituting δ and α for α and β respectively in (3) gives $P_{\delta|\alpha} = 0$. From $P_\delta > 0$ it follows that $P_\alpha < 1$.

Also by AIV (sufficiency),

$P_{\alpha|\beta} = P_{\alpha|\delta} = P_{\delta|\alpha} \cdot (P_\alpha / P_\delta) = 0.$ (8)

Since $P_{\alpha|\beta} = 0$ and $P_\alpha > 0$, then by (3) we have $P_{\alpha|\alpha} = 1$, so that $1 = P_{\alpha|\alpha} > P_\alpha > P_{\alpha|\beta} = 0$ and case L2 holds.

Alternatively if we assume $P_\alpha = 0$ then by (4), for all $\beta \neq \alpha$

$$P_{\alpha|\beta} = 0. \tag{9}$$

By AI (probability)

$$\sum_{\gamma \in I_m} P_{\gamma|\alpha} = 1. \tag{10}$$

For any term in (10) with $P_\gamma = 0$, substituting γ for β in (3) gives $P_{\gamma|\alpha} = 0$. By (9) any term in (10) with $P_\gamma = 0$ and $\gamma \neq \alpha$ is zero, so that $1 = P_{\alpha|\alpha} > P_\alpha = P_{\alpha|\beta} = 0$, and case L2 holds.

<u>C3</u>. If $P_\alpha > 0$ and $\alpha = \delta$, then by virtue of L3 and (3) for $\alpha \neq \beta$,

$$1 > P_{\alpha|\alpha} = P_\alpha = P_{\alpha|\beta} > 0 \tag{11}$$

and case L3 holds.

If $P_\alpha > 0$ and $\alpha \neq \delta$, then $P_\delta > 0$ implies that $P_\alpha < 1$. Substituting δ for α in (3), $P_\delta = P_{\delta|\beta}$ for $\beta \neq \delta$, and by AIV (sufficiency), $P_\delta = P_{\delta|\alpha}$. Multiplying both sides of the latter by P_α / P_δ, we have $P_\alpha = P_{\alpha|\delta}$. By AIV (sufficiency), $P_\alpha = P_{\alpha|\beta}$. Thus

$$1 > P_\alpha = P_{\alpha|\beta} > 0. \tag{12}$$

By (12) and (3), $1 > P_{\alpha|\alpha} = P_\alpha = P_{\alpha|\beta} > 0$ and case L3 holds.

If we assume $P_\alpha = 0$, then by (4) $P_{\alpha|\beta} = 0$. Also by AI (probability)

$$\sum_{\gamma \in I_m} P_{\gamma|\alpha} = 1. \tag{13}$$

But since

$$\sum_{P_\gamma > 0} P_\gamma = 1$$

then by (11) and (12)

$$\sum_{P_\gamma > 0} P_{\gamma|\alpha} = 1. \tag{14}$$

Comparing (13) and (14) shows that $P_{\alpha|\alpha} = 0$ if $P_\alpha = 0$. Thus case L3 holds.

<u>C4</u>. By AIV (sufficiency) $P_{\delta|\beta}$ is constant for all $\beta \neq \delta$. Suppose $P_{\delta|\beta} = 1$. Then if $P_\alpha > 0$ it follows that $\alpha = \delta$ and case L4 holds. If $P_\alpha = 0$, then $P_{\alpha|\beta} = 0$. Since $P_{\delta|\alpha} = 1$, then $P_{\alpha|\alpha} = 0$. Thus case L4 holds.

Alternatively if we assume $1 > P_{\delta|\beta} > 0$ for all $\beta \neq \delta$, it follows similarly that case L5 holds. If we assume that $P_{\delta|\beta} = 0$ for all $\beta \neq \delta$, it can be shown that case L6 holds. Lemma 4 is proven. □

For $\beta \in I_m$, $i \in N$ and $0 \le k_\beta \le k \le n-1$ let e be the event that of k specified players other than player i, exactly k_β are using strategy β. Then we define

$$f_\beta(k,k_\beta) = Pr_G(s_{i\beta}|e)$$

the probability that player i is using strategy β given the evidence of e. The function f_β is well-defined since by AII (symmetry), it is independent of the reference player i, and by AIV (sufficiency) it is independent of the particular strategy choices of the $k - k_\beta$ players not choosing strategy type β.

<u>Lemma 5</u>. If Pr_G satisfies AI (probability), AII (symmetry) and AIV (sufficiency) then for $k \le n - 1$

1) if there exists an $\alpha \in I_m$ such that $f_\alpha\,(k-1,0) > 0$, then for all $\beta \in I_m$, $\beta \neq \alpha$, and all k, k_β such that $0 \le k_\beta < k \le n-1$,

$$f_\beta(k,k_\beta) = \frac{f_\alpha(k,0)}{f_\alpha(k-1,0)}\; f_\beta(k-1,k_\beta),$$

2) if there exists an α such that $f_\alpha(k-1,1) > 0$, then for all $\beta \in I_m$, $\beta \neq \alpha$, and all k, k_β such that $0 \le k_\beta \le k - 1$, and $k \le n - 1$

$$f_\beta(k,k_\beta) = \frac{f_\alpha(k,1)}{f_\alpha(k-1,1)}\; f_\beta(k-1,k_\beta).$$

<u>Proof of Lemma 5</u>: Consider the probability

$$P_{\alpha\beta|\beta\ldots\beta\gamma\ldots\gamma} \qquad (15)$$

in which α, β and γ are distinct and in which k-1 strategies appear in the conditioning event, k_β of which are type β strategies and the remaining $k-1-k_\beta$ are type γ. Probability (15) may be expanded in two alternative ways by AI, 4), by including α or β in the conditioning event. Equating the two ways gives

$$P_{\alpha|\beta\beta\ldots\beta\gamma\ldots\gamma}\, P_{\beta|\beta\ldots\beta\gamma\ldots\gamma} = P_{\beta|\alpha\beta\ldots\beta\gamma}\, P_{\alpha|\beta\ldots\beta\gamma\ldots\gamma}$$

where β appears $k_\beta + 1$ times in the conditioning event of the first probability, k_β times in the second, third and fourth.

Therefore,

$$f_\alpha(k,0)\ f_\beta(k-1,k_\beta) = f_\beta(k,k_\beta)\ f_\alpha(k-1,0) \qquad (16)$$

for $\quad 0 \le k_\beta < k \le n-1.$

Under the assumption that $f_\alpha(k-1,0) > 0$, Lemma 5, case 1) follows. Lemma 5, case 2) may be proved by expanding the probability $P_{\alpha\beta|\alpha\beta\ldots\beta\gamma\ldots\gamma}$ in a similar fashion. □

<u>Proof of Theorem 1</u> For some α, i, j, $\lambda(Pr_G)$ is well-defined, since for some α, $P_\alpha > 0$. Thus λ takes on at least one value.

We next show that λ takes no more than one value. In case Pr_G is such that Lemma 4, L1 or L5 holds, λ where defined equals $P_{\alpha|\beta}/(P_\alpha - P_{\alpha|\beta})$. By AIV (sufficiency), λ is independent of β. To show it is independent of α, multiply numerator and denominator by the positive quantity P_β/P_α. It follows that $\lambda = P_{\beta|\alpha}/(P_\beta - P_{\beta|\alpha})$, which is independent of α by AIV (sufficiency). Therefore Theorem 1 is true for these cases.

In the event of Lemma 4, L2 and L6 clearly $\lambda=0$ wherever it is defined. Likewise for Lemma 4, L3 and L4, $\lambda=\infty$ where it is defined, and the theorem is proved. □

<u>Proof of Theorem 2</u> If G is a game in $\mathscr{G}^2_{UC}$ we may add a "dummy" strategy, one which all players are sure will not be used, to each player's set so as to transform G to a game in $\mathscr{G}^3_{UC}$. By AVI (extension) the values of Pr_G restricted to events involving the original strategies will be unaltered. We can therefore assume from now on that $m \ge 3$.

1. Assume that Pr_G is such that Lemma 4, L1 holds. Let $T \subset I_m$ be the set of strategy types δ that appear in $\bar{s}_{-i}$ i.e., $n_\delta > 0$, and that have unconditional probability zero, $Pr_G(s_{i\delta}) = 0$. The formula for $Pr_G(\bar{s}_{-i} | s_{i\alpha})$ will be derived for any α under each of the following conditions, which are mutually exclusive and exhaustive.

i) some $\delta \neq \alpha$ is in T,

ii) T is empty,

iii) α alone is in T, and $n = 2$,

iv) α alone is in T, and $n > 2$.

In case of condition i), $P_{\delta|\alpha} = 0$, so $Pr_G(\bar{s}_{-i} | s_{i\alpha}) = 0$ by Lemma 3. Theorem 2 is satisfied with $\lambda \in [0,\infty)$ since the product in formula (1) contains a zero factor.

In case of condition ii), let $\gamma \in I_m$ be such that $n_\gamma > 0$. Then $P_\gamma = f_\gamma(0,0) > 0$. We will show as a first step, that

$$f_\gamma(k,0) > 0 \qquad (17)$$

for all k, $0 \le k \le n - 1$. Substituting γ for α in Lemma 5, 1) implies that for any $\beta \in I_m$

$$f_\gamma(k,0)\ f_\beta(k-1,k-1) = f_\beta(k,k-1)\ f_\gamma(k-1,0). \qquad (18)$$

This is true even if $f_\gamma(k-1,0) = 0$, since both sides of (18) will equal zero by Lemma 3.

If we assume that $f_\gamma(k,0) = 0$ we can derive a contradiction. If $f_\gamma(k,0) = 0$ then either $f_\gamma(k-1,0) = 0$ or $f_\beta(k,k-1) = 0$ for all β. The latter is impossible since by Lemma 1 for $\beta \neq \delta$ we can consider an event e as evidence in which k-1 players use type β strategies, 1 uses type δ, and no other strategy types are used. Then

$$f_\beta(k,k-1) + f_\delta(k,1) + \sum_{\xi \in I_m - \{\beta,\delta\}} f_\xi(k,0) = 1$$

and since by AIII (positivity) if $f_\beta(k,k-1) = 0$ for all β, then

$$f_\delta(k,1) = f_\xi(k,0) = 0.$$

The alternative possibility, that $f_\gamma(k-1,0) = 0$ can be analyzed in a manner similar to the above to show that

$$f_\gamma(k-2,0) = f_\gamma(k-3,0) = \dots = f_\gamma(0,0) = 0.$$

A contradiction is reached since $f_\gamma(0,0) > 0$. Thus $f_\gamma(k,0) > 0$ for all k, $0 \le k \le n-1$. We may apply Lemma 5, 1), so that for $\beta \in I_m$, $0 \le k_\beta < k \le n-1$

$$f_\beta(k,k_\beta) = f_\beta(k-1,k_\beta) \cdot f_\gamma(k,0)/f_\gamma(k-1,0). \qquad (19)$$

From Lemma 1, assuming $\gamma \ne 1,2$

$$f_1(k, k-1) + f_2(k,1) + \sum_{\delta=3}^{m} f_\delta(k,0) = 1. \qquad (20)$$

Setting $\beta = 1$ to m, successively in (19) and substituting each in (20)

$$\frac{f_\gamma(k,0)}{f_\gamma(k-1,0)} f_1(k-1,k-1) + \frac{f_\gamma(k,0)}{f_\gamma(k-1,0)} f_2(k-1,1) + \frac{f_\gamma(k,0)}{f_\gamma(k-1,0)} \sum_{\delta=3}^{m} f_\delta(k,0) = 1.$$

Solving the above for $f_\gamma(k,0)$,

$$f_\gamma(k,0) = f_\gamma(k-1,0)/[f_1(k-1,k-1) + f_2(k-1,1) + \sum_{\delta=3}^{m} f_\delta(k-1,0)]. \qquad (21)$$

Formula (21) is a recursive expression in k for $f_\gamma(k,0)$ assuming $f_\gamma(0,0) > 0$. Values of $f_\gamma(k,k_\gamma)$ for $k_\gamma > 0$ may be calculated as follows. Choose $\beta \in I_m$ such that $P_\beta > 0$ and $\beta \ne \delta$. Next show in a manner similar to the derivation of (17) that $f_\beta(k-1,0) > 0$. Finally apply Lemma 5, 1) to derive for $1 \le k_\gamma \le k-1$,

$$f_\gamma(k,k_\gamma) = f_\gamma(k-1,k_\gamma) f_\beta(k,0)/f_\beta(k-1,0). \qquad (22)$$

Formulae (21) and (22) must be supplemented by an expression for the case $k_\gamma = k$. The following is a consequence of Lemma 1:

$$f_\gamma(k,k) = 1 - \sum_{\beta \in I_m - \{\gamma\}} f_\beta(k,0). \qquad (23)$$

Initial conditions for f_γ can be stated as follows:

$$f_\gamma(0,0) = P_\gamma \tag{24}$$

$$f_\gamma(1,0) = P_\gamma\lambda/(1+\lambda) \tag{25}$$

$$f_\gamma(1,1) = (1+P_\gamma\lambda)/(1+\lambda). \tag{26}$$

Expression (25) follows from the definition of λ given in Theorem 1. Expression (26) follows from that definition together with Lemma 1.

It may be verified that the following function (27) satisfies the recursive formulae (21), (22) and (23) and the initial conditions (24), (25) and (26). Since these completely determine f_γ, the function (27) is the only one to satisfy them.

$$f_\gamma(k,k_\gamma) = (k_\gamma + P_\gamma\lambda)/(k+\lambda) \tag{27}$$

Function (27) will now be used to calculate an expression for $Pr_G(\bar{s}_{-i} \mid s_{i\alpha})$. We may assume that the players in $N-\{i\}$ are ordered by the strategies they choose, i.e., the first n_1 players use strategy type 1, the next n_2 use type 2, etc. By Lemma 2

$$\begin{aligned} Pr_G(\bar{s}_{-i} \mid s_{n\alpha}) \\ = \; & f_1(n-1,n_1-1) \times f_1(n-2,n_1-2) \quad \cdots \quad \times f_1(n-n_1,0) \\ & \times f_2(n-n_1-1,n_2-1) \quad \cdots \quad \times f_2(n-n_1-n_2,0) \\ & \cdots \qquad\qquad\qquad\qquad \cdots \\ & \times f_\alpha(n-1-\textstyle\sum_{\beta=1}^{\alpha-1} n_\beta,\, n_\alpha) \quad \cdots \quad \times f_\alpha(n-1-\textstyle\sum_{\beta=1}^{\alpha} n_\beta,\, 1) \\ & \cdots \qquad\qquad\qquad\qquad \cdots \\ & \times f_m(n-1-\textstyle\sum_{\beta=1}^{n-1} n_\beta,\, n_m-1) \quad \cdots \quad \times f_m(1,0) \end{aligned} \tag{28}$$

where factors f_γ appear in (28) only for those strategy types γ such that $n_\gamma > 0$.

Substituting (27) in (28) gives the formula of Theorem 2, 1).

In case of condition iii), $n = 2$ so that determination of $Pr_G(\bar{s}_{-i} \mid s_{i\alpha})$ involves only the values $P_{\alpha|\alpha}$ and $P_{\beta|\alpha}$.

If $P_\beta = 0$ and $\beta \neq \alpha$ then $P_{\beta|\alpha} = 0$ by (4) and therefore $P_{\alpha|\alpha} = 1$.

If $P_\beta > 0$ there exists some $\delta \neq \beta$ with $P_\delta > 0$ by Lemma 4, 1). Then $P_{\beta|\alpha} = P_{\beta|\gamma}$ by AIV (sufficiency) and we can then define λ as in (23) to get

$$P_{\beta|\alpha} = P_\beta \lambda/(1+\lambda). \tag{29}$$

To determine $P_{\alpha|\alpha}$, we have by Lemma 1,

$$\begin{aligned} P_{\alpha|\alpha} &= 1 - \sum_{\beta\neq\alpha} P_{\beta|\alpha} \\ &= 1 - \sum_{P_\beta>0} P_{\beta|\alpha} \quad . \end{aligned} \tag{30}$$

By (25),

$$P_{\alpha|\alpha} = 1/(1+\lambda). \tag{31}$$

These values of $P_{\beta|\alpha}$ and $P_{\alpha|\alpha}$, (29) and (31) respectively are consistent with Theorem 2, with $\lambda \in (0,\infty)$.

Under condition iv), for any δ with $n_\delta > 0$ and $\delta \neq \alpha$ a formula may be derived for $f_\delta(k,k_\delta)$, $k_\delta \geq 0$, analogous to (27). For the purpose of substitution in (28), we also require a formula for $f_\alpha(k,k_\alpha)$, $k_\alpha \geq 1$. Lemma 5, 2) may be used as in the derivation of (21) to give

$$\begin{aligned} f_\alpha(k,1) = f_\alpha(k-1,1)/[&f_1(k-1,\ k-1) + f_2(k-1,1) \\ &+ \sum_{\delta=3}^{m} f_\delta(k,0)]. \end{aligned} \tag{32}$$

For $k_\alpha > 1$, $f_\alpha(k,k_\alpha)$ may be determined by using (22) and (23) substituted for γ.

Initial values of f_α are determined as follows.

$$f_\alpha(1,1) = 1 - \sum_{P_\beta>0} P_{\beta|\alpha}$$

as shown by (30).

Choosing β,γ such that P_β, $P_\gamma > 0$ we may define λ as in (23). Then $P_{\beta|\alpha} = P_{\alpha|\gamma} = P_\beta \lambda/(1+\lambda)$, so that $f_\alpha(1,1) = 1/(1+\lambda)$.

Thus $f_\alpha(k,k_\alpha)$ is completely determined for $k_\alpha \geq 1$ and it may be verified that (27) satisfies the initial conditions and the recursive formulae. In a manner similar to case ii), (28) is applied to show that Theorem 2 holds with $\lambda \in (0,\infty)$.

2. Assume Lemma 4, L2 holds. If $\bar{s}_{-i}$ contains a strategy type $\beta \neq \alpha$ then by Lemma 3, $Pr_G(\bar{s}_{-i} | s_{i\alpha}) = 0$. If $\bar{s}_{-i}$ contains only strategies of type α, then $Pr_G(\bar{s}_{-i} | s_{i\alpha}) = 1$ since the probabilities of all possible $\bar{s}_{-i}$ conditional on $s_{i\alpha}$ must sum to 1.

Thus the assumption of Lemma 4, L2 leads to Theorem 2 with $\lambda = 0$.

3. Assume that Pr_G is such that Lemma 4, L3 holds. If $P_\beta = 0$ for some β such that $n_\beta > 0$, then $Pr_G(\bar{s}_{-i} | s_{i\alpha}) = 0$ by Lemma 3. If $P_\beta > 0$ for all strategy types β that appear at least once in $\bar{s}_{-i}$ then we can deduce (21), (22) and (23) as above. Initial conditions $f_\beta(0,0) = f_\beta(1,0) = f_\beta(1,1) = P_\beta$ lead to Theorem 2 with $\lambda = \infty$.

4. Assume that Lemma 4, L4 holds. Let δ be the unique strategy type such that $P_\delta = 1$. By assumption of Lemma 4, L4, $P_{\beta|\alpha} = 0$ for $\beta \neq \delta$ and thus by Lemma 3, $Pr_G(\bar{s}_{-i} | s_{i\alpha}) = 0$ if $n_\delta \neq n-1$. Since the probabilities of all $\bar{s}_{-i}$ conditional on $s_{i\alpha}$ must sum to 1, then $Pr_G(\bar{s}_{-i} | s_{i\alpha}) = 1$ if $n_\delta = 1$. These values are consistent with Theorem 2 with $\lambda = \infty$.

5. Assuming that Lemma 4, L5 holds, let δ be the unique strategy such that $P_\delta = 1$. If $\delta = \alpha$ and $n_\alpha = n-1$, then $Pr_G(\bar{s}_{-i} | s_{i\alpha}) = 1$. If $n_\alpha < n-1$, $Pr_G(\bar{s}_{-i} | s_{i\alpha}) = 0$. These values follow from arguments similar to those of case 4.

If $\delta \neq \alpha$ and if $n_\alpha + n_\delta < n-1$, then $Pr_G(\bar{s}_{-i} | s_{i\alpha}) = 0$ by Lemma 3. If $\delta \neq \alpha$ and $n_\alpha + n_\delta = 1$ we may proceed as in case 1) to deduce

$$Pr_G(\bar{s}_{-i} | s_{i\alpha}) = n_\alpha! / [(n-1+\lambda)\dots(n-n_\alpha+\lambda)] .$$

This formula and the other values of Pr_G given previously are consistent with Theorem 2, with $\lambda \in (0,\infty)$.

6. Assume Lemma 4, L6 holds. If $\bar{s}_{-i}$ contains a strategy δ with $\delta \neq \alpha$ then $Pr_G(\bar{s}_{-i}|s_{i\alpha}) = 0$ by Lemma 3. The sum of the probabilities of all $\bar{s}_{-i}$ conditioned on $s_{i\alpha}$ must equal 1, so that if $n_\alpha = n-1$, then $Pr_G(\bar{s}_{-i}|s_{i\alpha}) = 1$.

These values are consistent with Theorem 2, with $\lambda = \infty$.

Theorem 2 is proven. □

Proof of Theorem 3

It will be shown that whether 1), 2), or 3) holds depends on whether λ is less than, equal to or greater than $\frac{n}{(n-1)a - 1}$.

First we assume

$$\lambda = \frac{n}{(n-1)a - 1} \tag{33}$$

and derive case 2).

Let $E_n(C)$ and $E_n(D)$ be the expected utilities of a player using the cooperation or defection strategies, respectively. Then

$$E_n(D) = \sum_{j=0}^{n-1} \binom{n-1}{j} Pr_G(j \text{ specified players use C}|\text{player i uses D}) \times U_D(j). \tag{34}$$

Using the value for Pr_G given in Theorem 2,

$$E_n(D) = \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(j-1+P_C\lambda)\ldots(P_C\lambda)(n-j-1+P_D\lambda)\ldots(P_D\lambda)}{(n-1+\lambda)\ldots(1+\lambda)} \frac{j}{n-1} \frac{\lambda+1}{\lambda}. \tag{35}$$

We will assume that $n \geq 3$. Abbreviating the lengthy second factor in (35) as $f(j-1,n-j-1,n-1)$ and combining the first and third factors into the single binomial coefficient $\binom{n-2}{j-1}$ yields

$$E_n(D) = \sum_{j=1}^{n-1} \binom{n-2}{j-1} f(j-1,n-j-1,n-1) \frac{\lambda+1}{\lambda}.$$

Expanding the binomial coefficient as the sum of two binomial coefficients,

$$E_n(D) = \sum_{j=1}^{n-2} \binom{n-3}{j-1} f(j-1,n-j-1,n-1) \frac{\lambda+1}{\lambda}$$
$$+ \sum_{j=2}^{n-1} \binom{n-3}{j-1} f(j-1,n-j-1,n-1) \frac{\lambda+1}{\lambda} .$$

We now change the index of summation of the second sum from j to j+1, and perform the following two substitutions that are derived from the definition of f(j-1,n-j-1,n-1):

$$f(j-1,n-j-1,n-1) = \frac{n-j-1+P_D\lambda}{n-1+\lambda} f(j-1,n-j-2,n-2)$$
$$f(j,n-j-2,n-1) = \frac{j+P_C\lambda}{n-1+\lambda} f(j-1,n-j-2,n-2).$$

This gives

$$E_n(D) = \sum_{j=1}^{n-2} \frac{n-j-1+P_D\lambda}{n-1+\lambda} \binom{n-3}{j-1} f(j-1,n-j-2,n-2) \frac{\lambda+1}{\lambda}$$
$$+ \sum_{j=1}^{n-2} \frac{j+P_C\lambda}{n-1+\lambda} \binom{n-3}{j-1} f(j-1,n-j-2,n-1) \frac{\lambda+1}{\lambda} .$$

Combining the two sums into one, and substituting $P_D = 1-P_C$ yields

$$E_n(D) = \sum_{j=1}^{n-2} \binom{n-3}{j-1} f(j-1,n-j-2,n-2) \frac{\lambda+1}{\lambda}$$
$$= E_{n-1}(D).$$

Thus the expected utility of a player who defects is constant, independent of the number of players in the game. Substituting n=2 in (35) gives $E_2(D) = P_C$, and thus all $E_n(D)$ are equal to that value.

It can be shown in a similar way that $E_n(C) = P_C$, under the assumption of (33). This proves the claim of Case 2) of the theorem that all players have expected value P_C. To derive $P_{C|D}$ and $P_{D|C}$, we use (33) and the definition of λ. and to derive the expression for $Pr_G (n_C = i)$ we substitute (33) in Theorem 2 and use

Pr_G(exactly n_C players cooperate)

$=$ Pr_G(exactly n_C-1 other than player i cooperate|player i cooperates) $\times$ Pr_G(player i cooperates)

$+$ Pr_G(exactly n_C other than player i cooperates|player i defects) $\times$ Pr_G(player i defects)

If λ is assumed less than or greater than the critical value in (33), the two other cases of Theorem 3, can be derived. $\square$

<u>Proof of Theorem 4</u>

The probabilities of moves by single players, P_C, P_D, $P_{D|C}$, and $P_{C|D}$, can be derived as limiting values of the probabilities in Theorem 3, as $n \to \infty$.

It must be shown that the probability of proportion x, as derived from the formula of Theorem 3, 2), approaches that of Theorem 4, 2).

From Theorem 3, 2), for finite n and letting $\lambda = \frac{n}{(n-1)a - 1}$,

$$f(x) = \frac{n!}{(xn)!\left[(1-x)n\right]!} \; \frac{(xn-1 + P_C\lambda)\ldots(P_C\lambda) \; ((1-x)n-1 + P_D\lambda)\ldots(P_D\lambda)}{(n-1+\lambda)\ldots(\lambda)}$$

$$= k \; \frac{(xn-1+P_C\lambda)\ldots(P_C\lambda)((1-x) \; n-1+P_D\lambda)\ldots(P_D\lambda)}{(xn)! \; \left[(1-x)n\right]!}$$

where k is a constant independent of x.

Using $\Gamma(z) = (z - 1)!$,

$$f(x) = k_1 \; \frac{\Gamma(xn + P_C\lambda) \; \Gamma((1-x)n + P_D\lambda)}{(xn + 1) \; (\Gamma((1-x)n + 1)} \; .$$

Using Stirling's approximation $\Gamma(z) \sim (2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}} e^{-z}$, for large n,

$$f(x) \sim k_2 \frac{(xn + P_C\lambda)^{xn + P_C\lambda - \frac{1}{2}} ((1-x)n + P_D\lambda)^{(1-x)n + P_D\lambda - \frac{1}{2}}}{(xn + 1)^{xn + \frac{1}{2}} ((1-x)n + 1)^{(1-x)n + \frac{1}{2}}}$$

$$\sim k_3 \frac{\left(1 + \frac{P_C\lambda}{xn}\right)^{xn + P_C\lambda - \frac{1}{2}}}{\left(1 + \frac{1}{xn}\right)^{xn + \frac{1}{2}}} x^{P_C\lambda - 1}$$

$$\times \frac{\left(1 + \frac{P_D\lambda}{(1-x)n}\right)^{(1-x)n + P_D\lambda - \frac{1}{2}}}{\left(1 + \frac{1}{(1-x)n}\right)^{(1-x)n + \frac{1}{2}}} (1-x)^{P_D\lambda - 1}.$$

Since $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{n + c} = e^x$, we have

$$f(x) \sim k_4 x^{P_C\lambda - 1} (1-x)^{P_D\lambda - 1}$$

where k_1, k_2, k_3 and k_4 do not depend on x.

Thus f(x) is a Beta distribution. Substituting $\lambda = 1/a$ gives proportionality as stated in Theorem 4, 2). □

Proof of Theorem 5

The mixed strategy extension of G, G′ is defined as the game in which each strategy set S_i in G is augmented by all mixed strategy combinations of the strategies in S_i. The payoff function of G′, $\bar{U}'$, is defined such that each element u_i' is the unique extension of u_i that is linear in the weights determining the mixed strategies.

Nash's well-known theorem (Nash, 1951, Theorem 1) shows that G possesses an equilibrium point. If $G \notin \mathcal{S}_{UC}$, let $\bar{s}_n$ be an equilibrium point that reflects the symmetries of the game. It will always be possible to choose such an equilibrium (Nash, 1951, Theorem 2). Let P be a function on $\mathcal{S}_G \times \mathcal{S}_G - \varphi$ giving the relative frequency probabilities of the various conditional events (involving pure strategies) when the players use $\bar{s}_n$.

It can be verified that Axioms I to VI are satisfied by P by virtue of the symmetry of $\bar{s}_n$ or by the latter's equilibrium point properties, so that P is strategic for G. □

Footnotes

[1]The general principle we are using here, equivalent to A3, is that if player 1's strategy s_i has maximal expected utility according to the probability matrix, and player 2's strategy s_j does not, then $Pr(s_j|s_i) = 0$.

[2]The following formulae calculated from Matrices 6 and 7 give the unconditional probabilities and payoffs for the game of Matrix 1.

$$Pr(R_1 \text{ and } C_1) = (1+4q)(1-q)/(7-8q) \qquad 3/8 \le q \le \frac{3}{4}$$
$$= 1 \qquad \frac{3}{4} \le q \le 1$$

$$Pr(R_2 \text{ and } C_2) = q(3-4q)/(7-8q) \qquad \frac{3}{8} \le q < 3/4$$
$$= 0 \qquad 3/4 \le q < 1$$

$$Pr(R_1 \text{ and } C_2) = Pr(R_2 \text{ and } C_1) = (3-4q)(1-q)/(7-8q)$$
$$= 0 \qquad \frac{3}{4} \le q \le 1$$

$$\text{Payoff} = 4(1-q) \qquad \frac{3}{8} \le q \le 3/4$$
$$= 1 \qquad \frac{3}{4} \le q < 1$$

[3]This is an overly narrow concept of what constitutes evidence of one move for another. A development of the theory would include other factors, such as the idea that if the payoffs of players were numerically close to being symmetrical we would expect a relationship of evidence to hold. Another way to expand the concept of evidence would be to base it on symmetry not in payoffs but in the best reply structure of the game, the mapping of each strategy n-tuple onto the set of strategies n-tuples where each player uses a best reply to the original n-tuple.

[4]The notation "λ" was introduced by Carnap (1952) who proved theorems like our theorems 1 and 2 but in the context of inductive logic. A typical problem in inductive logic deals with an investigator who samples a series of objects each of which has one of n different possible properties. The investigator wishes to determine the evidential probability that an object outside the sample has a given property.

The Greek letter λ stands for the weight given the logical possibility that the next object will be a new and different one, compared to the weight the investigator puts on the empirical evidence that it will be the same as the objects found in the sample.

Carnap dealt with the case in which the *a priori* probabilities of all properties (in our context, of all strategies of a player) are equal. The theorem is here extended to the case in which $Pr_G(s_{i\alpha}|U)$ may vary with α, including the case in which this probability is zero for some α. Zero probabilities like this were rejected by Carnap as inappropriate for inductive logic on the grounds that an investigator should not have a closed mind toward some logical possibility. Surprisingly, in our context of the theory of games, the situation that for some α, $Pr_G(s_{i\alpha}|U) = 0$, arises regularly as one of the possible consequences of AVI (rationality). A player must assess conditional probabilities on all moves including those that have suboptimal utility and are therefore certain not to occur.

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