DISCUSSION PAPER NO. 328

ON THE RELATIONSHIP BETWEEN CONDITIONS THAT INSURE A PL MAPPING IS A HOMEOMORPHISM

by

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May 1978

The work of both authors was supported by Grant MCS77-03472 from the National Science Foundation.

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ABSTRACT

In this note we consider two conditions which insure that a piecewise linear mapping is a homeomorphism. One of these is that the leading principal minors of certain matrices be positive, while the other is that all real eigenvalues be positive. We show that the latter condition is weaker than the former.

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§1. Introduction

Let R^n be the n-dimensional Euclidean space. Now, given a finite class Σ of closed convex polyhedral subsets of R^n, which partition R^n, and a continuous mapping F from R^n into R^n, we say that F is piecewise linear on the subdivision Σ if for each piece σ in Σ, $F_\sigma \equiv F|\sigma$ is an affine mapping, i.e., for some $n \times n$ matrix A_σ and an n vector a_σ, $F_\sigma(x) = A_\sigma x - a_\sigma$.

Several authors have contributed to the study of such mappings; see, for example, [1],[2],[3],[4],[7],[9],[11]. One of the important results established is that a necessary condition for F to be a homeomorphism is

$$\text{sign det } A_\sigma = \text{constant}, \quad \text{for all } \sigma \text{ in } \Sigma \tag{1.1}$$

and it can be readily demonstrated that this condition is not sufficient (see Figure 1.1. Here $F(x) = y$ has exactly three solutions for each $y \neq (0,0)$). It is also known that the above condition implies that the mapping is onto; see, for example, Chein and Kuh [1], Rheinboldt and Vandergraft [11]. Also, there are two sets of sufficient conditions that exist, one set produced by Fujisawa and Kuh [4], and the other by Kojima and Saigal [7]. We now discuss these conditions in some detail.

For an $n \times n$ matrix A, let A(k) be the submatrix consisting of the first k rows and k columns of A. Then, Fujisawa and Kuh [4] proved that:

<u>Theorem 1.1</u>: Let $A^1,\ldots,A^m$ be the Jacobians of the pieces of linearity of F. Then, F is a homeomorphism if

$$\text{sign det } A^j(k) = \varepsilon_k \tag{1.2}$$

for each $k = 1,\ldots,n$ and $j = 1,\ldots,m$, where $\varepsilon_k = +1,-1$.

In a recent paper, Kojima and Saigal [7] established another set of sufficient conditions. Let $\sigma_1,\ldots,\sigma_r$ be the unbounded pieces in Σ, and let $A^1,\ldots,A^r$ be the Jacobian of F on these pieces. Then, the following theorem was proved in [7]:

Theorem 1.2: Assume that the Jacobians of the pieces of linearity of F satisfy condition (1.1). Also, let there exist a matrix B such that $(1-t)B + tA^i$ is nonsingular for each t in [0,1] and $i = 1,\ldots,r$. Then, F is a homeomorphism.

In comparing Theorem 1.1 with Theorem 1.2, we see that the main condition of Theorem 1.2 is put only on the unbounded pieces. In this paper, we will relax the condition (1.2) to unbounded pieces, and thus weaken the sufficiency condition of [4]. Our approach is based on a homotopy introduced by Kojima [6] and studied in Saigal [12]. We also show that the condition of Theorem 1.1 implies the condition of Theorem 1.2, and by an example show that they are not equivalent.

As is evident, the condition of Theorem 1.2 is a global condition, and since we are dealing with a subdivision of R^n such a condition can be readily used locally. We also prove such a theorem, and show that this condition is weaker than the sufficiency condition of Eaves [2]. By putting a condition on the pieces, we also show that, in this case, a necessary and sufficient condition is that the Jacobians of the pieces of linearity have positive determinant.

In section 2 we present the weakening of the condition of [4] and in section 3 we show that the condition of Theorem 1.2 is weaker than that of Theorem 1.1. In section 4 we obtain a result under a local version of the condition of Theorem 1.2.

§2. Extension of the Fujisawa-Kuh Theorem

In this section, we present an extension of the theorem of Fujisawa and Kuh [4]. The extension we prove requires the sign property (1.2) only on the unbounded pieces of the mapping.

For an $n \times n$ matrix A, let A(k) be the principal submatrix consisting of the first k rows and first k columns of A, for each $k = 1,2,\ldots,n$. We call the determinants of A(k), $k = 1,\ldots,n$, the leading principal minors of A. Also, given a set of matrices $A^1,\ldots,A^r$, we say they satisfy the sign property if for each $j = 1,\ldots,r$ and $k = 1,\ldots,n$, (1.2) holds. We observe that we can, without loss of generality, assume that $\varepsilon_k = +1$ for each $k = 1,\ldots,n$. This can be readily seen since the matrices $A^1,\ldots,A^r$ have the sign property if and only if $DA^1,\ldots,DA^r$ have all leading principal minors positive, where D is a diagonal matrix with $D_{kk} = \varepsilon_{k-1}\varepsilon_k$, where $\varepsilon_0 = 1$.

Given any two matrices A^j and B, for each t in [0,1], define the $n \times n$ matrices

$$C_k^j(t) = (A_1^j, A_2^j, \ldots, A_{k-1}^j, tA_k^j + (1-t)B_k, B_{k+1}, \ldots, B_n), \tag{2.1}$$

where A_i is the i^{th} row of A. Then, we can prove:

<u>Lemma 2.1</u>: The matrices $C_k^j(t)$, $k = 0,1,\ldots,n$ and $t \in [0,1]$ are nonsingular if and only if A^jB^{-1} has all leading principal minors positive.

<u>Proof</u>: Let $D_k(t) = C_k(t)B^{-1}$, $0 \le t \le 1$, and some k. Then $D_k(t)$ has the partition:

$$\left[\begin{array}{c|c} A_{11}(t) & A_{12}(t) \\ \hline 0 & I \end{array}\right]$$

where det $A_{11}(t)$ is a principal minor of a $(k-1)\times(k-1)$ and a $k\times k$ principal submatrix of AB^{-1} for $t = 0$ and $t = 1$ respectively. To see the "if" part, note that $\det(D_k(t)) > 0$ for $t = 0, 1$. Also $\det D_k(t) = \det A_{11}(t)$. Since the last row of $A_{11}(t)$ is $tA_kB^{-1} + (1-t)u_k$ (where u_k is the k^{th} unit vector), and the other rows are independent of t, det $D_k(t)$ is a convex combination of two positive numbers, and is thus nonzero for each t. The "only if" part follows since if the two proper principal minors change sign, then det $D_k(t)$ must be zero for some t.

To prove our theorem, we shall need the following lemma from Kojima and Saigal [7]:

Lemma 2.2. Let $\hat{x}$ in R^n be such that $\det(A_\sigma) > 0$ (negative) for every σ containing $\hat{x}$. Then, there exists an $\varepsilon > 0$ such that $\deg(F, B_\delta(\hat{x}), F(\hat{x})) \geq 1$ (≤ -1) for each δ in $(0,\varepsilon)$, where $B_\delta(\hat{x}) = \{x : \|x - \hat{x}\| < \delta\}$.

Proof: See Theorem 3.3, [7].

Using Lemma 2.2, we can weaken the Fujisawa-Kuh sufficiency condition. Let $\sigma_1,\ldots,\sigma_r$ be the unbounded pieces in Σ and let A^j, $j = 1,\ldots,r$ be the Jacobians of the pieces of linearity of F in σ_j, $j = 1,\ldots,r$. Then, we can prove:

Theorem 2.3: Assume that the Jacobian matrix of each piece of linearity of F has a positive determinant. Also, let there exist a matrix B such that A^jB^{-1}, $j = 1,\ldots,r$ have all leading principal minors positive. Then F is a homeomorphism.

Proof: Let y be arbitrary. Then, consider the homotopy: For $x \in \sigma_j$, $(k-1)/n \leq t \leq k/n$,

$$H(x,t) = C_k^j(nt-k+1)x - a_k^j(nt-k+1), \tag{2.2}$$

where $a_k^j(s) = (a_1^j,\ldots,a_{k-1}^j, (1-s)a_k^j + sy_k, y_{k+1},\ldots,y_n)$. Now, $H(x,t)$ is continuous. Also, C_k^j is defined by (2.1). We claim that $H^{-1}(0)$ has no unbounded component. This is true since the contrary implies that for some σ_j, we can find a sequence $(x^p,t_p) \in H^{-1}(0)$, $p = 1,2,\ldots$ such that $x^p \in \sigma_j$ and $\|x^p\| \to +\infty$. Also, on some subsequence

$x^p/\|x^p\| \to x^*$, $t_p \to t^*$, $t^* \in [0,1]$ and $x^* \neq 0$. Also, there exists a k and a sufficiently large p' such that for all $p \geq p'$, $\frac{k-1}{n} \leq t_p \leq \frac{k}{n}$. Hence $C_k^j(nt_p - k+1)x^p - \underline{a}_k^j(nt_p - k+1) = 0$ for all $p \geq p'$. Dividing by $\|x^p\|$, and taking limits, since $C_k^j(t)$ is continuous in t, we have $C_k^j(nt^* - k+1)x^* = 0$. From Lemma 2.1, this is a contradiction. To see that it is one-to-one and onto, we observe that since $H^{-1}(0)$ is bounded for each y, and $\det(B) > 0$, the degree of $F(x) - y$ is 1. Also, since the set $\{x: F(x) = y\}$ has isolated points, the result follows from the Poincare-Hopf theorem [8], and Lemma 2.2.

This proof is similar to the proof of Theorem 5.1 of [7]. It differs from it in the use of the homotopy (2.2). The homotopy (2.2) is based on H_3 suggested by Kojima [6] for use when the mappings may be separable, and follows the notation of Saigal [12]. Here the row-wise separability is used.

§3. Relationship between the two sufficiency conditions

The aim of this section is to show that the sufficiency condition of [7] is weaker than that of [4]. For this, we introduce the following from Saigal [13]:

Lemma 3.1: Let A and B be two real matrices. Then $(1-t)A + tB$ is non-singular if and only if AB^{-1} has no real eigenvalues negative.

Proof: See Lemma 3.1.1 [13].

We now prove a proposition, before establishing the relationship between Theorems 1.1 and 1.2.

Proposition 3.2: Let A have all proper principal minors positive. Then, there exists an $\varepsilon^* > 0$ such that for every $\varepsilon \in (0,\varepsilon^*)$,

$$\det(AE(\varepsilon) - \lambda I) > 0 \quad \text{for all } \lambda \leq 0,$$

where

$$E(\varepsilon) = \begin{bmatrix} \varepsilon & & & & \\ & \varepsilon^2 & & & \\ & & \varepsilon^{2^2} & & \\ & & & \ddots & \\ & & & & \varepsilon^{2^{n-1}} \end{bmatrix}$$

Proof: We will prove that there exists an $\varepsilon^* > 0$, $b_1, \ldots, b_n > 0$ such that

$$\det[AE(\varepsilon) - \lambda I] \geq (-\lambda)^n + \varepsilon^1 b_1(-\lambda)^{n-1} + \varepsilon^{2^2-1} b_2(-\lambda)^{n-2} + \cdots + \varepsilon^{2^{n-1}-1} b_{n-1}(-\lambda) + \varepsilon^{2^n-1} b_n$$

for all ε in $(0,\varepsilon^*)$ and $\lambda \leq 0$.

This result is clearly true for $n = 1$, for which

$$\det(AE(\varepsilon) - \lambda I) = -\lambda + \varepsilon a_{11}$$

and we can take $b_1 = a_{11}$. Now, assume that the result is true for $n = 1,2,\ldots,r$ and consider the case $n = r+1$. Let $A = \left[\begin{array}{c|c} \bar{A} & b \\ \hline a & \gamma \end{array}\right]$. Then

$$\det[AE(\varepsilon) - \lambda I] = \det\left[\begin{array}{c|c} \bar{A}\bar{E}(\varepsilon) - \lambda I & \varepsilon^{2^r} b \\ \hline a\bar{E}(\varepsilon) & \varepsilon^{2^r}\gamma - \lambda \end{array}\right]$$

$$= (-\lambda)\det(A\bar{E}(\varepsilon) - \lambda I) + \det\left[\begin{array}{c|c} \bar{A}\bar{E}(\varepsilon) & \varepsilon^{2^r} b \\ \hline a\bar{E}(\varepsilon) & \varepsilon^{2^r}\gamma \end{array}\right]$$

$$= (-\lambda)\det(\bar{A}\bar{E}(\varepsilon) - \lambda I) + \phi(\lambda,\varepsilon)$$

and, from the induction hypothesis, there exists an ε^*, $b_1, \ldots, b_r > 0$ such that for each ε in $(0,\varepsilon^*)$

$$\det(\bar{A}\bar{E}(\varepsilon) - \lambda I) \geq (-\lambda)\{(-\lambda)^r + \varepsilon b_1(-\lambda)^{r-1} + \cdots + \varepsilon^{2^{r-1}-1} b_{r-1}(-\lambda) + \varepsilon^{2^r-1} b_r\}$$

and we can write

$$\phi(\lambda,\varepsilon) = \varepsilon^{2^{r+1}-1} \det A + \varepsilon^{2^r} \bar{\phi}(\lambda,\varepsilon)$$

where $\bar{\phi}(0,\varepsilon) = 0$, and $\bar{\phi}(\lambda,\varepsilon)$ is a polynomial of degree r, and it is a continuous function of ε. Hence, we obtain

$$\det(AE(\varepsilon) - \lambda I) \geq (-\lambda)^{r+1} + \varepsilon^1 \frac{b_1}{2} (-\lambda)^r$$
$$+ \cdots + \varepsilon^{2^r - 1} \frac{b_r}{2} (-\lambda)^r + \varepsilon^{2^{r+1}-1} \det A$$

for all sufficiently small $\varepsilon > 0$ and every $\lambda < 0$, and we are done.

Now, let F be a piecewise linear function on the subdivision Σ, and let $\sigma_1, \sigma_2, \ldots, \sigma_r$ be the unbounded pieces in Σ. Also, let A^i be the Jacobian of F on σ_i, $i = 1,\ldots,r$, respectively. Then we can prove that:

<u>Theorem 3.3</u>: Assume there exists a matrix B such that $A^i B^{-1}$, $i = 1,\ldots,r$, has all leading principal minors positive. Then, there exists a $\hat{B}$ such that $A^i \hat{B}^{-1}$, $i = 1,\ldots,r$, has no negative real eigenvalue.

<u>Proof</u>: From proposition 3.2, there exists an $\varepsilon_i^* > 0$ such that for every ε in $(0,\varepsilon_i^*)$, $A^i B^{-1} E(\varepsilon)$ has no negative real eigenvalue. Hence, choosing an $\varepsilon \leq \varepsilon_i^*$ for $i = 1,\ldots,k$, we define $\hat{B} = E(\varepsilon)^{-1} B$ and the result follows.

A consequence of Theorem 3.3 is that the conditions of Theorem 2.3 imply those of Theorem 1.2. In figure 3.1 we present a homeomorphism satisfying the conditions of Theorem 1.2 with

$$B = \begin{bmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Since I and -I appear as Jacobians of the pieces of linearity, there exists no matrix B for which the conditions of Theorem 2.3 are satisfied.

§4. A local version of Theorem 1.2

In this section we consider the conditions of Theorem 1.2 that if A^i, $i = 1,\dots,r$ are the Jacobians of the pieces of linearity of the unbounded pieces then there is a matrix B such that $(1-t)B + tA^i$ is nonsingular for each t in $[0,1]$, and $i = 1,\dots,r$. Since this condition is only put on the unbounded pieces, it is global in nature, and the result is then proved by a degree theoretic argument. We now show that there is a local version in which the result can be proved as a corollary to the Theorem of Palais:

Theorem 4.1: Let F map R^n into R^n, and let F be a local homeomorphism. Then, if $\|F(x)\| \to \infty$ as $\|x\| \to \infty$, F maps R^n homeomorphically onto R^n.

We now state our condition:

Condition 4.1: Let x be such that it lies in pieces $\sigma_1,\dots,\sigma_r$, and let $A^1,\dots,A^r$ be the Jacobians of the pieces of linearity of F respectively. There exists a B such that $(1-t)B + tA^i$ is nonsingular for each t in $[0,1]$ and $i = 1,\dots,r$.

We now prove our main theorem:

Theorem 4.2: Let F map R^n into R^n and satisfy condition 4.1 for every x in R^n. Then F maps R^n homeomorphically onto R^n.

Proof: Our claim is that for every x in R^n, if x lies in $\sigma_1,\dots,\sigma_r$, and only these pieces, then there exists an extension of the pieces σ_i to σ_i' such that σ_i', $i = 1,\dots,r$, subdivide R^n. From Theorem 1.2 the mapping is thus a local homeomorphism at x. Since any PL mapping satisfying the above condition is norm coercive (i.e., if $\|x\| \to \infty$ then $\|f(x)\| \to \infty$), our result now follows from Theorem 4.1.

We now show that the Theorem 13.2 of Eaves [2] follows as a corollary to the above Theorem 4.2.

Let $F_i(x)$ be the i^{th} component function of F, and let a generic piece of linearity of this function be σ^i. Now, for each x in R^n define $\sigma_1^i, \ldots, \sigma_{r_i}^i$ to be the pieces in which x lies, for each $i = 1, \ldots, n$, and let a_j^i be the gradients of F_i on σ_j^i for each $j = 1, \ldots, r_i$, respectively. Then, define $\partial F_i(x) = \text{hull}\{a_j^i : j = 1, \ldots, r_i\}$ and

$$\partial F(x) = \partial F_1(x) \times \partial F_2(x) \times \cdots \times \partial F_n(x).$$

We note that each element of $\partial F(x)$ can be associated with a matrix whose i^{th} row belongs to $\partial F_i(x)$. Hence, we will consider the elements of $\partial F(x)$ as matrices with this property.

We now state the condition of Theorem 13.2, [2].

Condition 4.2: For each x in R^n, each H in $\partial F(x)$ is nonsingular.

Theorem 4.3: Condition 4.2 implies condition 4.1.

Proof: This follows readily by observing that a piece of linearity σ of F is the intersection of certain pieces of linearity σ^i of F_i, $i = 1, \ldots, n$. Then, for $x \in \sigma$, if A is the Jacobian of F(x), $A_i \in \partial F(x_i)$. The result then follows by observing that $\partial F(x)$ is convex.

Using Lemma 2.1, it can also be shown that condition 4.2 implies the sign property (1.2) and thus Theorem 13.2 [2] also follows as a corollary to Theorem 2.3.

As another corollary to Theorem 4.2, we now put conditions on the pieces of linearity and then prove a necessary and sufficient condition for F to be a homeomorphism. We will say that a subdivision of R^n is regular if for any two pieces σ_1 and σ_2, either $\sigma_1 \cap \sigma_2 = \phi$ or σ_1 and σ_2 meet on a common facet (a $(n-1)$ dimensional face). An example of a regular subdivision of R^2 is given in Figure 4.1. We can then prove:

Theorem 4.4: Let F be a piecewise linear mapping from R^n into R^n, and let the subdivision generated by the pieces of linearity of F be regular. Then, a necessary and sufficient condition that F map R^n homeomorphically onto R^n is that the determinants of the Jacobians of the pieces of linearity of F have the same sign.

Proof: We now show that condition 4.1 is satisfied, and thus the sufficiency would follow from Theorem 4.2. Let $x \in \sigma_1 \cap \cdots \cap \sigma_r$ and let the Jacobians of the pieces of linearity be $A^1,\ldots,A^r$. We now show that $(1-t)A^1 + tA^i$ is nonsingular for all t in [0,1] and i = 2,...,r. Since the subdivision is regular, $\sigma_1 \cap \sigma_i$ is a common facet. Hence

$$A^i = A^1 + a_i b_i^T$$

for some column vectors a_i and b_i. Thus

$$(1-t)A^i + tA^1 = A^1 + ta_i b_i^T$$

and, using standard arguments, $\det((1-t)A^1 + tA^i) = \det(A^1 + ta_i b_i^T) = 1 + tb_i^T(A^i)^{-1}a_i$. But, since $\det A^i > 0$, $1 + b_i^T(A^i)^{-1}a_i > 0$, hence we have our result since $t \le 1$.

To see the necessity, assume that the determinants of the Jacobians do not have the same sign. Then, there exist two pieces σ_1 and σ_2 such that $\det(A^1 \;\; A^2) < 0$, and $\sigma_1 \cap \sigma_2$ is a common facet. Thus, $[(1-t)A^1 + tA^2]d = 0$ for some t and $d \neq 0$. We claim, without loss of generality, that there exists $x_1 \in \sigma_1$, $x_2 \in \sigma_2$ such that $x_2 - x_1 = d$ and $x_1 + td \in \sigma_1 \cap \sigma_2$. Now, $F(x_1) - F(x_2) = A^1x_1 - a_1 - A^2x_2 + a_2$. Since $A^1(x_1 + td) - a_1 = A^2(x_2 + td) - a_2$, we see that $F(x_1) = F(x_2)$, a contradiction.

We reproduce an example from [7] to show that any necessary and sufficient condition for a mapping to be a homeomorphism must also include some information of the pieces, and thus these would be of the Theorem 4.4 type. This example is given in Figure 4.2. The matrices

$$\begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$

are Jacobians of the pieces of linearity of the mapping of Figure 1.1, which is a non-homeomorphism, and thus from Theorem 1.2, there exists no matrix B satisfying condition 4.1 at $x = 0$. As can be readily verified, this example is a homeomorphism of R^n, and this is a counterexample to any necessary and sufficient conditions put only on the Jacobians.

§5. Acknowledgment: Section 4 of this paper benefited from discussions with A. Mas-Colell.

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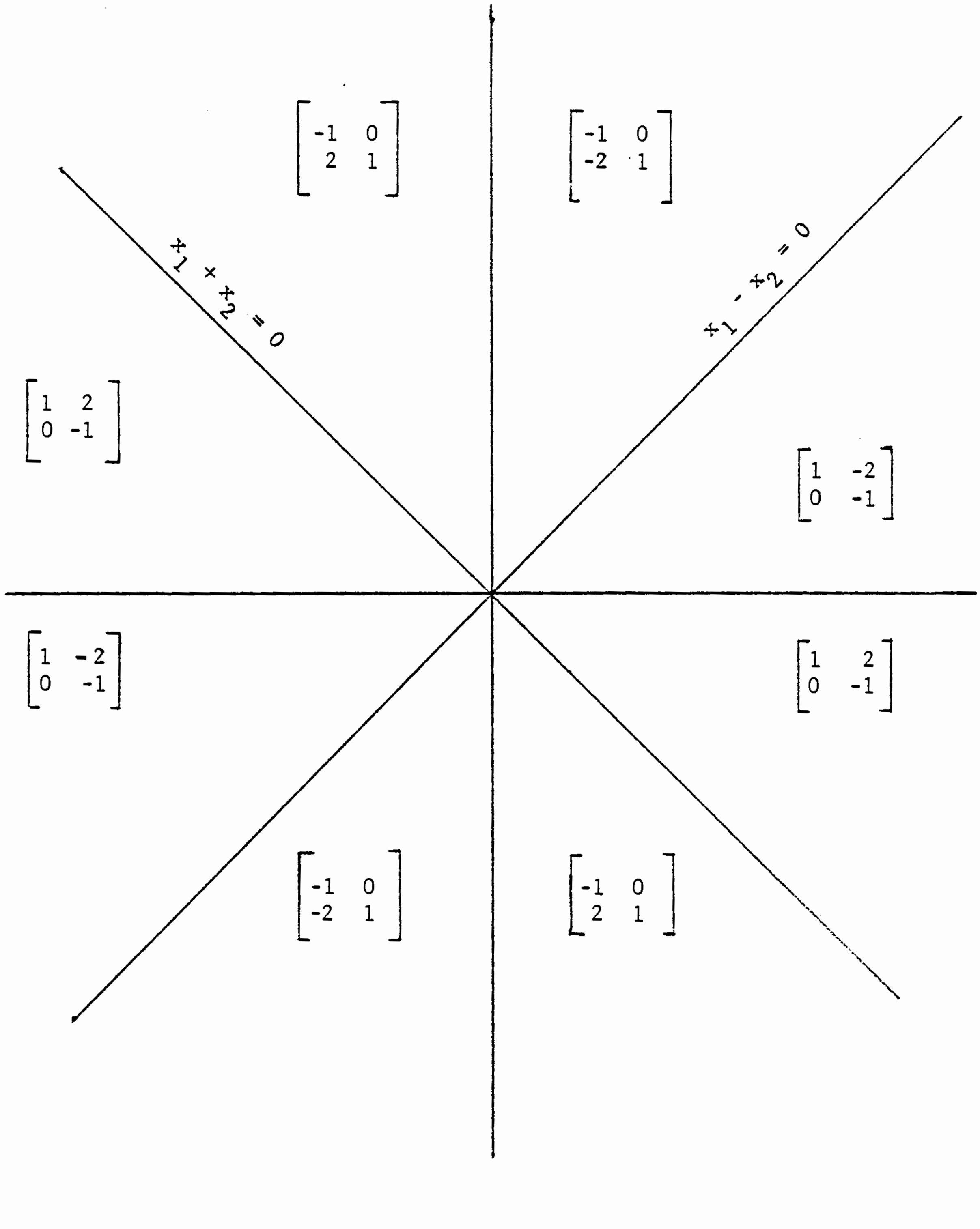


Figure 1.1

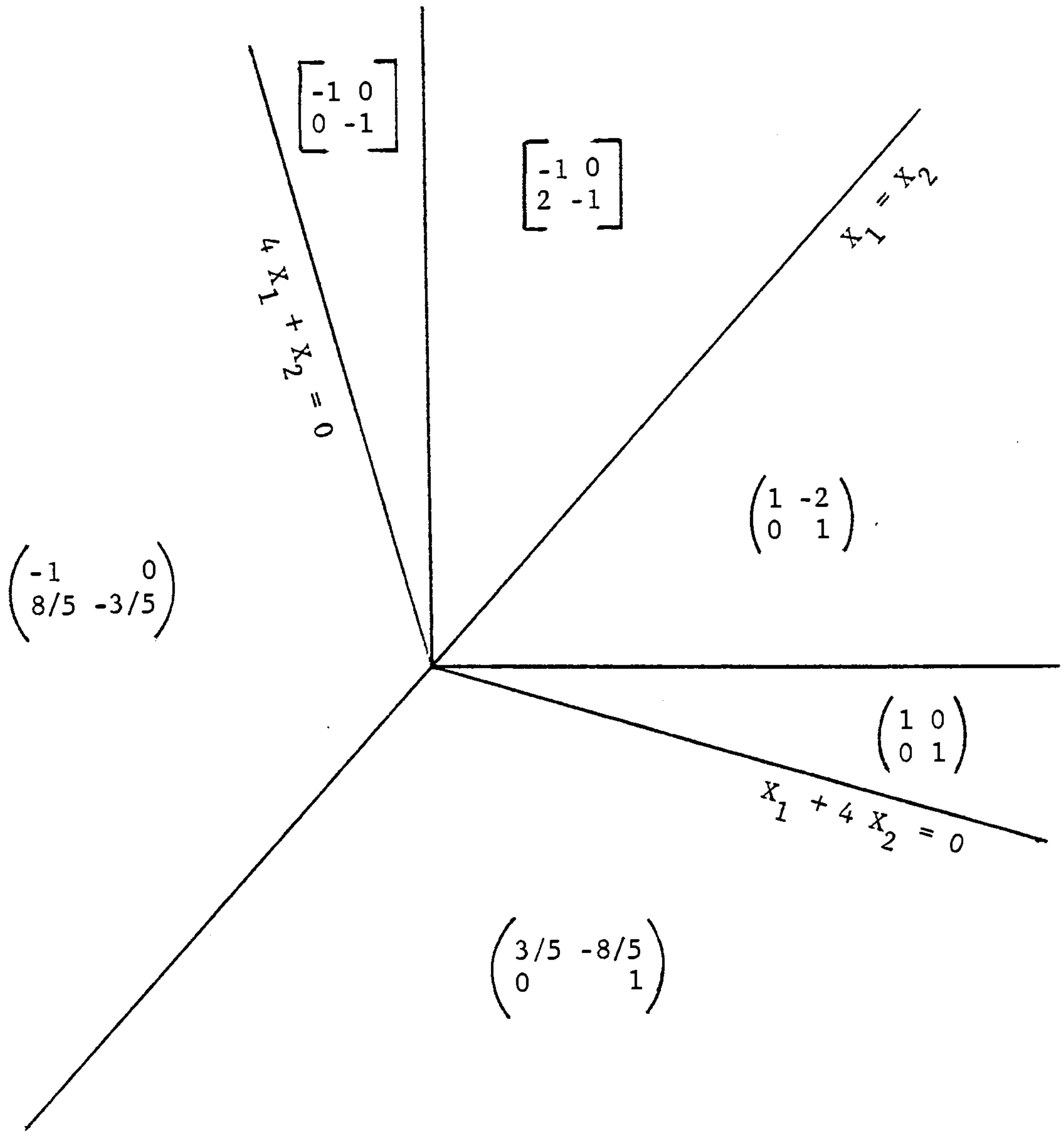


Figure 3.1

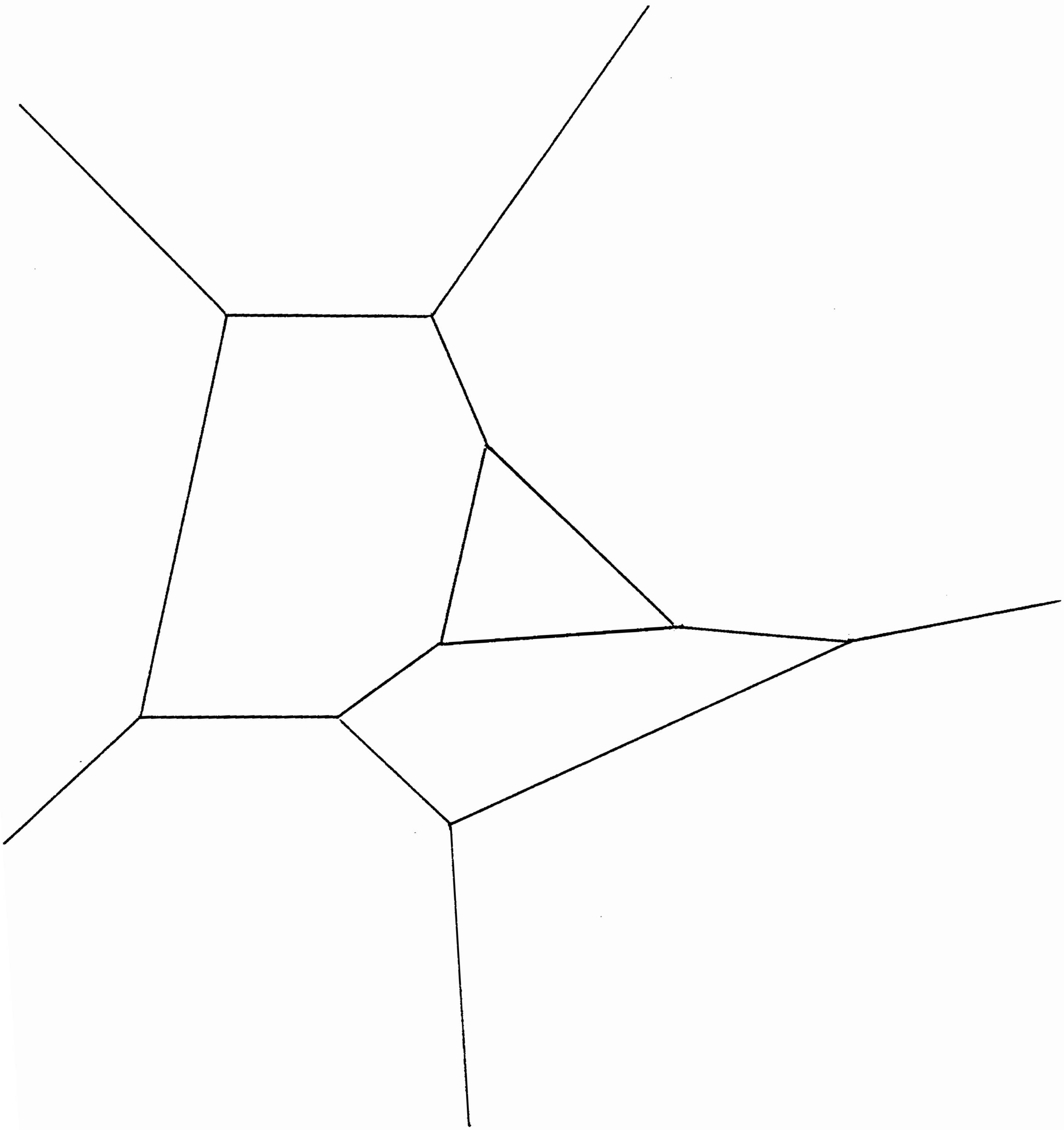
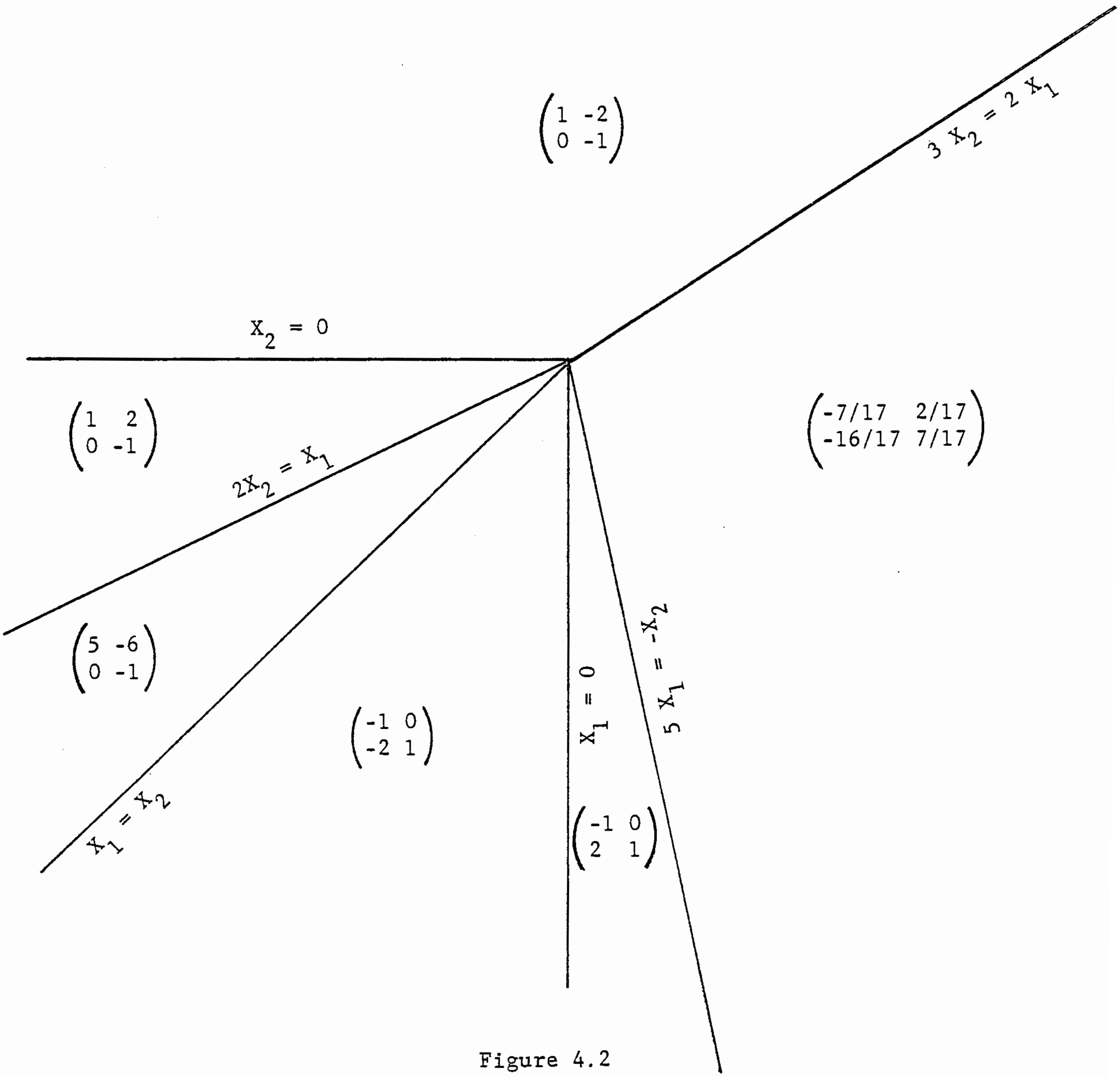


Figure 4.1



$\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$
$3 X_2 = 2 X_1$
$X_2 = 0$
$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$
$2X_2 = X_1$
$\begin{pmatrix} -7/17 & 2/17 \\ -16/17 & 7/17 \end{pmatrix}$
$\begin{pmatrix} 5 & -6 \\ 0 & -1 \end{pmatrix}$
$5 X_1 = -X_2$
$X_1 = 0$
$\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$
$X_1 = X_2$
$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$

Figure 4.2