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ASYMPTOTIC BIAS OF ORDINARY LEAST SQUARES ESTIMATOR FOR MULTIVARIATE AUTOREGRESSIVE MODELS

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Summary

Asymptotic bias is derived, as a relatively simple matrix function of the true parameters, for the ordinary least squares estimator in multivariate autoregressive models. The result is obtained through a convenient asymptotic expansion of the estimator.

Some key words: Asymptotic bias; Asymptotic expansion; Autoregressive model; Ordinary least squares estimator.

1. Introduction and Model

Kendall (1954) obtained the bias of the ordinary least squares estimator to order n^{-1} for the first order autoregressive model with a constant term, while White (1961) to orders higher than n^{-1} for the one without a constant term. In the present note, we derive the asymptotic bias to order n^{-1} for the ordinary least squares estimator of a multivariate autoregressive model with a constant term. It can be reduced to a model without a constant term as a special case, which corrects an error in Groenewald and de Waal (1979). While the asymptotic bias, whose precise definition is given in Section 2, is conceptually different from the approximation of the finite bias by Kendall and White, they are shown to be identical to order n^{-1} for both types of the first order multivariate autoregressive models.

Let us consider the p-variate first order autoregressive process

$$\underset{\sim}{y}_t = \underset{\sim}{B}\underset{\sim}{y}_{t-1} + \underset{\sim}{b}_o + \underset{\sim}{v}_t, \qquad (1)$$

where $\underset{\sim}{y}_t = [y_{t1}, y_{t2}, \ldots, y_{tp}]'$, $\underset{\sim}{v}_t = [v_{t1}, v_{t2}, \ldots, v_{tp}]'$, $\underset{\sim}{B} = [\beta_{ij}]$ is the pxp coefficient matrix and $\underset{\sim}{b}_o = [\beta_{io}]$ is the p x 1 coefficient vector. We assume that $\underset{\sim}{v}_t$ is independently identically distributed as $N(\underset{\sim}{0}, \underset{\sim}{\Omega}_v)$ and all characteristic roots of $\underset{\sim}{B}$ are less than unity in absolute value. Following Anderson (1959), the process is alternatively expressed as

$$\underset{\sim}{z}_t = \underset{\sim}{A}\underset{\sim}{z}_{t-1} + \underset{\sim}{u}_t, \qquad (2)$$

where $\underset{\sim}{z}_t = [\underset{\sim}{y}_t', 1]'$, $\underset{\sim}{u}_t = [\underset{\sim}{v}_t', 0]$ with $\underset{\sim}{\Omega}_u = E(\underset{\sim}{u}_t \underset{\sim}{u}_t')$, and $\underset{\sim}{A}$ is the (p+1)x(p+1) matrix

$$\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{B} & \underset{\sim}{b}_o \\ \underset{\sim}{0} & 1 \end{bmatrix} .$$

We can express $\underset{\sim}{z}_t$ and $E(\underset{\sim}{z}_t \underset{\sim}{z}_t')$ as

$$\underset{\sim}{z}_t = \underset{\sim}{d} + \sum_{i=0}^{\infty} \underset{\sim}{A}^i \underset{\sim}{u}_{t-i} \qquad , \underset{\sim}{\Gamma} = E(\underset{\sim}{z}_t \underset{\sim}{z}_t') = \underset{\sim}{d}\underset{\sim}{d}' + \sum_{i=0}^{\infty} \underset{\sim}{A}^i \underset{\sim}{\Omega}_u \underset{\sim}{A}'^i ,$$

where $\underset{\sim}{A}^o = \underset{\sim}{I}_{p+1}$, $\underset{\sim}{I}_k$ is the identity matrix of rank k, and $\underset{\sim}{d}$ is the (p+1)th column of $\underset{\sim}{A}^\infty$, i.e., $\underset{\sim}{d} = \underset{\sim}{A}^\infty \underset{\sim}{z}_{-\infty} = [((\underset{\sim}{I}_p - \underset{\sim}{B})^{-1} \underset{\sim}{b}_o)', 1]'$.

2. Asymptotic Bias of Ordinary Least Squares Estimator

For given observations $\underset{\sim}{y}_o, \underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_n$, the ordinary least squares estimator of $\underset{\sim}{A}$ is given by

$$\hat{\underset{\sim}{A}} = \left(\sum_{t=1}^{n} \underset{\sim}{z}_t \underset{\sim}{z}'_{t-1}\right)\left(\sum_{t=1}^{n} \underset{\sim}{z}_{t-1} \underset{\sim}{z}'_{t-1}\right)^{-1} . \tag{5}$$

Here we assume that the initial observation $\underset{\sim}{y}_o$ obeys the same multivariate normal distribution as $\underset{\sim}{y}_t$ for $t \geq 1$. As a generalization of Akahira (1979), we can expand $\hat{\underset{\sim}{A}} - \underset{\sim}{A}$ as

$$\begin{aligned} \hat{\underset{\sim}{A}} - \underset{\sim}{A} &= \left(\sum_{t=1}^{n} \underset{\sim}{u}_t \underset{\sim}{z}'_{t-1}\right) \left(\sum_{t=1}^{n} \underset{\sim}{z}_{t-1} \underset{\sim}{z}'_{t-1}\right)^{-1} \\ &= \underset{\sim}{W}\{(n^{\frac{1}{2}} \underset{\sim}{I}_{p+1} + \underset{\sim}{V})\underset{\sim}{\Gamma}\}^{-1} \\ &= \underset{\sim}{W}\underset{\sim}{\Gamma}^{-1}(n^{-\frac{1}{2}} \underset{\sim}{I}_{p+1} - n^{-1}\underset{\sim}{V}) + o_p(n^{-1}) \end{aligned} \tag{6}$$

where $\underset{\sim}{W} = n^{-\frac{1}{2}} \sum_{t=1}^{n} \underset{\sim}{u}_t \underset{\sim}{z}'_{t-1}$

$$\underset{\sim}{V} = n^{\frac{1}{2}}\left(n^{-1} \sum_{t=1}^{n} \underset{\sim}{z}_{t-1} \underset{\sim}{z}'_{t-1} - \underset{\sim}{\Gamma}\right)\underset{\sim}{\Gamma}^{-1} = n^{-\frac{1}{2}}\left(\sum_{t=1}^{n} \underset{\sim}{z}_{t-1} \underset{\sim}{z}'_{t-1}\right)\underset{\sim}{\Gamma}^{-1} - n^{\frac{1}{2}} \underset{\sim}{I}_{p+1} .$$

Let us define the asymptotic bias of $\hat{\underset{\sim}{A}}$, denoted by ABIAS($\hat{\underset{\sim}{A}}$), as follows:

$$\text{ABIAS}(\hat{A}) = \text{AE}(\hat{A}-A)$$
$$= E\{W\Gamma^{-1}(n^{-\frac{1}{2}}I_{p+1}-n^{-1}v)\} + o(n^{-1}), \qquad (7)$$

where AE is the expectation of the asymptotic expansion of the distribution function of $\hat{A}$.

Theorem: The asymptotic bias of the ordinary least squares estimator (5) for the first order multivariate autoregressive model with a constant term (1) is given by

$$\text{ABIAS}(\hat{A}) = -n^{-1}\Omega_u[\sum_{k=0}^{\infty} A'^{k}\Gamma^{-1}\{d'\Gamma^{-1}d + dd'\Gamma^{-1} + (\sum_{i=0}^{\infty} A^{i}\Omega_u A'^{i})A'^{k+1}\Gamma^{-1} + \text{tr}(A^{k+1}(\sum_{i=0}^{\infty} A^{i}\Omega_u A'^{i})\Gamma^{-1})\}] + o(n^{-1}), \qquad (8)$$

Proof: See Appendix.

As easily seen, the above result includes a univariate p-th order autoregressive model as a special case. Specifically, for the univariate first order autoregressive model, we have

$$A = \begin{bmatrix} \beta_1 & \beta_0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \omega^2(1-\beta_1^2)^{-1} + c^2 & c \\ c & 1 \end{bmatrix},$$

and $d = [c,1]'$, $c = \beta_0/(1-\beta_1)$, $\Sigma_{i=0}^{\infty} A^{i}\Omega_u A'^{i} = \omega^2(1-\beta_1^2)^{-1}M$, $\omega^2 = E(u_t^2)$, $M = ee'$ and $e = [1,0]'$. Then, the first row of the first term in (8) reduces to $(1+\beta_1)[1,-c]$, while that of the second term vanishes. The first rows of the third and fourth terms are simplified to the same vector $\beta_1[1,-c]'$. Thus, for the univariate first order autoregressive model with a constant term, we get

$$ABIAS(\hat{\beta}_1,\hat{\beta}_o) = n^{-1}(1+3\beta_1)[-1,\beta_o/(1-\beta_1)] + o(n^{-1}).$$

The result of ABIAS($\hat{\beta}_1$) conforms with Kendall (1954), while, as far as we know, ABIAS($\hat{\beta}_o$) has not previously been obtained.

The result (8) is also simplified for the case of no constant term.

Corrollary: Let $\hat{\underset{\sim}{B}}$ be the ordinary least squares estimator for the first order multivariate autoregressive model in (1) with a priori knowledge of $\underset{\sim}{b}_o = \underset{\sim}{0}$. Then, the asymptotic bias of $\hat{\underset{\sim}{B}}$ is given by

$$ABIAS(\hat{\underset{\sim}{B}}) = -n^{-1}\underset{\sim}{\Omega}_v\{\sum_{k=0}^{\infty}\underset{\sim}{B}'^{2k+1}\underset{\sim}{\Gamma}^{-1} + \sum_{k=0}^{\infty}\underset{\sim}{B}'^{k}\underset{\sim}{\Gamma}^{-1}tr(\underset{\sim}{B}^{k+1})\}$$

$$+ \quad o(n^{-1}),$$

where $\hat{\underset{\sim}{B}} = (\sum_{t=1}^{n}\underset{\sim}{y}_t\underset{\sim}{y}'_{t-1})(\sum_{t=1}^{n}\underset{\sim}{y}_{t-1}\underset{\sim}{y}'_{t-1})^{-1}$, $\underset{\sim}{\Gamma} = E(\underset{\sim}{y}_t\underset{\sim}{y}'_t) = \sum_{i=0}^{\infty}\underset{\sim}{B}^i\underset{\sim}{\Omega}_v\underset{\sim}{B}'^i$.

Proof: Since $\underset{\sim}{d} = 0$ in this case, the first two terms of (8) are dropped. Further, replacing $\underset{\sim}{A}$ and $\underset{\sim}{\Omega}_u$ by B and $\underset{\sim}{\Omega}_v$ in (8), and noting that $\underset{\sim}{\Gamma} = \sum_{i=0}^{\infty}\underset{\sim}{B}^i\underset{\sim}{\Omega}_v\underset{\sim}{B}'^i$ in this case, the proof is completed.

We note that, while the second term of (9) is the same as the second term of equation (3.13) in Groenewald and de Waal (1979), the first term is different from their first term.

For the case of the univariate first order autoregressive model without a constant term, we have $\underset{\sim}{B} = \underset{\sim}{B}' = \beta_1$ and $\underset{\sim}{\Gamma} = \omega^2/(1-\beta_1^2)$. Thus the above result is reduced to

$$ABIAS(\hat{\beta}_1) = -(2\beta_1)/n \quad ,$$

which conforms with White (1961) to order n^{-1}.

Finally, it is easily seen from the proof in Appendix that all the results remain to hold, even if we replace the assumption on $\underset{\sim}{y}_o$ by $\underset{\sim}{y}_o = \underset{\sim}{0}$.

Appendix: Proof of Theorem

By the structure of $\underset{\sim}{V}$, we can rewrite (6) as

$$\hat{\underset{\sim}{A}} - \underset{\sim}{A} = n^{-\frac{1}{2}}\{2\underset{\sim}{W}\Gamma^{-1}\} - n^{-1}\underset{\sim}{W}\Gamma^{-1}\{n^{-\frac{1}{2}}(\sum_{t=0}^{n-1} \underset{\sim}{z}_t \underset{\sim}{z}_t')\underset{\sim}{\Gamma}^{-1}\} + o_p(n^{-1}).$$

Since $E(\underset{\sim}{w}) = 0$, ABIAS$(\hat{\underset{\sim}{A}})$ is reduced to

$$\text{ABIAS}(\hat{\underset{\sim}{A}}) = -n^{-1}E\{n^{-1}\sum_{t=1}^{n}\underset{\sim}{u}_t\underset{\sim}{z}_{t-1}'\underset{\sim}{\Gamma}^{-1}(\sum_{t=0}^{n-1}\underset{\sim}{z}_t\underset{\sim}{z}_t')\underset{\sim}{\Gamma}^{-1}\} + o(n^{-1}). \qquad \text{(A.1)}$$

Using the expression of $\underset{\sim}{z}_t$ in (4), we can express $\Sigma_{t=1}^{n}\underset{\sim}{u}_t\underset{\sim}{z}_{t-1}'$ and $\Sigma_{t=0}^{n-1}\underset{\sim}{z}_t\underset{\sim}{z}_t'$ as

$$\sum_{t=1}^{n} \underset{\sim}{u}_t\underset{\sim}{z}_{t-1}' = \sum_{t=1}^{n}(\underset{\sim}{g}_t + \underset{\sim}{h}_t), \qquad \text{(A.2)}$$

$$\sum_{t=0}^{n-1} \underset{\sim}{z}_t\underset{\sim}{z}_t' = \sum_{t=0}^{n-1} (\underset{\sim}{d}\underset{\sim}{d}' + \underset{\sim}{F}_t + \underset{\sim}{G}_t + \underset{\sim}{G}_t' + \underset{\sim}{H}_t + \underset{\sim}{H}_t'),$$

where $\underset{\sim}{g}_t = \underset{\sim}{u}_t\underset{\sim}{d}'$

$$\underset{\sim}{h}_t = \sum_{i=1}^{\infty} \underset{\sim}{h}_{t,i} = \sum_{i=1}^{\infty} \underset{\sim}{u}_t\, \underset{\sim}{u}_{t-i}'\, \underset{\sim}{A}'^{\,i-1},$$

$$\underset{\sim}{F}_t = \sum_{i=0}^{\infty} \underset{\sim}{A}^i\underset{\sim}{u}_{t-i}\, \underset{\sim}{u}_{t-i}'\, \underset{\sim}{A}'^{\,i},$$

$$\underset{\sim}{G}_t = \sum_{i=0}^{\infty} \underset{\sim}{G}_{t,i} = \sum_{i=0}^{\infty} \underset{\sim}{d}\underset{\sim}{u}_{t-i}'\, \underset{\sim}{A}'^{\,i},$$

$$\underset{\sim}{H}_t = \sum_{i=0}^{\infty} \underset{\sim}{H}_{t,i} = \sum_{i=0}^{\infty} \underset{\sim}{A}^i\underset{\sim}{u}_{t-i}(\sum_{j=1}^{\infty} \underset{\sim}{u}_{t-i-j}'\, \underset{\sim}{A}'^{\,i+j}).$$

Noting that $E\{(\underset{\sim}{u}_s\underset{\sim}{z}_{s-1}')(\underset{\sim}{z}_t\underset{\sim}{z}_t')\} = 0$ for $s > t$, the expectation for given s is given by

$$E\{\underset{\sim}{u}_s\underset{\sim}{z}_{s-1}'\underset{\sim}{\Gamma}^{-1}(\sum_{t=0}^{n-1} \underset{\sim}{z}_t\underset{\sim}{z}_t')\underset{\sim}{\Gamma}^{-1}\}$$

$$= E\{\underset{\sim}{u}_s\underset{\sim}{z}_{s-1}'\underset{\sim}{\Gamma}^{-1}(\sum_{t=s}^{\infty} \underset{\sim}{z}_t\underset{\sim}{z}_t')\underset{\sim}{\Gamma}^{-1}\} - E\{\underset{\sim}{u}_s\underset{\sim}{z}_{s-1}'\underset{\sim}{\Gamma}^{-1}(\sum_{t=n}^{\infty}\underset{\sim}{z}_t\underset{\sim}{z}_t')\underset{\sim}{\Gamma}^{-1}\},$$

$$s=1,2,\ldots,n. \qquad \text{(A.3)}$$

We first evaluate the first term of the above. Since the expectation exists only when the time indices of $\underset{\sim}{u}_t$'s are equal or pairwise equal, it can be expressed by (A.2) as

$$E\{\underset{\sim}{u}_s \underset{\sim}{z}'_{s-1} \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \underset{\sim}{z}_t \underset{\sim}{z}'_t) \underset{\sim}{\Gamma}^{-1}\}$$

$$= E\{\underset{\sim}{g}_s \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \underset{\sim}{G}_t) \underset{\sim}{\Gamma}^{-1} + \underset{\sim}{g}_s \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \underset{\sim}{G}'_t) \underset{\sim}{\Gamma}^{-1} \qquad (A.4)$$

$$+ \underset{\sim}{h}_s \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \underset{\sim}{H}_t) \underset{\sim}{\Gamma}^{-1} + \underset{\sim}{h}_s \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \underset{\sim}{H}'_t) \underset{\sim}{\Gamma}^{-1}\}.$$

The first term of the above can be reduced to

$$E\{\underset{\sim}{g}_s \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \underset{\sim}{G}_t) \underset{\sim}{\Gamma}^{-1}\} = E\{\underset{\sim}{g}_s \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \sum_{i=0}^{\infty} \underset{\sim}{G}_{t,i}) \underset{\sim}{\Gamma}^{-1}\}$$

$$= E\{\underset{\sim}{g}_s \underset{\sim}{\Gamma}^{-1} (\sum_{k=0}^{\infty} \underset{\sim}{G}_{s+k,k}) \underset{\sim}{\Gamma}^{-1}\}$$

$$= E\{\underset{\sim}{u}_s \underset{\sim}{d}' \underset{\sim}{\Gamma}^{-1} \sum_{k=0}^{\infty} \underset{\sim}{d} \underset{\sim}{u}'_s \underset{\sim}{A}'^k \underset{\sim}{\Gamma}^{-1}\}.$$

Since $\underset{\sim}{d}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{d}$ is a scalor, we get

$$E\{\underset{\sim}{g}_s \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \underset{\sim}{G}_t) \underset{\sim}{\Gamma}^{-1}\} = \underset{\sim}{\Omega}_u (\sum_{k=0}^{\infty} \underset{\sim}{A}'^k) \underset{\sim}{\Gamma}^{-1} (\underset{\sim}{d}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{d}). \qquad (A.5)$$

Similarly, the second term of (A.4) is reduced to

$$E\{\underset{\sim}{g}_s \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \underset{\sim}{G}'_t) \underset{\sim}{\Gamma}^{-1}\} = E\{\underset{\sim}{g}_s \underset{\sim}{\Gamma}^{-1} (\sum_{k=0}^{\infty} \underset{\sim}{G}'_{s+k,k}) \underset{\sim}{\Gamma}^{-1}\}$$

$$= E\{\underset{\sim}{u}_s \underset{\sim}{d}' \underset{\sim}{\Gamma}^{-1} \sum_{k=0}^{\infty} \underset{\sim}{A}^k \underset{\sim}{u}_s \underset{\sim}{d}' \underset{\sim}{\Gamma}^{-1}\}.$$

Since $\underset{\sim}{d}' \underset{\sim}{\Gamma}^{-1} \Sigma_{k=0}^{\infty} \underset{\sim}{A}^k \underset{\sim}{u}_s$ is a scalor, first transposing it and then taking the expectation, we get

$$E\{\underset{\sim}{g}_s \underset{\sim}{\Gamma}^{-1} (\sum_{t=s}^{\infty} \underset{\sim}{G}'_t) \underset{\sim}{\Gamma}^{-1}\} = \underset{\sim}{\Omega}_u (\sum_{k=0}^{\infty} \underset{\sim}{A}'^k) \underset{\sim}{\Gamma}^{-1} \underset{\sim}{d} \underset{\sim}{d}' \underset{\sim}{\Gamma}^{-1}. \qquad (A.6)$$

The third term of (A.4) can be written as

$$E\{\underset{\sim}{h}_s\underset{\sim}{\Gamma}^{-1}(\sum_{t=s}^{\infty}\underset{\sim}{H}_t)\underset{\sim}{\Gamma}^{-1}\} = E\{\underset{\sim}{h}_s\underset{\sim}{\Gamma}^{-1}(\sum_{t=s}^{\infty}\sum_{i=0}^{\infty}\underset{\sim}{H}_{t,i})\underset{\sim}{\Gamma}^{-1}\}$$

$$= E\{\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\underset{\sim}{h}_{s,i}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{H}_{s+k,k}\underset{\sim}{\Gamma}^{-1}\}$$

$$= E\{\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\underset{\sim}{u}_s\underset{\sim}{u}'_{s-i}\underset{\sim}{A}'^{i-1}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{A}^k\underset{\sim}{u}_s\underset{\sim}{u}'_{s-i}\underset{\sim}{A}'^{k+i}\underset{\sim}{\Gamma}^{-1}\} .$$

Since $\underset{\sim}{u}'_{s-i}\underset{\sim}{A}'^{i-1}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{A}^k\underset{\sim}{u}_s$ is a scalor, first transposing it and then taking the expectation, we get

$$E\{\underset{\sim}{h}_s\underset{\sim}{\Gamma}^{-1}(\sum_{t=s}^{\infty}\underset{\sim}{H}_t)\underset{\sim}{\Gamma}^{-1}\} = \underset{\sim}{\Omega}_u\{\sum_{k=0}^{\infty}\underset{\sim}{A}'^k\underset{\sim}{\Gamma}^{-1}(\sum_{i=0}^{\infty}\underset{\sim}{A}^i\underset{\sim}{\Omega}_u\underset{\sim}{A}'^i)\underset{\sim}{A}'^{k+1}\}\underset{\sim}{\Gamma}^{-1} . \qquad (A.7)$$

The fourth term of (A.4) can be similarly written as

$$E\{\underset{\sim}{h}_s\underset{\sim}{\Gamma}^{-1}(\sum_{t=s}^{\infty}\underset{\sim}{H}'_t)\underset{\sim}{\Gamma}^{-1}\} = E\{\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\underset{\sim}{h}_{s,i}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{H}'_{s+k,k}\underset{\sim}{\Gamma}^{-1}\}$$

$$= E\{\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\underset{\sim}{u}_s\underset{\sim}{u}'_{s-i}\underset{\sim}{A}'^{i-1}\underset{\sim}{\Gamma}^{-1}\underset{\sim}{A}^{k+i}\underset{\sim}{u}_{s-i}\underset{\sim}{u}'_s\underset{\sim}{A}'^k\underset{\sim}{\Gamma}^{-1}\}.$$

Since $\underset{\sim}{u}'_{s-i}\underset{\sim}{A}'^{i-1}\underset{\sim}{\Gamma}^{-1}A^{k+1}\underset{\sim}{u}_{s-i}$ is a scalor, moving it to the end and then taking the expectation, we get

$$E\{\underset{\sim}{h}_s\underset{\sim}{\Gamma}^{-1}(\sum_{t=s}^{\infty}\underset{\sim}{H}'_t)\underset{\sim}{\Gamma}^{-1}\} = \underset{\sim}{\Omega}_u[\sum_{k=0}^{\infty}\underset{\sim}{A}'^k\underset{\sim}{\Gamma}^{-1}tr\{\underset{\sim}{A}^{k+1}(\sum_{i=0}^{\infty}\underset{\sim}{A}^i\underset{\sim}{\Omega}_u\underset{\sim}{A}'^i)\underset{\sim}{\Gamma}\}]. \qquad (A.8)$$

Thus, the first term of (A.3) is

$$E\{\underset{\sim}{u}_s\underset{\sim}{z}'_{s-1}\underset{\sim}{\Gamma}^{-1}(\sum_{t=0}^{\infty}\underset{\sim}{z}_t\underset{\sim}{z}'_t)\underset{\sim}{\Gamma}^{-1}\}$$

$$= \underset{\sim}{\Omega}_u[\sum_{k=0}^{\infty}\underset{\sim}{A}'^k\underset{\sim}{\Gamma}^{-1}\{\underset{\sim}{d}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{d} + \underset{\sim}{d}\underset{\sim}{d}'\underset{\sim}{\Gamma}^{-1} + (\sum_{i=0}^{\infty}\underset{\sim}{A}^i\underset{\sim}{\Omega}_u\underset{\sim}{A}'^i)\underset{\sim}{A}'^{k+1}\underset{\sim}{\Gamma}^{-1}$$

$$+ \; tr(\underset{\sim}{A}^{k+1}(\sum_{i=0}^{\infty}\underset{\sim}{A}^i\underset{\sim}{\Omega}_u\underset{\sim}{A}'^i)\underset{\sim}{\Gamma}^{-1})\}], \qquad s = 1,2,\ldots,n. \qquad (A.9)$$

It is easily seen that the second term of (A.3) is $0(\underset{\sim}{B}^{n-s})$ for $s=1,2,\ldots,n$, because $\underset{\sim}{A}^i\underset{\sim}{u}_t = [(\underset{\sim}{B}^i\underset{\sim}{v}_t)',0]'$. Since $0(B^{n-s}) = 0(\rho^{n-s})$ where $0 \leq \rho < 1$ by the assumption on the characteristic roots, we have $\Sigma_{s=1}^n 0(B^{n-s}) = 0(1)$. Thus,

$$E\{n^{-1}\sum_{t=1}^{n} \underset{\sim}{u}_t \underset{\sim}{z}'_{t-1} \underset{\sim}{\Gamma}^{-1}(\sum_{t=0}^{n-1} \underset{\sim}{z}_t \underset{\sim}{z}'_t)\underset{\sim}{\Gamma}^{-1}\}$$

$$= E\{\underset{\sim}{u}_s \underset{\sim}{z}'_{s-1} \underset{\sim}{\Gamma}^{-1}(\sum_{t=0}^{\infty} \underset{\sim}{z}_t \underset{\sim}{z}'_t)\underset{\sim}{\Gamma}^{-1}\} + O(n^{-1})$$

From (A.1) and (A.9), the proof is completed.

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