# SIMULTANEOUS ASCENDING AUCTIONS WITH COMMON COMPLEMENTARITIES

by

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**Abstract**

Competitive equilibria are shown to exist in two-object exchange economies with indivisibilities and additive complementarities in agent valuations between objects, provided that complementarities are common across agents. We further investigate whether the competitive equilibrium can be obtained as an outcome of a simultaneous English-type auction mechanism under non-strategic (honest) bidding.

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## 1 Introduction

This paper addresses the problem of allocating two heterogeneous objects among a number of agents in an environment where there exist positive complementarities in agent valuations between objects. The problem of allocating heterogeneous objects in the presence of complementarities emerged, for example, in the recent sale of spectrum licenses by the Federal Communications Commission (Ausubel et. al., 1997). There are two issues of interest. The first is the existence of competitive equilibrium. The second is whether the competitive equilibrium can be obtained as an outcome of a relatively simple auction mechanism.

It is well-known that competitive (Walrasian) equilibria may not exist in environments with indivisible objects. Kelso and Crawford (1982) and further Gul and Stachetti (1999) show that the competitive equilibrium exists in environments with indivisibilities if the gross substitute condition is satisfied, that is, if there is a certain substitutability in agent valuations across objects. In the presence of complementarities, examples of non-existence of competitive equilibrium are easily generated (e.g., Bykowsky et al., 2000). It is then of interest to investigate whether competitive equilibria exist in some special classes of environments with complementarities.

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Brusco and Lopomo (BL, 1999) consider two-object environments with either no complementarities or large additive complementarities in their study of bidder collusion in multi-unit ascending price auctions. Although they focus on collusive equilibria of these auctions, they demonstrate that there also exist Perfect Bayesian Equilibria (PBE) that can be easily shown to be competitive (Walrasian) equilibria in the neoclassical sense. However, BL do not consider the case of moderate complementarities.[1] We show that, at least in the special case when complementarities are common to all bidders, the competitive equilibrium exists and is efficient irrespective of the magnitude of the complementarity term.

The second issue of interest is whether the competitive equilibrium outcomes may be implemented via a simple auction mechanism. Demange, Gale and Sotomayor (DGS; 1986) introduced a progressive auction mechanism that, under the assumption of honest (non-strategic) bidding, achieves efficient allocations and minimal Walrasian equilibrium prices. They consider a multi-unit allocation problems with heterogeneous goods, in which each bidder is constrained to get at most one object. Gul and Stachetti (2000) suggest a progressive auction mechanism that achieves efficient allocations and minimal Walrasian equilibrium prices in environments where bidders are not constrained to buy only one good, but the gross substitute condition of Kelso and Crawford (1982) on agent preferences is satisfied. We investigate whether the competitive equilibrium (CE) outcomes may be achieved by honest bidders under a simultaneous (non-combinatorial) progressive auction in the common complementarity case. Honest bidding rules prescribe bidding on an object or a package only if it maximizes bidder payoff at current prices. While honest bidding may or may not be an equilibrium strategy, it may correspond to naive price-taking behavior by unsophisticated bidders. We show that in the case of two bidders, a certain variation of the simultaneous ascending price auction mechanism ensures that honest bidding leads to minimal CE prices and efficient allocations for any value of complementarity (moderate or large). The variation we propose is analogous to the "exact" progressive auction mechanism of DGS, where bidders are required to report their full demand sets at each price. In contrast, the "approximate" bidding mechanism analogous to simultaneous English auction, such as the one considered by BL, may result in prices above the competitive equilibrium levels and bidder losses due to the exposure problem (Bykowsky et al., 2000). With more than two bidders, both the exact and the approximate mechanisms may result in disequilibrium allocations and prices. However, we show that such problems never arise and both mechanisms perform well if the complementarity is large.

[1] BL note that in general, in this case both the competitive and collusive PBE are hard to characterize.

## 2 Characterization of competitive equilibrium in the common complementarity case

The framework is similar to BL (1999). There are two objects, $A$ and $B$, and a set $N$ of $n$ agents (bidders), $n < \infty$. Let $a_i$ be bidder $i$'s value for object $A$, and $b_i$ be bidder $i$'s value for object $B$, with $a_i, b_i \in [0, \bar{v}]$. Then $i$'s value for the package AB is given by

$$u_i(AB) = a_i + b_i + k,$$

where $k$ is the common additive complementarity term, $k \geq 0$.[2] Let $W$ be the set of possible packages that can be sold to a bidder, $W \equiv \{\emptyset, A, B, AB\}$, and let $w$ be an element of $W$. We assume that bidders have quasilinear utilities in packages and money, and are not budget constrained. Then bidder $i$'s utility of buying a package $w$ given prices $p = (p_a, p_b)$ is $i$'s net value of the package, or his surplus: $S_i(w;p) = u_i(w) - \sum_{j \in w} p_j$, where $j$ is the object index, $j \in \{a, b\}$. Specifically,

$$S_i(\emptyset; p) = 0 \tag{1}$$

$$S_i(A; p) = a_i - p_a \tag{2}$$

$$S_i(B; p) = b_i - p_b \tag{3}$$

$$S_i(AB; p) = a_i - p_a + b_i - p_b + k \tag{4}$$

For any price vector $(p_a, p_b)$, let $i$'s demand set be the set of packages that maximize $i$'s surplus at this price:

$$D_i(p) = \{w \in W | S_i(w;p) = max_{v \in W} S_i(v;p)\}. \tag{5}$$

We employ a standard Walrasian notion of competitive equilibrium. A price $p = (p_a, p_b)$ is a competitive equilibrium price if, given $p$, there is an allocation of objects to bidders $\mu : \{A, B\} \to N$ such that each bidder gets a package in their demand set, i.e., there is no excess demand. Such price and allocation pair $(p, \mu)$ is called a competitive equilibrium. Given that bidders' values for objects are non-negative, $a_i, b_i \geq 0$, the equilibrium also requires no excess supply, i.e., both objects are allocated to bidders.

[2]BL allow the additive complementarity term to vary across bidders, so that $u_i(AB) = a_i + b_i + k_i$, with $k_i \in K$, where $K$ is either $\{0\}$ (no complementarity) or an interval $[\underline{k}, \bar{k}]$ with $\underline{k} > \bar{v}$ (large complementarity). Assuming that the object values are drawn independently across bidders from the same probability distribution, and the objects are allocated using a simultaneous ascending bid auction, BL show that with either no complementarity or with large complementarities there exists a PBE of this auction that lead to a CE outcome in the neoclassical sence; the resulting allocation is efficient. With no complementarity, the bidders with the highest values for each object buy the objects at the prices equal to the second highest values for that objects. With large complementarities, the two objects are allocated to the bidder with the highest value for the package, at the price equal to the second highest valuation for the package.

In the presence of complementarity, efficiency and equilibrium conditions and prices will differ depending on whether the objects are allocated to the same or to different bidders. We will say that "packaging" is efficient if it is efficient to allocate both items to the same bidder $i \in N$. "Splitting" is efficient if it is efficient to allocate the items to two different bidders.

**Proposition 1** *For any finite number of bidders, $n < \infty$, and any common complementarity term, $k \geq 0$, the set of CE prices and allocations is non-empty, and any CE allocation is efficient.*[3] *The set of CE prices is characterized as follows:*

- *Suppose that allocating both items to one bidder, or packaging, is efficient:*

$$a_i + b_i + k \geq \max\{\max_{j \in N} a_j + \max_{j \in N} b_j,\ \max_{j \neq i}(a_j + b_j) + k\} \tag{6}$$

  *for some $i \in N$. Then the set of CE prices is given by $(p_a, p_b)$ such that*

$$\begin{aligned} \max_{j \neq i}(a_j + b_j) + k \ &\leq\ p_a + p_b \ \leq\ a_i + b_i + k; && (7)\\ \max_{j \neq i} a_j \ &\leq\ p_a \ \leq\ a_i + k; && (8)\\ \max_{j \neq i} b_j \ &\leq\ p_b \ \leq\ b_i + k. && (9) \end{aligned}$$

- *Suppose that splitting of items between bidders is efficient:*

$$a_i + b_j \geq \max\{\max_{l \in N} a_l + \max_{l \in N} b_l,\ \max_{l \in N}(a_l + b_l) + k\}, \tag{10}$$

  *for some $i, j \in N$, $i \neq j$. Then the set of CE prices is given by $(p_a, p_b)$ such that*

$$\begin{aligned} \max_{l \neq i \neq j}(a_l + b_l) + k \ &\leq\ p_a + p_b; && (11)\\ \max\{a_j + k, \max_{l \neq i \neq j} a_l\} \ &\leq\ p_a \ \leq\ a_i; && (12)\\ \max\{b_i + k, \max_{l \neq i \neq j} b_l\} \ &\leq\ p_b \ \leq\ b_j. && (13) \end{aligned}$$

Before turning to the proof of proposition 1, it is useful to write out explicitly conditions under which a package $w \in W$ is demanded by a bidder $i \in N$. Let $p = (p_a, p_b)$ be a price vector. Applying definitions 1-4 and 5, we obtain:

- $AB \in D_i(p)$ if and only if:

$$\begin{aligned} a_i + b_i + k \ &\geq\ p_a + p_b && (14)\\ b_i + k \ &\geq\ p_b && (15)\\ a_i + k \ &\geq\ p_a. && (16) \end{aligned}$$

[3] If CE exists, then efficiency follows from the First Welfare Theorem. We re-establish efficiency here for the sake of completeness.

- $A \in D_i(p)$ if and only if:

$$\begin{aligned} a_i &\geq p_a && (17) \\ b_i + k &\leq p_b. && (18) \end{aligned}$$

(The third condition, $a_i - p_a \geq b_i - p_b$, follows from 17-18 and is therefore redundant.)

- $B \in D_i(p)$ if and only if:

$$\begin{aligned} b_i &\geq p_b && (19) \\ a_i + k &\leq p_a. && (20) \end{aligned}$$

(The third condition, $b_i - p_b \geq a_i - p_a$, follows from 19-20 and is therefore redundant.)

- $\emptyset \in D_i(p)$ if and only if:

$$\begin{aligned} a_i + b_i + k &\leq p_a + p_b && (21) \\ a_i &\leq p_a && (22) \\ b_i &\leq p_b. && (23) \end{aligned}$$

The following efficiency conditions will be also useful:

- Efficiency condition 6 holds, i.e., it is efficient to allocate the package $AB$ to bidder $i \in N$, if and only if:

$$\begin{aligned} a_i + b_i &\geq a_j + b_j \quad \text{for all } j \in N && (24) \\ a_i + b_i + k &\geq a_j + b_l \quad \text{for all } j, l \neq i && (25) \\ b_i + k &\geq b_j \quad \text{for all } j \neq i && (26) \\ a_i + k &\geq a_j \quad \text{for all } j \neq i. && (27) \end{aligned}$$

- Efficiency condition 10 holds, i.e., it is efficient to allocate item $A$ to bidder $i \in N$, and item $B$ to bidder $j \in N$, $i \neq j$, if and only if:

$$\begin{aligned} a_i &\geq a_l \quad \text{for all } l \in N && (28) \\ b_j &\geq b_l \quad \text{for all } l \in N && (29) \\ b_j &\geq b_i + k && (30) \\ a_i &\geq a_j + k. && (31) \\ a_i + b_j &\geq \max_{l \neq i \neq j}(a_l + b_l) + k. && (32) \end{aligned}$$

*Proof of proposition 1* The sets of CE prices are derived by solving for the no excess demand equilibrium conditions. Let $(\mu, p)$ be a CE price and allocation pair. Suppose under allocation $\mu$ each bidder $i \in N$ is assigned a package $w_i \in W$, so that $\cup_i w_i = \{A, B\}$, $w_i \cap w_j = \emptyset$ for all $i, j \in N$, $i \neq j$. The no excess demand conditions are:

$$S_i(w_i; p) \geq S_i(v; p) \quad \text{for any } v \in W. \tag{33}$$

There may be only two types of equilibrium allocations: either both items in $\{A, B\}$ are given to one of the bidders, or the items are split between the bidders. Consider equilibrium conditions for each of the two cases in turn.

CASE 1: Suppose that, in equilibrium, the package $AB$ is assigned to bidder $i \in N$. The no excess demand conditions are conditions 14-16 for bidder $i$, and conditions 21-23 for all other bidders $j \neq i$. Combining the inequalities, we obtain the characterization of the set of CE prices as given in 7-9. Note that a price vector satisfying the ineqalities 7-9 exists if and only if conditions 24-27 hold: Obviously, if conditions 24-27 are satisfied, we can find prices $(p_a, p_b)$ that satisfy 7-9. Conversely, suppose there exists a price vector $(p_a, p_b)$ satisfying 7-9. Then 7 implies 24, 8 implies 27, 9 implies 26; finally, adding 8 and 9, we obtain $\max_{j \neq i} a_j + \max_{l \neq i} b_l \leq p_a + p_b$, which, together with 7, implies 25. Hence we obtain that a set of CE prices supporting the allocation of the package $AB$ to bidder $i$ is non-empty if and only if such allocation is efficient.

CASE 2: Now suppose that, in equilibrium, item $A$ is assigned to bidder $i$, and item $B$ is assigned to bidder $j$, for some $i, j \in N$, $j \neq i$. Hence $A \in D_i(p)$, $B \in D_j(p)$, and $\emptyset \in D_l(p)$ for all $l \neq i \neq j$; that is, inequalities 17-18 hold for $i$, inequalities 19-20 hold for $j$, and inequalities 21-23 hold for all other bidders $l \neq i \neq j$. Combining these inequalities, we obtain the characterization of the set of equilibrium prices as given by 11-13. As in the previous case, it is straightforward to show that a price vector satisfying ineqalities 11-13 exists if and only if efficiency conditions 28-32 hold. $\square$

We will say that a CE price $p$ is a minimal CE price if for any other CE price $\tilde{p}$, $p_a + p_b \leq \tilde{p}_a + \tilde{p}_b$. Let us compare the minimal CE prices in the common complementarity case with two benchmarks. The first is the prices that result from the separate English auctions (SEA) run for each object if there are no complementarities between objects. Obviously, the SEA then implement an efficient outcome, with prices equal to the second highest values for each object: Let $a_i = \max_{l \in N} a_l$, and $b_j = \max_{l \in N} b_l$; then $(p_a^{SEA}, p_b^{SEA}) \equiv (\max_{l \neq i} a_l, \max_{l \neq j} b_l)$. Observe that these are the minimal CE prices for the no complementarity case.

The second benchmark is the prices that would result if $A$ and $B$ are bundled and sold in an English auction as a package, without the option of splitting the objects between the bidders. The two objects are then allocated to the bidder with the highest value for the package, at the price equal to the second highest valuation. We will call this price the Vickrey price for the package: Let $(a_i + b_i) = \max_{j \in N}(a_j + b_j)$; then $p_{ab}^{Vick} \equiv \max_{j \neq i}(a_j + b_j) + k$.

**Corollary 1** *1. In the presence of a positive complementarity, $k > 0$, any CE price is no lower then the minimal competitive (SEA) price in the no complementarity case: $(p_a, p_b) \geq (p_a^{SEA}, p_b^{SEA})$.*

*2. If the complementarity is large, $k > \bar{v}$, then packaging of items is always efficient, and the minimal CE price for the package is equal to the Vickrey price:*

$$p_a + p_b = \max_{j \neq i}(a_j + b_j) + k, \tag{34}$$

*where $i \in N$ is such that $(a_i + b_i) \geq (a_j + b_j)$ for all $j \neq i$.*

Statement (1) of the corollary follows from conditions 7-9 and 11-13 of proposition 1. Statement (2) is obtained by observing that 10 cannot hold as a strict inequality if $k > \bar{v}$, and by further checking that in this case $\max_{j \neq i}(a_j + b_j) + k \geq \max_{j \neq i} a_j + \max_{j \neq i} b_j$, where $i$ is the bidder with the highest value for the package; hence the lower bound on the sum of CE prices is determined from inequality 7.

The following characterization of CE prices in the case of two bidders will be used in further analysis.

**Corollary 2** *Let there be two bidders, $n = 2$, denoted by indexes $i, j \in N$, with $i \neq j$.*

- *Suppose that allocating both items to bidder $i$, or packaging, is efficient. Then the set of CE prices is characterized by the following constraints:*

$$a_j + b_j + k \;\leq\; p_a + p_b \;\leq\; a_i + b_i + k; \tag{35}$$

$$a_j \;\leq\; p_a \;\leq\; a_i + k; \tag{36}$$

$$b_j \;\leq\; p_b \;\leq\; b_i + k. \tag{37}$$

*The minimal CE prices are as follows:*

*1. If $a_i \geq a_j$ and $b_i \geq b_j$, then the set of minimal CE price vectors is given by:*

$$p_a = a_j + \lambda k \tag{38}$$

$$p_b = b_j + (1 - \lambda)k \tag{39}$$

*for any $\lambda \in [0, 1]$. In particular, $(a_j, b_j + k)$ and $(a_j + k, b_j)$ are minimal CE price vectors.*

2. *If $a_i > a_j$ and $b_i < b_j$, then the set of minimal CE price vectors is given by:*

$$p_a = a_j + \lambda k + (1-\lambda)(b_j - b_i) \tag{40}$$
$$p_b = b_j + (1-\lambda)k - (1-\lambda)(b_j - b_i) \tag{41}$$

*for any $\lambda \in [0,1]$. In particular, $(a_j + b_j - b_i, b_i + k)$ and $(a_j + k, b_j)$ are minimal CE price vectors.*

3. *If $a_i < a_j$ and $b_i > b_j$, then the set of minimal CE price vectors is given by:*

$$p_a = a_j + (1-\lambda)k - (1-\lambda)(a_j - a_i) \tag{42}$$
$$p_b = b_j + \lambda k + (1-\lambda)(a_j - a_i) \tag{43}$$

*for any $\lambda \in [0,1]$. In particular, $(a_i + k, a_j + b_j - a_i)$ and $(a_j, b_j + k)$ are minimal CE price vectors.*

- *Suppose that splitting of items between bidders, such that bidder $i$ is allocated $A$, and bidder $j$ is allocated $B$, is efficient. Then the set of CE prices is given by $(p_a, p_b)$ such that*

$$a_j + k \leq p_a \leq a_i \tag{44}$$
$$b_i + k \leq p_b \leq b_j. \tag{45}$$

*The minimal CE price is given by:*

$$p_a = a_j + k \tag{46}$$
$$p_b = b_i + k. \tag{47}$$

The above also implies that in the case of two bidders, the minimal CE price equals the Vickrey price of the package for any value of $k$, as long as the packaging is efficient:

**Corollary 3** *Let there be two bidders, $n = 2$. Suppose that packaging of items is efficient, $a_i + b_i + k \geq \max\{a_i + b_j, a_j + b_i, a_j + b_j + k\}$, for some $i, j \in N$, $i \neq j$. Then, for any value of complementarity term, $k > 0$, the minimal CE price for the package equals to the Vickrey price: $p_a + p_b = a_j + b_j + k$.*

# 3 Performance of the exact simultaneous auction mechanism under honest bidding

We next investigate whether competitive equilibrium outcomes may be achieved by honest bidders under a simultaneous English-type auction (SIMEA) in the common complementarity case. As in DGS (1986), we will consider two variants of the SIMEA mechanism

– the exact and the approximate mechanisms. We discuss the exact mechanism in this section. The auction starts with an initial price vector $(p_a^0, p_b^0) = (0, 0)$ announced by the auctioneer. We assume that all values are discrete. Specifically, all prices are integers, and all bidder valuations are even integers (in general, for the case of two objects, if all valuations are multiples of $\delta$, prices are required to be multiples of $\delta/2$). Each bidder announces which packages $w \in W$ are in her demand set at this price. It is required that all bidders report all packages in their demand sets. If it is possible to assign items $\{A, B\}$ to bidders so that each bidder gets a package in her demand set, then the prices must be at a CE, and the auction stops. If no such assignment exists, then the auctioneer raises prices by one unit on items in $\{A, B\}$ which are overdemanded. An items is overdemanded at price $p$ if it is necessary to increase the supply of this item (and, possibly, some other items) to find an assignment so that each bidder gets a package in their demand set. For example, in the case of two bidders, if bidder 1's and 2's demand sets at price $p$ are $D_1(p) = \{AB\}$, $D_2(p) = \{AB\}$, then the overdemanded set is $O(p) = \{A, B\}$. If bidder 1's and 2's demand sets at price $p$ are $D_1(p) = \{AB, A\}$, $D_2(p) = \{AB\}$, then the overdemanded set is $O(p) = \{A\}$.[4] After the prices are raised, the bidders report their new demand sets, and the procedure continues until a price vector is reached at which no excess demand exists. We first show that with two bidders, two objects and a common complementarity term $k \geq 0$, if all bidders report their demand sets honestly (that is, they follow the honest bidding strategy), the exact SIMEA mechanism converges to a minimal CE price and leads to an efficient allocation.

**Proposition 2** *Suppose there are 2 bidders, $n = 2$. If bidders follow the honest bidding strategy, then, for any value of the common complementarity term $k \geq 0$, the exact SIMEA mechanism converges to a minimal CE price, and the resulting allocation is efficient.*

It is clear that under honest bidding, the iterations will stop at some point, since prices are bounded by bidder valuations: $p_a, p_b \leq \bar{v} + k$. It is also obvious that in the no complementarity case, $k = 0$, the mechanism will converge to the SEA prices. To establish the case of positive complementarity, $k > 0$, we employ the following properties of bidder demands under honest bidding:

**Lemma 1 (No switching)** *Suppose bidders bid honestly in the exact SIMEA, and let $k > 0$. For any bidder $i$, any price $p$, and any items $v, w \in \{A, B\}$ with $v \neq w$, if*

[4] If bidder 1's and 2's demand set at price $p$ are $D_1(p) = \{AB, A, B\}$, $D_2(p) = \{AB\}$, then the overdemanded set is either $O(p) = \{A\}$, or $O(p) = \{B\}$, but not $\{A, B\}$. The mechanism then prescribes to raise the price of either $A$ or $B$, but not both. This creates an indeterminacy in the mechanism. Lemma 1 below shows, however, that under honest bidding, a bidder's demand set may never consist of $\{AB, A, B\}$, given $k > 0$. For $k = 0$, $\{AB, A, B\} \subseteq D_i(p)$ implies $\emptyset \in D_i(p)$.

*$v \in D_i(p)$, then $w \notin D_i(\tilde{p})$ for all and $\tilde{p} \geq p$. That is, a bidder does not demand two packages containing two separate items at the same time, and does not switch from one separate item to the other as the prices rise.*

*Proof* Suppose that, for some bidder $i \in N$, $A \in D_i(p)$ at some price $p = (p_a, p_b)$. Then, from 18, $p_b \geq b_i + k$, which obviously implies that $\tilde{p}_b \geq b_i + k$ for any $\tilde{p}_b \geq p_b$. But, from 19, we may have $B \in D_i(p)$ only if $p_b \leq b_i$. □

**Lemma 2 (Bidder demands with and without complementarities)** *Suppose bidders bid honestly in the exact SIMEA. For any price $p$, let $D_i^0(p)$ denote bidder $i$'s demand set in the no complementarity case, $k = 0$, and let $D_i^+(p)$ denote bidder $i$'s demand set in the positive complementarity case, $k > 0$. If, at some price $p$, $AB \in D_i^0(p)$, then $D_i^+(p) = \{AB\}$; if $v \in D_i^0(p)$ for some $v \in \{A, B\}$, and $\emptyset \notin D_i^0(p)$, then $D_i^+(p) \subseteq \{v, AB\}$. That is, a positive complementarity induces bidders to seek higher aggregations.*

*Proof* Suppose, for some price $p$, $AB \in D_i^0(p)$. This implies that $a_i \geq p_a$ and $b_i \geq p_b$, and, hence, $D_i^+(p) = \{AB\}$ for any $k > 0$. Now suppose that $A \in D_i^0(p)$ and $\emptyset \notin D_i^0(p)$. This implies that $a_i > p_a$ and, hence, for any $k > 0$, either $D_i^+(p) = \{A\}$ (if $p_b > b_i + k$), or $D_i^+(p) = \{AB\}$ (if $p_b < b_i + k$), or $D_i^+(p) = \{A, AB\}$ (if $p_b = b_i + k$). □

The above proof also demonstrates that if, at some price $p$, there is excess demand when $k = 0$, then there is excess demand when $k > 0$. Hence, we obtain:

**Corollary 4 (Lower bound on prices)** *If bidders bid honestly in the exact SIMEA, then final auction prices are at least as high as the SEA prices in the no complementarity case: $(p_a, p_b) \geq (p_a^{SEA}, p_b^{SEA})$.*

We now prove proposition 2.

*Proof of proposition 2* It is sufficient to consider the positive complementarity case, $k > 0$. Since $a_i \geq 0$ and $b_i \geq 0$ for all $i \in N$, we observe that both bidders will initially demand package $\{AB\}$ only, and therefore the prices will rise on both items simultaneously. Let $t$ be the last iteration at which both bidders demand $\{AB\}$ only; let the corresponding price be $p_a^t = p_b^t = \underline{p}$. From 14-16, this implies that for each $i \in N$:

$$a_i + b_i + k > 2\underline{p} \tag{48}$$

$$a_i + k > \underline{p} \tag{49}$$

$$b_i + k > \underline{p}. \tag{50}$$

Suppose, at iteration $t+1$, bidder $i$ reports some other package in her demand sets. Given honest bidding, we note that $\underline{p}+1 \geq \min\{a_1, a_2, b_1, b_2\}$. There are two possibilities:

CASE I: $\emptyset \in D_i(\underline{p}+1)$. This implies that:

$$a_i + b_i + k \leq 2\underline{p} + 2 \tag{51}$$

$$a_i \leq \underline{p} + 1 \tag{52}$$

$$b_i \leq \underline{p} + 1. \tag{53}$$

From inequalities 48 and 51 (and given that the values are even intergers), we obtain that $a_i + b_i + k = 2\underline{p} + 2$. The auction stops at the price $p_a = p_b = \underline{p} + 1$, and package $AB$ is allocated to bidder $j \neq i$.

It is left to demonstrate that this allocation is efficient. Since $j$ demanded $\{AB\}$ at iteration $t$, we obtain that $a_j + b_j + k \geq 2\underline{p} + 2 = a_i + b_i + k$, i.e., $a_j + b_j \geq a_i + b_i$. We next need to show that $a_j + b_j + k \geq \max\{a_i + b_j, a_j + b_i\}$, or:

$$a_i \leq a_j + k \tag{54}$$

$$b_i \leq b_j + k. \tag{55}$$

But these follow from inequalitites 49 and 52, 50 and 53 implied by $j$'s demand at $\underline{p}$ and $i$'s demand at $\underline{p}+1$:

$$a_j + k \geq \underline{p} + 1 \geq a_i \tag{56}$$

$$b_i + k \geq \underline{p} + 1 \geq b_i. \tag{57}$$

We note that the resulting price satisfies all the requirements 35-37 of CE prices and is, by corollary 2, a minimal CE price: $p_a + p_b = a_i + b_i + k$.

CASE II: $\emptyset \notin D_i(\underline{p}+1)$. Suppose, without loss of generality, that $A \in D_i(\underline{p}+1)$. By lemma 1, $B \notin D_i(\underline{p}+1)$. (The case when $B \in D_i(\underline{p}+1)$, $A \notin D_i(\underline{p}+1)$ is analogous.) Similarly to Case I above, given that all values are even intergers, we obtain $AB \in D_i(\underline{p}+1)$. Hence, $D_i(\underline{p}+1) = \{A, AB\}$. $A \in D_i(\underline{p}+1)$ implies:

$$a_i > \underline{p} + 1 \tag{58}$$

$$b_i + k = \underline{p} + 1 \tag{59}$$

If follows that $a_i > b_i + k$. The auction may follow several scenarios depending on the demand of bidder $j$:

1. If $\emptyset \in D_j(\underline{p}+1)$, then the auction stops, and the package $AB$ is allocated to bidder $i$. This is Case I considered above.

2. If $\emptyset \notin D_j(\underline{p}+1)$ and $B \notin D_j(\underline{p}+1)$, then either $D_j(\underline{p}+1) = \{AB\}$, or $D_j(\underline{p}+1) = \{A, AB\}$ and the price will further rise on item $A$, but not on $B$. The following two points are worth noting. First, for any $\tilde{p} = (\tilde{p}_a, \tilde{p}_b)$ such that $\tilde{p}_a > \underline{p}+1$, $\tilde{p}_b = \underline{p}+1$, if $S_j(AB; \underline{p}+1) > S_j(A; \underline{p}+1)$, then $S_j(AB; \tilde{p}) > S_j(A; \tilde{p})$. If $S_j(AB; \underline{p}+1) = S_j(A; \underline{p}+1)$, then $S_j(AB; \tilde{p}) = S_j(A; \tilde{p})$. That is, bidder $j$'s preferences between packages $A$ and $AB$ do not change when the price rises on $A$, but not on $B$; this follows from the relation between $j$'s surpluses from $A$ and $AB$ identified by conditions 15 and 18. Similarly, bidder $i$ will demand either $\{A, AB\}$, or nothing, when the price rises on $A$ but not on $B$. (By lemma 1, she will not demand $B$). Second, since bidder $j$ still demands package $AB$ at the price $(\underline{p}+1, \underline{p}+1)$, we obtain, using 59, that $b_j + k \geq \underline{p} + 1 = b_i + k$, and hence $b_j \geq b_i$.

   There are only three possible cases:

   (a) Bidder $j$ keeps demanding $AB$ (and, possibly, $A$, but not $B$), until $(p_a, p_b) = (a_i, b_i + k)$. At this point bidder $i$ reports $\emptyset$ as part of her demand set, and the auction stops, with package $AB$ allocated to bidder $j$. Let us show that this allocation is efficient. Since $AB$ is demanded by $j$ at the price $(p_a, p_b) = (a_i, b_i + k)$, we have:

   $$a_j + b_j + k \geq a_i + b_i + k \tag{60}$$
   $$b_j + k \geq b_i + k \tag{61}$$
   $$a_j + k \geq a_i. \tag{62}$$

   The above three inequalities imply that it is efficient to allocate package $AB$ to bidder $j$, as given by efficiency conditions 24-27. From corollary 2, the resulting price $(p_a, p_b) = (a_i, b_i + k)$ is a minimal CE price.

   (b) Bidder $j$ keeps demanding $AB$ and then demands $B$ when the price of $A$ reaches some level $p_a$, $b_i + k < p_a \leq a_i$. At this point the auction stops, with item $A$ allocated to bidder $i$ at the above price $p_a$, and item $B$ allocated to bidder $j$ at $p_b = b_i + k$. Let us show that this allocation is efficient. Since $j$ demands $B$ at the price $(p_a, b_i + k)$, we have:

   $$b_j \geq b_i + k \tag{63}$$
   $$a_j + k = p_a. \tag{64}$$

   From 64, $a_j + k = p_a \leq a_i$, and therefore $a_j + k \leq a_i$. Hence we obtain that $b_j > b_i + k$ and $a_i \geq a_j + k$, which are equivalent to the efficiency conditions 28-32 adopted for the case of two bidders. Hence, the items are allocated efficiently at the minimal CE price $(p_a, p_b) = (a_j + k, b_i + k)$.

(c) Bidder $j$ keeps demanding $AB$ (and, possibly, $A$), and then demands $\emptyset$ when the price of $A$ reaches some level $p_a$, $b_i + k < p_a \leq a_i$. At this point the auction stops, with package $AB$ allocated to bidder $i$ at the price $(p_a, b_i + k)$, where $p_a$ is defined as above. Let us show that such allocation is efficient. Since $j$ demands $\emptyset$ at the price $(p_a, b_i + k)$, we have:

$$p_a + b_i + k = a_j + b_j + k \tag{65}$$

$$p_a \geq a_j \tag{66}$$

$$b_i + k \geq b_j. \tag{67}$$

Since $p_a \leq a_i$, from 65 we obtain $a_i + b_i \geq a_j + b_j$. From 66, $a_j \leq p_a \leq a_i$, and hence $a_j \leq a_i$. Finally, 67 states that $b_i + k \geq b_j$. These inequalities together establish the efficiency conditions 24-27. From corollary 2, the resulting price $(p_a, p_b) = (a_j + b_j - b_i, b_i + k)$ is a minimal CE price (note that $b_j \geq b_i$).

3. If $\emptyset \notin D_j(\underline{p} + 1)$ and $B \in D_j(\underline{p} + 1)$, then the auction stops. Item $A$ is allocated to bidder $i$, and item $B$ is allocated to bidder $j$ at prices $p_a = p_b = \underline{p} + 1$. To show that this allocation is efficient, we note that since $B \in D_j(\underline{p} + 1)$, then:

$$b_j \geq \underline{p} + 1 \tag{68}$$

$$a_j + k = \underline{p} + 1 \tag{69}$$

From 58 and 59, we know that $a_i > \underline{p} + 1$ and $b_i + k = \underline{p} + 1$. Hence, $b_j \geq b_i + k$, and $a_i \geq a_j + k$, which shows that the conditions 28-32 holds, i.e., the resulting allocation is efficient. From corollary 2, the resulting price $(p_a, p_b) = (a_j + k, b_i + k)$ is the minimal CE price.

This exhausts all possible honest bidding scenarios for $n = 2$. $\square$

Unfortunately, the desirable properties of the exact SIMEA do not generalize to the case of more than two bidders if no additional constraints are imposed on the range of values of the complementarity term $k$.[5] The following example demonstrates that with more than two bidders, $n > 2$, the exact SIMEA mechanism may lead to inefficient allocations and prices out of equilibrium range.

**Example 1** Let there be three bidders, $n = 3$, and let $a_1 = b_1 = 20$, $a_2 = 36$, $b_2 = 0$, $a_3 = b_3 = 16$, and $k = 20$. Hence it is efficient to allocate both items to bidder 1; from 7-9,

[5] This is in spite of the fact that the properties of bidder demands as given in lemmas 1-2 and corollary 4 apply irrespective of the number of bidders.

| $p_a$ | $p_b$ | $\max S_1(p)$ | $D_1(p)$ | $\max S_2(p)$ | $D_2(p)$ | $\max S_3(p)$ | $D_3(p)$ | $O(p)$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 60 | AB | 56 | AB | 52 | AB | A,B |
| ... | | | | | | | | |
| 20 | 20 | 20 | AB | 16 | A,AB | 12 | AB | A,B |
| 21 | 21 | 18 | AB | 15 | A | 10 | AB | A,B |
| ... | | | | | | | | |
| 26 | 26 | 8 | AB | 10 | A | 0 | AB,∅ | A |
| 27 | 26 | 7 | AB | 9 | A | 0 | ∅ | A |
| ... | | | | | | | | |
| 34 | 26 | 0 | AB,∅ | 2 | A | 0 | ∅ | ∅ |

Table 1: An example of failure of the exact SIMEA mechanism to reach a competitive equilibrium outcome with three bidders.

the set of CE prices is given by:

$$56 \leq p_a + p_b \leq 60 \tag{70}$$
$$36 \leq p_a \leq 40 \tag{71}$$
$$16 \leq p_b \leq 40 \tag{72}$$

Consider the bidding dynamics under the exact mechanism as illustrated in table 1. All three bidders will initially demand package $AB$ only, and therefore the prices will rise on both items simultaneously. At $p_a = p_b = 20$, bidder 2 switches his demand from $AB$ to $A$: $S_2(AB; p) = S_2(A; p) = 16$. However, bidders 1 and 3 keep demanding $AB$ only, and hence the prices rise on both items until they reach $p_a = p_b = 26$. At this point bidder 3 reports $\emptyset \in D_3(p)$, given $S_3(AB; p) = S_3(\emptyset; p) = 0$, and the price of $B$ stops rising. Now bidder 1 demands $AB$, and bidder 2 demands $A$, hence the price of $A$ keeps rising until the prices reach the level of $p_a = 34$, $p_b = 26$. At this point $S_1(AB; p) = S_1(\emptyset; p) = 0$, and bidder 1 reports $\emptyset \in D_1(p)$; bidder 2 still demands $A$, with $S_2(A; p) = 2$. Hence the auction stops with item $A$ allocated to bidder 2, and item $B$ not allocated; the resulting prices, $(p_a, p_b) = (34, 26)$, are out of the equilibrium range: $p_a < 36$.

However, we can show that in the case of a large complementarity, $k > \bar{v}$, the exact mechanism performs well with any number of bidders. We first observe the following:

**Lemma 3** *Suppose the complementarity is large, $k > \bar{v}$. Then, in the exact SIMEA mechanism, honest bidders never bid on individual items. That is, for any bidder $i \in N$, for any price $p$ that may result from honest bidding, $D_i(p) \subseteq \{AB, \emptyset\}$.*

*Proof* As before, we observe that all bidders will initially demand the package $AB$ only, and therefore the prices will rise on both items simultaneously. Let $t$ be the last iteration when all bidders demand $AB$ only; let the corresponding price be $p_a^t = p_b^t = \underline{p}$. From 15, we obtain $b_i + k > \underline{p}$ for all $i \in N$. Suppose some bidder $i$ demands $A$ when the prices rise to $(\underline{p} + 1)$. Then, from 17-18:

$$a_i \geq \underline{p} + 1 \tag{73}$$

$$b_i + k \leq \underline{p} + 1. \tag{74}$$

Hence, $a_i \geq b_i + k \geq k > \bar{v}$, a contradiction. $\square$

Further, from corollary 1, we know that when $k > \bar{v}$, packaging of items is always efficient, and the minimal equilibrium price is equal to the Vickrey price for the package. Hence we obtain:

**Proposition 3** *Suppose the complementarity is large, $k > \bar{v}$. Then for any number of bidders, $n \geq 2$, honest bidding under the exact SIMEA mechanism leads to an efficient allocation and minimal CE prices:*

$$p_a = p_b = p_{ab}^{Vick}/2.$$

# 4 Features of the approximate SIMEA meachanism under honest bidding

The approximate SIMEA mechanism is similar to an English-type open outcry auction, run simultaneously for both goods. It is completely analogous to the approximate mechanism in DGS (1986), except that bidders may bid for more than one item at any given time. As in the "exact" mechanism, the auction starts with an initial price vector $(p_a^0, p_b^0) = (0, 0)$ announced by the auctioneer. At this point any bidder may bid for any item or items, which means she commits herself to possibly buying the items at the announced prices; the items are temporarily assigned to this bidder. Then any uncommitted bidder may (i) bid for some unassigned item(s), in which case she becomes committed to them at their initial prices; (ii) she may bid for assigned items, in which case she becomes committed to them, and their prices increase by a fixed amount $\delta$ each; or (iii) she may drop out of the bidding. The auction stops when there are no more uncommitted bidders, at which point each committed bidder buys the items assigned to her at their current prices (see also DGS, p. 867).[6]

DGS show that in their setting, the approximate mechanism leads to prices arbitrary close to the minimal equilibrium prices, provided that the bid increment $\delta$ is set small

[6] This mechanism is also analogous to the simultaneous ascending bid auction described by BL (1999), except that the bidders here are constrained to bid up by a fixed increment $\delta$.

enough. We demonstrate that generally this is no longer true for our setting. In the presence of complementarities, the approximate mechanism occasionally suffers from an exposure problem (Bykowsky et al., 2000), which may lead to prices above the competitive equilibrium levels, bidder losses, and inefficient allocations. However, we show that this can never happen when complementarities are large, $k > \bar{v}$. Further, the results of our numerical simulations indicate that even with moderate complementarities, the occurrences of bidder losses are rare, and in most cases the approximate mechanism leads to efficient allocations and prices close to the minimal CE prices.

The exposure problem arises under the approximate mechanism because, unlike the exact mechanism, as part of the procedure bidders have to commit themselves to possibly buying the items at the announced prices. In the presence of complementarities, a bidder often needs to bid above the stand-alone value of objects to obtain the package, which may lead to "mutually destructive bidding" and generate losses when the desired packages do not materialize (Bykowsky et al., 2000; Kagel and Levin, 2001).[7]

Bykowsky et al. discuss two types of problems that may arise in environments with complementarities under simple (non-combinatorial) auction procedures similar to the approximate mechanism. First, if bidders have to bid above items' stand-alone values to obtain packages, they may drop out of bidding early due to the fear losses, and the efficient equilibrium may not be reached. However, this is not a problem if bidders are not afraid of financial exposure (in particular, if they bid honestly). Second, in some environments the competitive equilibrium may not exist at all, and it may be impossible to reach an efficient allocations via a non-combinatorial auction without inflicting losses on bidders. In our setting, the first problem is ruled out because we assume that bidders bid honestly and therefore do not suffer from loss avoidance; the second problem is ruled out because the competitive equilibrium exists (proposition 1). However, we show that even when the competitive equilibrium exists and bidders bid honestly, the approximate mechanism may lead to prices above the CE levels and bidder losses.

**Example 2** Consider the following example. Let there be two bidders, $n = 2$, and let $a_1 = 21$, $b_1 = 64$, $a_2 = 20$, $b_2 = 99$, and $k = 30$. Efficiency prescribes allocating both items to bidder 2; the minimal CE prices are given by: $p_a \in [21, 50]$, $p_b \in [65, 94]$, with $p_b = 115 - p_a$, and the maximal CE prices are $p_a \in [21, 50]$, $p_b \in [90, 119]$, with $p_b = 140 - p_a$. Consider the bidding dynamics under the approximate mechanism with the bid increment $\delta = 1$. Here we assume that bidders bid in the most aggressive manner

[7] Kagel and Levin find that the exposure problem is quite strong in experimental ascending auctions with homogeneous goods and variable complementarities. Kwasnica and Sherstyuk (2000) do not observe bidder losses in their experimental markets with common complementarities, but they report significant underbidding which may be due to the fear of such losses.

| round | $p_a$ | $p_b$ | $\mu_a$ | $\mu_b$ | bidder | S(AB) | S(A) | S(B) | S(∅) | BidA | BidB |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 1 | 113 | 20 | 63 | 0 | yes | yes |
| 2 | 1 | 1 | 1 | 1 | 2 | 145 | 18 | 97 | 0 | yes | yes |
| 3 | 2 | 2 | 2 | 2 | 1 | 109 | 18 | 61 | 0 | yes | yes |
| 4 | 3 | 3 | 1 | 1 | 2 | 141 | 16 | 95 | 0 | yes | yes |
| ... | | | | | | | | | | | |
| 20 | 19 | 19 | 1 | 1 | 2 | 109 | 0 | 79 | 0 | yes | yes |
| 21 | 20 | 20 | 2 | 2 | 1 | 73 | 0 | 43 | 0 | yes | yes |
| 22 | 21 | 21 | 1 | 1 | 2 | 105 | -2 | 77 | 0 | yes | yes |
| 23 | 22 | 22 | 2 | 2 | 1 | 69 | -2 | 41 | 0 | yes | yes |
| ... | | | | | | | | | | | |
| 48 | 47 | 47 | 1 | 1 | 2 | 53 | -28 | 51 | 0 | yes | yes |
| 49 | 48 | 48 | 2 | 2 | 1 | 17 | -28 | 15 | 0 | yes | yes |
| 50 | 49 | 49 | 1 | 1 | 2 | 49 | -30 | 49 | 0 | yes | yes |
| 51 | 50 | 50 | 2 | 2 | 1 | 13 | -30 | 13 | 0 | yes | yes |
| 52 | 51 | 51 | 1 | 1 | 2 | 45 | -32 | 47 | 0 | no | yes |
| 53 | 51 | 52 | 1 | 2 | 1 | 11 | -30 | n/a | n/a | holds | yes |
| 54 | 51 | 53 | 1 | 1 | 2 | 43 | -32 | 45 | 0 | no | yes |
| ... | | | | | | | | | | | |
| 63 | 51 | 62 | 1 | 2 | 1 | 1 | -30 | n/a | n/a | holds | yes |
| 64 | 51 | 63 | 1 | 1 | 2 | 33 | -32 | 35 | 0 | no | yes |
| 65 | 51 | 64 | 1 | 2 | 1 | -1 | -30 | n/a | n/a | holds | yes |
| 66 | 51 | 65 | 1 | 1 | 2 | 31 | -32 | 33 | 0 | no | yes |
| ... | | | | | | | | | | | |
| 92 | 51 | 91 | 1 | 1 | 2 | 5 | -32 | 7 | 0 | no | yes |
| 93 | 51 | 92 | 1 | 2 | 1 | -29 | -30 | n/a | n/a | holds | yes |
| 94 | 51 | 93 | 1 | 1 | 2 | 3 | -32 | 5 | 0 | no | yes |
| 95 | 51 | 94 | 1 | 2 | 1 | -31 | -30* | n/a | n/a | holds* | no |
| 96 | 51* | 94* | 1 | 2 | 2 | 3 | n/a | 5* | n/a | no | holds* |

Table 2: An example of bidder loss under honest bidding in the approximate mechanism. Bidder values are $a_1 = 21$, $b_1 = 64$, $a_2 = 20$, $b_2 = 99$, and $k = 30$.

consistent with honest bidding, i.e., they bid on both $A$ and $B$ whenever $AB$ is in their demand set. However, it is easy to check that the bidding dynamics and the resulting prices and allocations would stay essentially the same if bidders bid less aggressively (i.e., bid on an individual item whenever the item and the package are both in the demand set), or the bid increment is reduced (e.g., $\delta = 0.5$ or $\delta = 0.1$). The bidding dynamics is illustrated in table 2. Since under the approximate mechanism the two bidders will take turns bidding, the bidding may be described in rounds. For each round, the table shows the current prices, $p_a$ and $p_b$; assignments, $\mu_a$ and $\mu_b$[8]; the bidder whose turn it is to bid; this bidder's resulting surpluses from each package if she acquires this package; and the bidder's decision whether to bid or not on each item ("holds" indicates that the bidder is

[8] $\mu_a = \mu_b = 0$ in the first round indicates that both items are initially unassigned; $\mu_v = i$ in the later rounds indicates that item $v$ is currently assigned to bidder $i$.

currently committed to the item).

The bidding starts with both bidders bidding for both items in pursuit of the package $AB$; the prices increase accordingly. Note that the bidders continue bidding in this way even when the price of one of the items, $A$, exceeds its stand-alone value (rounds 23 and 22, for bidders 1 and 2, respectively), and bidders become exposed to financial losses. At round 52, the prices reach the levels where bidder 2 finds most profitable bidding on item $B$ only ($S_2(B) = 47 > 45 = S_2(AB)$); she therefore drops out of bidding on $A$. Bidders 1 then appears "stuck" with $A$, and keeps on bidding on $B$ in an attempt first to avoid the loss from holding $A$ alone (rounds 53-63), and later to minimize the loss from packages he may be committed to buy (rounds 65-93). The auction stops at round 95 when the price of $B$ rises to the level where buying $AB$ for bidder 1 would mean even a greater loss of 31 than the loss of 30 from holding $A$ alone. The resulting prices, bidder surpluses and allocations are marked with asterisks in the table. We note that the prices exceed the CE prices, $p_a + p_b = 51 + 94 = 145 > 140$, the joint bidder surplus is negative, $S_1(B) + S_2(A) = 5 - 30 = -25$, and the resulting allocation is inefficient: bidder 1 buys item $A$, and bidder 2 buys item $B$.

It therefore appears that the exposure problem may be quite severe even in a simple two-object two-bidder setting with common additive complementarity. However, we note the following. First, the reasoning behind Lemma 3 and Proposition 3 fully applies to the approximate mechanism, and hence we obtain that when complementarities are large, $k > \bar{v}$, the exposure problem never emerges, and the approximate mechanism leads to prices essentially equal to the minimal CE price.[9] Further, results of our numerical simulations indicate that even when the complementarity is moderate, $0 < k < \bar{v}$, the exposure problem emerges quite rarely. Table 3 reports the results of numerical simulations of honest bidding with two bidders and the bid increment $\delta = 1$ when bidder values are drawn independently from the uniform distribution, $v_i \sim U[0, 100]$, for $k \in \{10, 20, ..., 90, 101\}$. For every value of $k$, the resulting prices, on average, are very close to the minimal CE price; over 90% of allocations are efficient. The maximal price deviation and maximal bidder loss never exceed the value of the complementarity term $k$, and the exposure problem is the most noticable, both in terms of frequency and the size of bidder losses, when complementarities are small to intermediate in values, $30 \leq k \leq 50$.[10] Additional nu-

[9]BL show that with large complementarities, the honest bidding strategy profile forms a PBE in the simultaneous ascending bid auction; this leads to an efficient allocation and Vickrey price for the package. See also footnote 2.

[10]The latter observation is interesting in relation to the discussion of the exposure problem in FCC auctions. Ausubel et al. (1997) argue that the exposure problem in FCC auctions was not severe since the synergiesies were small. We find that in our setting, the exposure problem is most likely to surface precisely in the case of small to moderate synergies.

| k | number of obs. | % efficient allocations | Price deviation from minCE mean | min | max | % obs. with losses | Max loss per bidder |
|---|---|---|---|---|---|---|---|
| 10 | 1000 | 95.5 | 1.09 | -1 | 10 | 4.8 | 10 |
| 20 | 1000 | 92.8 | 1.42 | -1 | 20 | 7.6 | 20 |
| 30 | 1000 | 93.3 | 1.31 | -1 | 30 | 6.8 | 30 |
| 40 | 1000 | 93.6 | 1.17 | -1 | 34 | 6.1 | 34 |
| 50 | 1000 | 95.1 | 1.22 | -1 | 32 | 5.3 | 32 |
| 60 | 1000 | 97.6 | 0.74 | -1 | 31 | 2.2 | 31 |
| 70 | 1000 | 98.3 | 0.66 | -1 | 14 | 1.4 | 14 |
| 80 | 1000 | 99.3 | 0.5 | -1 | 9 | 0.3 | 9 |
| 90 | 1000 | 99.4 | 0.49 | -1 | 6 | 0.1 | 6 |
| 101 | 1000 | 99.5 | 0.44 | -1 | 2 | 0 | 0 |

Table 3: Results of numerical simulations of the approximate mechanism with two honest bidders. Bidder values are drawn from the uniform distribution on $[0, 100]$.

merical simulations conducted with five bidders indicate that increasing the number of bidders neither aggravates nor eliminates the problem. It appears that the disequilibrium problems of the SIMEA mechanisms manifest themselves quite infrequently, and, overall, the mechanisms perform quite well in terms of both prices and efficiency.

# 5 Conclusions

We have demonstrated that the competitive equilibrium exists in a simple class of environments with two indivisible objects and a common additive complementarity in bidder valuations between the objects. This raises an interesting question on whether the equilibrium existence may be established in a more general framework with positive complementarities. An obvious way to proceed is to allow a certain degree of variability in the complementarity term across bidders (as in BL), and to further generalize the setting to more than two objects. Such generalizations lead to quite complex problems which are beyond the scope of this paper.

We have also investigated to what extent simple non-combinatorial auctions may be able to implement competitive equilibrium outcomes in environments with common additive complementarity if bidders follow honest bidding rules. Here our findings generally support the viewpoint that applicability of such mechanisms in environments with complementarities is limited. A variation of the simultaneous English auction mechanism will lead to a competitive equilibrium outcome if there are only two bidders, or if the complementarity between objects is large enough to ensure that package bidding always dominates bidding for individual objects. However, in more general cases such auction may occasionally result in disequilibrium prices and allocations. Even though our numerical simulations indicate that such disequilibrium outcomes may be quite rare, the simultaneous English auction mechanism is not guaranteed to perform well in every case. Further, the problems may become more severe in more general environments. Our results therefore suggest that simultaneous English auctions may be used in environments with complementarities if simplicity of the auction is a major concern and occasional failures of the mechanism are admissible. In situations where achieving the efficient equilibrium outcome is critical, mechanism designers should turn to more complex combinatorial auctions that would allow for package bidding (such as in Bykowsky et al., 2000).

## References


[1] Ausubel, Lawrence, Peter Cramton, Preston McAfee, and John McMillan, 1997, Synergies in Wireless Telephony: Evidence from the Broadband PCS Auctions, Journal of Economics and Management Strategy 6(3): 497-527.

[2] Brusco, Sandro, and Guiseppe Lopomo, 1999, Collusion via signaling in simultaneous ascending bid auctions with heterogeneous objects, with and without and complementarities. Review of Economic Studies, forthcoming.

[3] Bykowsky, Mark, Robert Cull, and John Ledyard, 2000, Mutually destructive bidding: The FCC Auction Design Problem, Journal of Regulatory Economics 17(3): 205-228.

[4] Demange, Gablielle, David Gale, and Marilda Sotomayor, 1986, Mulit-Item Auctions, Journal of Political Economy, 94, 863-872.

[5] Gul, Faruk, and Ennio Staccetti, 1999, Walrasian Equilibrium with Gross Substitutes, Journal of Economic Theory 87: 95-124.

[6] Gul, Faruk, and Ennio Staccetti, 2000, The English Auction with Differentiated Commodities, Journal of Economic Theory 92: 66-95.

[7] Kagel, John, and Dan Levin, 2001, Multi-Unit Demand Auctions with Synergies: Behavior in Sealed Bid versus Ascending Bid Uniform Price Auctions, Mimeo, Ohio State University.

[8] Kelso, A., and V. Crawford, 1982, Job Matching, Coalition Formation, and Gross Substitutes, Econometrica 50: 1483-1504.

[9] Kwasnica, Anthony, and Katerina Sherstyuk, 2001, Collusion via Signaling in Multiple Object Auctions with Complementarities: an Experimental Test, Mimeo, Penn State University.