LINEAR UNBIASED ESTIMATORS AS A GLS CLASS AND THE GENERALIZED AITKEN THEOREM AS A COROLLARY

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Working Paper No. 90-23
February, 1990

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1. INTRODUCTION.

This paper introduces two new theorems: one establishes that all linear unbiased estimators are of the GLS class; the other derives the optimal matrix solution of the covariance matrix function. One important consequence is that the Generalized Aitken Theorem is a corollary of the two theorems.

2. THEOREMS.

THEOREM 1:

Let

(1) $$\underset{n\times 1}{y} = \underset{n\times k}{X}\ \underset{k\times 1}{\beta} + \underset{n\times 1}{u}, \qquad 0 < k \leq n$$

be the linear model with all the classical assumptions except $E(uu')=\Omega$, where Ω is an nxn p.s.d. matrix with $\rho(\Omega)\geq k$. Then all linear unbiased estimators can be expressed as a GLS-class estimator:

(2) $$b_g = b_g(\Sigma) = (X'\Sigma X)^{-1}X'\Sigma y,$$

where Σ is any nxn matrix with $\rho(\Sigma)\geq k$.

Proof:

Since $b_{ols}=(X'X)^{-1}X'y$ is an unbiased estimator of β in (1), any other linear unbiased estimator, b_{nols}, can be expressed as $b_{nols} = b_{nols}(A) = [(X'X)^{-1}X' + A]y$ where A can be any kxn non-null matrix such that $AX=0_{kxk}$ (Theil, 1971, p.120). Combining b_{ols} and b_{nols}, we can write any linear unbiased estimator as

(3) $$b_a = b_a(A) = [(X'X)^{-1}X' + A]y$$

where A, such that $AX=0_{kxk}$, now can be a null matrix.

Since $A=(X'X)^{-1}X'XA$, and also since $X'XAX=0_{kxk}$ directly follows from $AX=0_{kxk}$, we can rewrite b_a as:

$$\begin{aligned} b_a &= [(X'X)^{-1}X' + A]y \\ &= [(X'X)^{-1}X' + (X'X)^{-1}X'XA]y \\ &= (X'X)^{-1}X'[I_n + XA]y \\ &= (X'X + X'XAX)^{-1}X'[I_n + XA]y \\ &= [X'(I_n + XA)X]^{-1}X'(I_n + XA)y \\ &= (X'\Sigma X)^{-1}X'\Sigma y = b_g, \end{aligned}$$

where

$$(4) \qquad \Sigma \equiv I_n + XA.$$

Noting that $AX=0$, $\rho(\Sigma) \geq k$ easily follows from post-multiplying (4) by X: $\Sigma X = X + XAX = X$ so that $\rho(\Sigma X)=\rho(X)$; since $\rho(X)=k$ and $\rho(\Sigma X)=\min(\rho(\Sigma),\rho(X))$, $\min(\rho(\Sigma), k)=k$ which immediately implies that $\rho(\Sigma) \geq k$.

Let S_g a set of linear unbiased estimators in the form of b_g, and S_a the set of all linear unbiased estimators in the form of b_a. Since any linear unbiased estimators can be expressed in the form of b_a, S_a is the complete set of all linear unbiased estimators; therfore

$$S_g \subset S_a.$$

However, since any linear unbiased estimator in the form of b_a can be transformed into b_g, as shown above,

$$S_g \supset S_a.$$

Therefore

$$S_g = S_a.$$

Q.E.D.

Then, $Var(b_g) = (X'\Sigma X)^{-1}X'\Sigma\Omega\Sigma'X(X'\Sigma X)^{-1}$ can be optimized with respect to the variable matrix Σ, which is discussed as a separate theorem which may well be useful in some other contexts in econometric theory:

THEOREM 2:

Define $V = V(\Sigma) = (X'\Sigma X)^{-1}X'\Sigma\Omega\Sigma'X(X'\Sigma X)^{-1}$ where Σ, X, and Ω are all as defined in Theorem 1. Then, under $dV(\Sigma) = 0_{kxk}$, and $d^2V(\Sigma) \geq 0_{kxk}$ (i.e., p.s.d.); and $\Sigma=\Omega^+$, with Ω^+ denoting the Moor-Penrose inverse, is a unique solution for Σ of the first derivative.

Proof:

Let the differential operator d operate only on the immediately following matrix. Then, using $d(AB)=dAB+AdB$ and $dA^{-1} = -A^{-1}dAA^{-1}$ (Graham, 1980, p.79),

(7) $$dV = Q^{-1}X'\phi XQ'^{-1} + Q^{-1}X'\phi'XQ'^{-1}$$
$$= Q^{-1}[X'\phi X + (X'\phi X)']Q'^{-1}.$$

with the matrices defined as:

(8) $$Q = Q(\Sigma) = X'\Sigma X$$
$$\phi = \phi(\Sigma) = d\Sigma M\Omega\Sigma'.$$
$$M = M(\Sigma) = I - XQ^{-1}X'\Sigma.$$

For V in (7), $dV = 0_{kxk}$ obviously requires

(9) $$X'\phi X = 0_{kxk}.$$

Differentiating dV in (7) with respect to Σ,

(10) $$d^2V = d(dV)$$

$$= dQ^{-1}[X'\phi X + (X'\phi X)']Q'^{-1}$$

$$+ Q^{-1}[X'd\phi X + (X'd\phi X)']Q'^{-1}$$

$$+ Q^{-1}[X'\phi X + (X'\phi X)']dQ'^{-1}.$$

Substituting $d\phi = -d\Sigma XQ^{-1}X'\phi + d\Sigma M\Omega d\Sigma'$ (Appendix A) into (10), then reflecting (9), we can show that $d^2V(\Sigma)$ is p.s.d.:

(11) $$d^2V = Q^{-1}X'd\Sigma[M\Omega + \Omega'M']d\Sigma'XQ'^{-1}$$

$$= (Q^{-1}X'd\Sigma)(M\Omega + \Omega'M')(Q^{-1}X'd\Sigma)'$$

$$\geq 0_{kxk},$$

noting that both M, idempotent, and Ω are p.s.d. for all Σ.

Substituting ϕ defined in (8) into (9),

(12) $$X'd\Sigma M\Omega\Sigma'X = 0_{kxk}.$$

Since $d\Sigma$ is a non-null arbitrary matrix, it can not be that $X'd\Sigma \neq 0_{kxn}$. Hence, (12) requires

(13) $$d\Sigma M\Omega\Sigma'X = 0_{nxk}.$$

Substituting $d\Sigma = d(\Sigma G)\Sigma + \Sigma G d\Sigma$, G denoting a generalized inverse of Σ, into (13) yields (Appendix B)

(14) $\quad d(\Sigma G)\dot{M}\Sigma\Omega\Sigma'X = 0_{nxk}$

where $\dot{M} \equiv I - \Sigma X Q^{-1} X'$.

Since $\dot{M}\Sigma X = (I - \Sigma X(X'\Sigma X)^{-1}X')\Sigma X = 0_{nxk}$, (14) is satisfied if $\Sigma\Omega\Sigma'=\Sigma$ of which $\Sigma=\Omega^{+}$ is a unique solution for Σ (Appendix C).

Q.E.D.

GENERALIZED AITKEN THEOREM AS A COROLLARY:

For $c'b_g$ a linear combination of b_g in Theorem I where c is an arbitray kx1 column vector, we can easily show that

(15) $\quad q \equiv var(c'b_g) = c'V(\Sigma)c.$

Then, the first- and second-order conditions for minimization of q will be

$$(16) \qquad dq = c' dV(\Sigma) c = 0, \text{ and}$$

$$d^2q = c' d^2V(\Sigma) c \geq 0.$$

Both the first- and second-order conditions require $dV(\Sigma) = 0_{kxk}$ and $d^2V(\Sigma) \geq 0_{kxk}$, respectively, which in turn are satisfied uniquely by $\Sigma = \Omega^+$ as established in Theorem 2.

Substituting $\Sigma = \Omega^+$ into b_a in (2) yields

$$(17) \qquad b_{gls} = (X'\Omega^+X)^{-1}X'\Omega^+y$$

as the BLUE of β in (1), which proves the Generalized Aitken Theorem (Aitken, 1935; Theil, 1971, pp.278-280).

APPENDIX A:

Differenting M defined in (8) with respect to Σ,

(A.1) $$dM = -XdQ^{-1}X'\Sigma - XQ^{-1}X'd\Sigma$$

$$= XQ^{-1}X'd\Sigma XQ^{-1}X'\Sigma - XQ^{-1}X'd\Sigma$$

$$= -XQ^{-1}X'd\Sigma M.$$

Differentiating defined in (8), using (A.1), with respect to Σ,

(A.2) $$d\phi = d(d\Sigma M\Omega\Sigma')$$

$$= d^2\Sigma M\Omega\Sigma' + d\Sigma dM\Omega\Sigma' + d\Sigma M\Omega d\Sigma'$$

$$= -d\Sigma(XQ^{-1}X'd\Sigma M)\Omega\Sigma' + d\Sigma M\Omega d\Sigma'$$

$$= -d\Sigma XQ^{-1}X'\phi + d\Sigma M\Omega d\Sigma',$$

noting that $d^2\Sigma = 0_{nxn}$ since $d\Sigma$ is arbitrary.

APPENDIX B:

Denoting a generalized inverse of Σ by G, we can express Σ as $\Sigma = \Sigma G \Sigma$. Then, $d\Sigma$ easily follows:

$$(B.1) \qquad d\Sigma = d(\Sigma G \Sigma)$$
$$= d((\Sigma G)\Sigma)$$
$$= d(\Sigma G)\Sigma + \Sigma G d\Sigma.$$

Substituting (B.1) into (13),

$$(B.2) \qquad d\Sigma M\Omega\Sigma' X = [d(\Sigma G)\Sigma + \Sigma G d\Sigma] M\Omega\Sigma' X$$
$$= d(\Sigma G)\Sigma M\Omega\Sigma' X + \Sigma G(d\Sigma M\Omega\Sigma' X)$$
$$= d(\Sigma G)\dot{M}\Sigma\Omega\Sigma' X$$
$$= 0_{nxk}$$

noting that the third equality is based on $\Sigma M = \dot{M}\Sigma$ in which M and $\dot{M}$ are as defined in (8) and (14), respectively, and also equation (13) itself.

APPENDIX C:

Since Ω is p.s.d., so is $\Sigma\Omega\Sigma'=\Sigma$. Therefore, Σ and Σ^+ can be decomposed (Theil, 1971, p. 276) as

$$(C.1) \qquad \Sigma = F\Lambda F';$$

$$\Sigma^+ = FD^{-1}F'$$

where F is a nxr full-column matrix whose columns consisting of eigenvectors corresponding to the non-zero eigenvalues of Σ, so that $F'F=I_r$ where $r=\rho(\Sigma)\geq k$; and D is a rxr diagonal matrix consisting of the r non-zero eigenvalues of Ω. Hence, substituting (C.1) into $\Sigma\Omega\Sigma'=\Sigma$,

$$(C.2) \qquad FDF'\Omega FDF'=FDF'.$$

Pre- and post-multiplying $FD^{-1}F'$ to (C.2),

$$(C.3) \qquad \Omega = FD^{-1}F'=\Sigma^+;$$

subsequently

$$\Sigma = (\Sigma^{+})^{+} = \Omega^{+}. \tag{C.4}$$

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