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A Note on the Optimal Design of an Office Building

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Abstract. **This study examines the economics of the optimal footprint area, atrium area and height of an office building. We extend the work of Doiron, Shilling and Sirmans (1992) by incorporating realistic revenue and cost functions and reverting to the sufficient conditions of optimality.**

Introduction

This note develops a model for selecting key parameters of office building design. Doiron, Shilling and Sirmans (1992) (DSS) present a microeconomic model of optimal footprint area, atrium area and building height and investigate the results of the analysis empirically. DSS contains several peculiarities that this constructive note aims to correct. In order to optimally design an office building we draw upon (i) the research of Gat (1995), who demonstrates the convex nature of the cost function of a developer, (ii) the research of Brennan, Cannaday and Colwell (1984), who estimate a rent function, and (iii) the sufficiency conditions of an optimum (Chiang, 1984).

This study is organized as follows: section two identifies peculiarities in the optimal design of the model of DSS; the third section offers an alternative model along with a numerical example, while section four concludes this note.

Peculiarities in the DSS Model

The analysis of DSS is flawed in that it fails to incorporate realistic cost and revenue functions. Furthermore, the sufficient condition DSS impose on their rent function may lead to a silly solution.

First, the cost function selected by DSS is unreasonable. This is due to the fact that they assume:

(i) a fixed unit cost (p) of the overall space (qh) instead of usable space $(q-a)h$, where q is the footprint area, h is the height and a is the atrium area (i.e., development costs are unaffected by the size of the atrium). Thus, they are rebuilding the floor of the atrium (a) at each floor of the building (i.e., their concept of an atrium must be an open public space on each floor rather than a space that is open through all the floors).

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(ii) a flat cost per unit of footprint (q) instead of a concave function arising due to constructing the outside walls of the building. DSS implicitly assume the impossible: that wall length is proportional to floor area.
(iii) total cost as being a constant proportion of height rather than a convex function as could be inferred from the research of Gat (1995).[1]

Second, DSS assume that the tenants' bid rent curve (r = F((q-a)h,ah)) satisfies the following sufficient conditions:

$$F_q \equiv \frac{\partial F}{\partial q} < 0, F_a \equiv \frac{\partial F}{\partial a} > 0, F_h \equiv \frac{\partial F}{\partial h} < 0$$
$$F_{qq} \equiv \frac{\partial^2 F}{\partial q^2} > 0, F_{aa} \equiv \frac{\partial^2 F}{\partial a^2} < 0, and\ F_{hh} \equiv \frac{\partial^2 F}{\partial h^2} > 0,$$

where:

r = rent per unit of area,
q = area per floor of the building,
a = atrium area per floor, and
h = number of building stories.

This technical assumption is defective as it does not necessarily lead to a *unique* optimum. For example, if we assume the following:

(a) the concave rent function,

$$r = F((q-a)h, ah) = \frac{p_x\left[((q-a)h)^{\alpha} = (ah)^{1-\alpha}\right]}{(q-a)h},$$

which has all the attributes DSS specify;
(b) the price of the tenant's output, p_x=\$100/unit;
(c) the cost of the developer per square foot, p=\$50/square foot; and
(d) α =0.8,

then the profit function (π_D) of the developer is evaluated as:

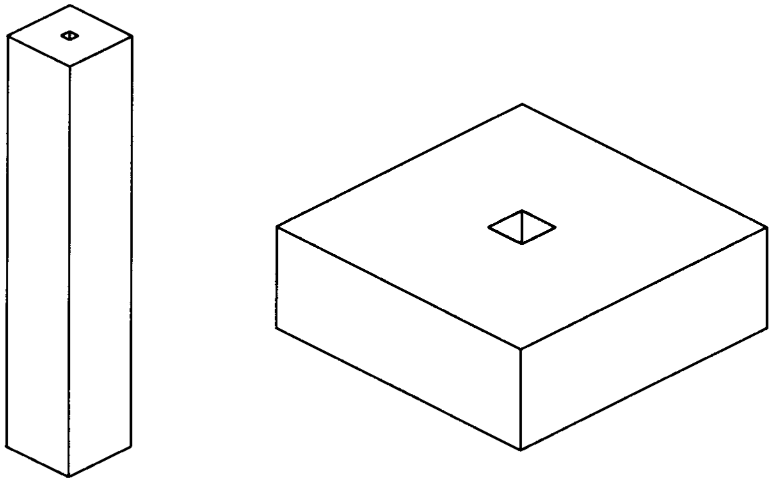
$$\pi_D = R(q,a,h) - C(q,h)$$
$$= F(q,a,h)\left[(q-a)h\right] - C(q,h).$$

This would be maximized by the following optimal solutions:

$$q* = 10.8039/h$$
$$\text{and } a* = 0.318108/h.$$

The optimal profit function equals \$194.69 irrespective of the height of the building. This is a very silly solution as the optimal solution implies that one can construct a squat building or a pencil-thin building as long as $q*/a* \approx 34$. Two such optimal design solutions are illustrated in Exhibit 1.

Exhibit 1



It is possible to avoid the above problematic assumptions of the mathematical characteristics of the bid rent curve and the cost function (imposed by DSS) if the profit function evaluated is a concave function with respect to q, a and h. At the optimum, the first-order necessary conditions (FONCs) need to be satisfied. That is, the partial derivative of the profit with respect to each of the endogenous variables (q,a,h) is zero. This implies that marginal revenue equals marginal cost with respect to each of the endogenous variables. In addition, there must be concavity (i.e., the second-order sufficient condition requires a negative definite Hessian matrix or negative eigenvalues of the Hessian matrix) (Chiang, 1984).

An Alternative Model

The first step here is to model the rent function appropriately. We draw upon the work of Brennan et al. (1984), who study office rent in the Chicago CBD. They determine that rental revenue function could be explained by the area included in a particular lease transaction and vertical height of the structure among other variables. The area paid for but not usable by tenants (similar to an atrium space) is also found to be significant. Other studies that support the finding that height and floor area (i.e., size) of office buildings have significant impact on its value include Mills (1992) and Shilton and Zaccaria (1994). One concave function that thus comes to mind is:

$$R(q,a,h) = p_x (q-a)^{a_1} h^{a_3} a^{a_2} \quad \text{where } a_i > \forall i,\text{[2]} \tag{1}$$

where:

p_x = price of the tenant's output x (see DSS);
$(q-a)$ = usable area per floor;
a = atrium area; and
h = height of the structure.

If all these defects in the DSS cost function are corrected, the cost function for a square building with a square atrium could look like the following:

$$C(q,a,h)=\left[p(q-a)+4\delta_1\sqrt{q}+4\delta_2\sqrt{a}\right]\left[h+\gamma h^2\right], \tag{2}$$

where $p(1=\gamma)$, $\delta_1(1-\gamma)$ and $\delta_2(1=\gamma)$ are the first-floor (i.e., h=1) unit-in-place cost of usable area (q–a), unit-in-place cost of first-floor exterior wall length ($4\sqrt{q}$), and unit-in-place cost of first floor walls of the atrium ($4\sqrt{a}$), respectively. The coefficient 'γ' imparts the convexity to the cost function with respect to the height of the structure. From (1) and (2), the profit function simplifies to:

$$\begin{aligned}\pi_D &= p_x*(q-a)^{\alpha_1}*h^{\alpha_3}*a^{\alpha_2}\\ &-\left[p(q-a)+4\delta_1\sqrt{q}+4\delta_2\sqrt{a}\right]\left[h+\gamma h^2\right].\end{aligned} \tag{3}$$

For this function to be concave w.r.t. q, a, h, we need $\alpha_1 \in (0,1)$, $\alpha_2 \in (0,0.5)$, and $\alpha_3 \in (0,2)$, respectively.

The FONCs for an optimum require:

$$\begin{aligned}
\partial\pi_D/\partial q &= p_x\alpha_1(q-a)^{\alpha_1-1}h^{\alpha_3}a^{\alpha_2}-\left[p+2\delta_1 q^{-1/2}\right]\left[h+\gamma h^2\right]=0\\
\partial\pi_d/\partial a &= p_x\left[-\alpha_1(q-a)^{\alpha_1-1}h^{\alpha_3}a^{\alpha_2}+\alpha_2(q-a)^{\alpha_1}h^{\alpha_3}a^{\alpha_2-1}\right]\\
&\quad-\left[-p+2\delta_2 a^{-1/2}\right]\left[h+\gamma h^2\right]=0\\
\partial\pi_d/\partial h &= p_x\left[(q-a)^{\alpha_1}a^{\alpha 2}\alpha_3 h^{\alpha_3-1}\right]\\
&\quad-\left[p(q-a)+4\delta_1\sqrt{q}+4\delta_2\sqrt{a}\right]\left[1+2\gamma h\right]=0
\end{aligned} \tag{4}$$

To solve for the optimal dimensional values of a building, we selected the following numerical parameters: p_x=\$50/sq.ft., p=\$25/sq.ft., α_1=0.93654, α_2=0.00044, α_3=1.06011, δ_1=1, δ_2=1.25, and γ=0.01.

The optimal parameters of the building design are solved from the FONCs as:

q^*=40,163.3 sq.ft., a^*=10,135.3 sq.ft., $h^*\approx$15 stories,

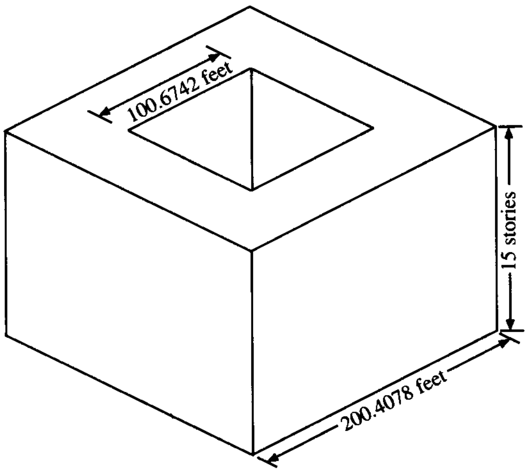
and the optimal rent per square foot is \$30.71. Exhibit 2 illustrates this optimal design solution.

The second-order conditions are investigated as follows:

(a) the Hessian matrix (**H**) is derived as:[3]

$$\begin{bmatrix} -0.000909561 & 0.000930432 & -2.02249\\ 0.000930432 & -0.00098725 & 2.01968\\ -2.02249 & 2.01968 & -11122.4\end{bmatrix}$$

Exhibit 2



It is negative semi-definite as:

$$\text{Det}[H1] - h_{11} = -0.000909561 < 0,$$

$$\text{Det}[H2] = \begin{vmatrix} -0.000909561 & 0.000930432 \\ 0.000930432 & -0.00098725 \end{vmatrix} = 0.000000032 > 0, \text{ and}$$

$$\text{Det}[H3] = \text{Det}[H] = -0.00021152463963 82955 < 0.$$

(b) the eigenvalues of the above Hessian are confirmed to be negative as they are:

$$-11122.4$$
$$-0.0011457$$
$$-0.0000165994.$$

Thus, the second-order conditions for a maximum are verified.

Summary

The focus of this constructive note is to correct the errors in the design of office buildings made by Doiron et al. in their 1992 study. If one avoids their errors in selection of a rent function, incorporates a realistic cost function and follows standard optimization principles, then one can model the major parameters in the design of office buildings. The parameters chosen in the alternative model are roughly realistic and the resultant optimal design is within the realm of possibility.

Notes

[1]Gat (1995) postulates a convex cost function of the structure. He quotes that "In the construction of a building on a given site, as more floor space is added to the site, vertical stacking of floors takes place and causes marginal cost to rise." The DSS cost function, *pqh*, is neither concave nor convex.
[2]The domain of α_i's is determined by requiring that the profit function (π_D) be concave as described below in the text.
[3]This can be generalized by writing the Hessian in symbols. Unfortunately, it is very lengthy and cumbersome. A copy of it can be obtained by writing to the authors.

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