A Note on Optimal Portfolio Selection and Diversification Benefits with a Short Sale Restriction on Real Estate Assets

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Abstract. **This paper develops an optimal portfolio selection technique when short sales on real estate assets are restricted. Using the well-known mean-variance efficient concept, we are able to derive the optimal weights for portfolios consisting of both financial assets and real estate assets. Our paper provides a simple but powerful tool for portfolio managers to correctly construct mean-variance portfolios under short sale constraints.**

Introduction

The diversification benefits of including real estate assets in portfolios have been a subject of recent studies. Hartzell et al. [4], [5] examine diversification benefits and diversification categories of real estate portfolios (portfolios consisting of only real estate assets). Webb et al. [9] extend the literature by examining the diversification benefits of including real estate assets in mixed-asset portfolios (portfolios consisting of both financial assets and real estate assets). These studies suggest that real estate assets provide diversification benefits when included in portfolios.

While the concept of including real estate assets into a portfolio, based on asset correlations, is important to portfolio managers, there are still numerous questions to be addressed. One of the most important issues is how to construct portfolios that consist of both financial and real assets using the mean-variance efficient concept.[1] So far, only a few studies have addressed this issue. However, these studies neglect the constraint on the short sales of real estate assets. When short sales of real estate assets are not allowed, the diversification benefits provided by real estate assets will be reduced.[2] In addition, portfolio managers may not be able to correctly construct mean-variance portfolios using the traditional portfolio construction technique, which permits short sales on all types of assets.

The reason why the short sale constraint has been largely ignored in the past is, in part, due to the difficulty in solving constrained nonlinear programming.[3] However, as pointed out by Lusht [6], the constraint on short sales is an important characteristic of real estate assets and it should not be neglected by researchers. Elton and Gruber [2, p. 53] also point out that most institutional investors do not short sell. Apparently, there is a need to address the short sale constraint in the portfolio selection process. In this

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paper we develop an easy-to-use optimal weight (mean-variance) selection technique for portfolios consisting of both financial assets and real estate assets. In the optimal portfolio, the weights for real estate assets are restricted to be non-negative. Section two derives the optimal portfolio selection technique and section three presents a numerical example. The last section contains the conclusions.

Model Development

To develop the model, we assume that there are two types of investment assets in a portfolio: financial assets and real estate assets. Short sales are allowed for financial assets but are restricted for real estate assets. The prices on financial assets and real estate assets are denoted by $\mathbf{P}_s$ and $\mathbf{P}_r$, respectively.[4] The investor's problem is how to allocate his wealth (W) over the available assets. Let $\mathbf{X}_{is}\mathbf{P}_{is}$ and $\mathbf{X}_{jr}\mathbf{P}_{jr}$ be, respectively, the amount of the investor's wealth (W) invested in a financial asset i and a real estate asset j. The budget constraint may be written as

$$W = \mathbf{X}_s\mathbf{P}_s + \mathbf{X}^T{}_r\,\mathbf{P}_r \text{ or}$$

$$1 = \mathbf{x}_s\mathbf{e}_s + \mathbf{x}^T{}_r\,\mathbf{e}_r, \tag{1}$$

where $\mathbf{x}_s$ and $\mathbf{x}_r$ are the vectors of the percentage of wealth invested in financial assets and real estate assets, respectively. The ith element in $\mathbf{x}_s(\mathbf{x}_r)$ is determined by the product of the ith element of $\mathbf{X}_s(\mathbf{X}_r)$ and $\mathbf{P}_s(\mathbf{P}_r)$, divided by wealth ($W$). The vectors, $\mathbf{e}_s$ and $\mathbf{e}_r$, are vectors with one in each entry. The vectors of the expected returns on financial assets and real estate assets are denoted by $\mathbf{R}_s$ and $\mathbf{R}_r$, respectively. The value of the investor's portfolio at the terminal period is stochastic, and can be written as

$$\begin{aligned} \tilde{\mathbf{R}}_x &= \mathbf{x}^T{}_s(\mathbf{e}_s + \tilde{\mathbf{R}}_s) + \mathbf{x}^T{}_r(\mathbf{e}_r + \tilde{\mathbf{R}}_r) \\ &= 1 + \mathbf{x}^T{}_s\tilde{\mathbf{R}}_s + \mathbf{x}^T{}_r\,\tilde{\mathbf{R}}_r \\ &= 1 + \mathbf{x}^T\tilde{\mathbf{R}}. \end{aligned} \tag{2}$$

The expected return is denoted by $\mathbf{R}^T$, where $\tilde{\mathbf{R}}^T = (\tilde{\mathbf{R}}^T{}_s, \tilde{\mathbf{R}}^T{}_r)$. The portfolio's expected return, R_x, and its variance, $\sigma^2{}_x$, are determined by:

$$R_x = 1 + \mathbf{x}^T\,\mathbf{R} \tag{3}$$

$$\sigma^2{}_x = \mathbf{x}^T\,\mathbf{V}\mathbf{x}, \tag{4}$$

where $\mathbf{x}^T = (\mathbf{x}^T{}_s, \mathbf{x}^T{}_r)$ and $\mathbf{V}$ is the variance-covariance matrix of financial assets and real estate assets. Short sale restrictions can be expressed by restricting the shares demanded for real estate assets to be non-negative. Under this constraint, the mean-variance efficient portfolio problem can be formulated as a constrained optimization problem that minimizes the portfolio risk subject to the expected terminal return and budget constraints, or[5]

$$\text{Min } \mathbf{x}^T \mathbf{V}\mathbf{x}/2, \tag{5}$$

subject to

$$\mathbf{x}^T \mathbf{R} = \mu, \tag{6}$$

$$\mathbf{x}^T \mathbf{e} = 1,$$

$$\mathbf{x}_r \geq 0. \tag{7}$$

Applying the Lagrangian multipliers κ, γ and $\mathbf{\Gamma}_r$ to the budget constraint (1), the expected return constraint (6), and the short sale constraint (7), respectively, the Lagrangian is given by:

$$L = \mathbf{x}^T \mathbf{V}\mathbf{x}/2 + \kappa(1 - \mathbf{x}^T\mathbf{e}) + \gamma(\mu - \mathbf{x}^T \mathbf{R}) - \mathbf{\Gamma}^T{}_r(\mathbf{x}_r). \tag{8}$$

The differentiation of (8) gives the first-order conditions:

$$\mathbf{V}\mathbf{x} = \kappa\mathbf{e} + \gamma\mathbf{R} + \mathbf{\Gamma}, \tag{9}$$

$$\mathbf{x}^T\mathbf{R} = \mu,$$

$$\mathbf{x}^T\mathbf{e} = 1,$$

$$\mathbf{\Gamma}_{ir}\mathbf{x}_{ir} = 0 \text{ for real estate asset } i, \tag{10}$$

where $\mathbf{\Gamma}^T = (0, \mathbf{\Gamma}^T{}_r)$. Equation (10) is the Kuhn-Tucker condition, which ensures that the weights of real estate assets in a portfolio are non-negative. Since Lagrangian (8) is a quadratic function, equations (9) and (10) represent both the necessary and sufficient conditions for the optimal portfolio decision. If the variance-covariance matrix ($\mathbf{V}$) is invertible, the optimal mean-variance portfolio when short sales are not allowed for real estate assets is determined by

$$\mathbf{x} = \kappa\mathbf{V}^{-1}\mathbf{e} + \gamma\mathbf{V}^{-1}\mathbf{R} + \mathbf{V}^{-1}\mathbf{\Gamma}. \tag{11}$$

The first two terms on the right-hand side of equation (11) represent the demand for financial assets and real estate assets when short sales are allowed. The effect of short sale restrictions on the demand for financial assets and real estate assets is determined by the last term of equation (11). This term is a net zero investment vector that ensures the investment in real estate assets is non-negative.

The procedure to obtain the optimal portfolio solution can be solved in two steps. The first step solves κ and γ in terms of $\mathbf{\Gamma}$. The second step applies equation (11) and the Kuhn-Tucker conditions to solve for $\mathbf{x}$ and $\mathbf{\Gamma}$. Specifically, κ and γ can be determined in terms of Γ from equations (1), (6) and (9):

$$\kappa=[(C-B\mu)-(C\mathbf{e}^{T}-B\mathbf{R}^{T})\mathbf{V}^{-1}\mathbf{\Gamma}]/\mathbf{\Delta} \tag{12}$$

$$\gamma=[(A\mu-B)-(A\mathbf{R}^{T}-B\mathbf{e}^{T})\mathbf{V}^{-1}\mathbf{\Gamma}]/\mathbf{\Delta}, \tag{13}$$

where

$A=\mathbf{e}^{T}\mathbf{V}^{-1}\mathbf{e}>0,$

$B=\mathbf{e}^{T}\mathbf{V}^{-1}\mathbf{R},$

$C=\mathbf{R}^{T}\mathbf{V}^{-1}\mathbf{R}>0,$ and

$\mathbf{\Delta}=AC-B^{2}>0.$

Substituting (12) and (13) into (11) yields:

$$\begin{aligned}\mathbf{x}&=[(C-B\mu)-(C\mathbf{e}^{T}-B\mathbf{R}^{T})\mathbf{V}^{-1}\mathbf{\Gamma}]\mathbf{V}^{-1}\mathbf{e}/\mathbf{\Delta}+[(A\mu-B)-(A\mathbf{R}^{T}-B\mathbf{e}^{T})\mathbf{V}^{-1}\mathbf{\Gamma}]\mathbf{V}^{-1}\mathbf{R}/\mathbf{\Delta}+\mathbf{V}^{-1}\mathbf{\Gamma}\\&=[(C-B\mu)\mathbf{V}^{-1}\mathbf{e}+(A\mu-B)\mathbf{V}^{-1}\mathbf{R}]/\mathbf{\Delta}-[\mathbf{V}^{-1}(C\mathbf{e}\mathbf{e}^{T}+A\mathbf{R}\mathbf{R}^{T}-B\mathbf{e}\mathbf{R}^{T}-B\mathbf{R}\mathbf{e}^{T})/\mathbf{\Delta}-\mathbf{I}]\mathbf{V}^{-1}\mathbf{\Gamma}\\&=\mathbf{y}(\mu)+\mathbf{\Omega}\mathbf{\Gamma},\end{aligned} \tag{14}$$

where **I** is the identity matrix with one on the diagonal and zero on the off-diagonal.

$$\mathbf{y}(\mu)=(C-B\mu)\mathbf{V}^{-1}\mathbf{e}+(A\mu-B)\mathbf{V}^{-1}\mathbf{R}]/\mathbf{\Delta},\text{ and }\mathbf{\Omega}=-[\mathbf{V}^{-1}(C\mathbf{e}\mathbf{e}^{T}+A\mathbf{R}\mathbf{R}^{T}-B\mathbf{e}\mathbf{R}^{T}-B\mathbf{R}\mathbf{e}^{T})/\mathbf{\Delta}-\mathbf{I}]\mathbf{V}^{-1}.$$

$\mathbf{\Omega}$ must be a symmetric matrix that satisfies $\mathbf{e}^{T}\mathbf{\Omega}=0$. When μ (the mean return of a portfolio) is given, the optimal portfolio solution can be solved easily using the linear programming technique (maximizing $\mathbf{x}\mathbf{R}^{T}$ subject to equations 7 and 14).

A Numerical Example

To demonstrate the technique, we construct an example using data reported in Table 1 and Table 3 of Webb et al.'s study [9]. Webb's data set includes ten different types of assets (six financial assets and four real estate assets). To simplify the presentation (but without loss of generality), we only include five assets (three financial assets and two real estate assets) in our portfolio. The three financial assets are the NYSE stock index, the OTC stock index, and Treasury bills. The two real estate assets included are the Farmland Index and the WAC Index.[6]

Using the variance and correlation matrix reported by Webb et al. [9], we construct the variance-covariance matrix of these five asset types. The variance-covariance matrix and the mean returns (from Table 1 of Webb et al.) of these five asset types are reported in Panel A of Exhibit 1. Panel B of Exhibit 1 reports the inverse matrix of the variance-covariance matrix.

Exhibit 1

	NYSE Stock (NS)	OTC Stock (OC)	Treasury Bills (TB)	Farmland Index (FL)	WAC Index (WC)	Mean Return
Panel A: Variance-covariance matrix (V)						
NS	410.87	444.05	−2.55	−60.90	−8.20	11.22
OC	444.05	531.76	−.73	−86.60	13.33	13.47
TB	−2.55	−.73	9.92	−17.24	14.57	8.14
FL	−60.90	−86.60	−17.24	115.13	−22.33	14.49
WC	−8.20	13.33	14.57	−22.33	33.41	11.80
Panel B: Inverse matrix (V^{-1}):						
NS	.0781	−.0713	−.1904	−.0178	.1187	
OC	−.0713	.0676	.1838	.0189	−.1120	
TB	−.1904	.1838	.8370	.0791	−.4322	
FL	−.0178	.0189	.0791	.0187	−.0339	
WC	.1187	−.1120	−.4322	−.0339	.2695	

*See Tables 1 and 3 of Webb et al. [9], from which data are derived.

The optimal solution of mean-variance portfolios can be obtained in two steps. The first step solves for $\mathbf{y}(\mu)$ and $\mathbf{\Omega}$. From equation (14), we know that $\mathbf{y}(\mu)=[(C-B\mu)\mathbf{V}^{-1}\mathbf{e}+(A\mu-B)\mathbf{V}^{-1}\mathbf{R}]/\Delta$, and $\mathbf{\Omega}=[\mathbf{V}^{-1}(C\mathbf{e}\mathbf{e}^{T}+A\mathbf{R}\mathbf{R}^{T}-B\mathbf{e}\mathbf{R}^{T}-B\mathbf{R}\mathbf{e}^{T})/\Delta-\mathbf{I}]\mathbf{V}^{-1}$. The notations, A, B, C and Δ are defined in equations (12) and (13). Panel A of Exhibit 2 reports the values of A, B, C and Δ using the variance-covariance matrix reported in Exhibit 1. For convenience, we set μ to be equal to a random number, 9.7491%.

Panel B of Exhibit 2 first reports the values of the $(C-B\mu)\mathbf{V}^{-1}\mathbf{e}/\Delta$ column vector and the $(A\mu-B)\mathbf{V}^{-1}\mathbf{R}/\Delta$ column vector. The summation of these two column vectors is the $\mathbf{y}(\mu)$ column vector, which is reported in the last column of Panel B. It should be noted

Exhibit 2
Solution for $\mathbf{y}(\mu)$ when $\mu=9.7491$

Panel A: Solving for A, B, C, Δ using $\mathbf{V}$ and $\mathbf{V}^{-1}$

$$A=\mathbf{e}^{T}\mathbf{V}^{-1}\mathbf{e} = .3568374$$
$$B=\mathbf{e}^{T}\mathbf{V}^{-1}\mathbf{R} = 2.8316268$$
$$C=\mathbf{R}^{T}\mathbf{V}^{-1}\mathbf{R} = 24.4537220$$
$$\Delta=AC-B^{2} = .7078929$$

Panel B: Solving for $\mathbf{y}(\mu)$

$$\mathbf{y}(\mu)^{*}=[(C-B\mu)\mathbf{V}^{-1}\mathbf{e}+(A\mu-B)\mathbf{V}^{-1}\mathbf{R}]/\Delta$$

$$= \begin{pmatrix} .368700 \\ -.387500 \\ -2.125898 \\ -.289515 \\ .845194 \end{pmatrix} + \begin{pmatrix} -.450393 \\ .511179 \\ 2.925906 \\ .521215 \\ -.918886 \end{pmatrix} = \begin{pmatrix} -.081690 \\ .123679 \\ .800007 \\ .231699 \\ -.073690 \end{pmatrix}$$

*$\mathbf{Y}(\mu)$, $(C-B\mu)\mathbf{V}^{-1}\mathbf{e}$, and $(A\mu-B)\mathbf{V}^{-1}\mathbf{R}$ are column vectors.

that the last element (element 5) of the $\mathbf{y}(\mu)$ column vector is negative. This negative number indicates that, when short sales on real estate assets are allowed, the optimal portfolio requires the short selling of the real estate WAC Index. This is why the $\boldsymbol{\Omega}$ matrix is important in this optimization process; it ensures that there are no negative weights for real estate assets.

To construct the $\boldsymbol{\Omega}$ matrix, we first estimate values for the $C\mathbf{ee}^T$, $A\mathbf{RR}^T$, $B\mathbf{eR}^T$, $B\mathbf{Re}^T$ matrixes. These four matrixes are reported in Panels A, B, C, and D of Exhibit 3. The $\boldsymbol{\Omega}$ matrix equals $-[\mathbf{V}^{-1}(C\mathbf{ee}^T+A\mathbf{RR}^T-B\mathbf{eR}^T-B\mathbf{Re}^T)/\boldsymbol{\Delta}-\mathbf{I}]\mathbf{V}^{-1}$ and must be a symmetric matrix that satisfies $\mathbf{e}^T\boldsymbol{\Omega}=0$ (see equation 14). Panel E reports the values of the $\boldsymbol{\Omega}$ matrix. Note that $\boldsymbol{\Omega}$ is a symmetric matrix that satisfies $\mathbf{e}^T\boldsymbol{\Omega}=0$.

The second step is a straightforward application of the standard linear programming technique. The optimal weights of the portfolio can be derived by maximizing the

Exhibit 3
Solution for $\boldsymbol{\Omega}$ matrix

	NS	OC	TB	FL	WC
Panel A: $C\mathbf{ee}^T$ matrix					
NS	24.5	24.5	24.5	24.5	24.5
OC	24.5	24.5	24.5	24.5	24.5
TB	24.5	24.5	24.5	24.5	24.5
FL	24.5	24.5	24.5	24.5	24.5
WC	24.5	24.5	24.5	24.5	24.5
Panel B: $A\mathbf{RR}^T$ matrix					
NS	44.9	53.9	32.6	58.0	47.2
OC	53.9	64.7	39.1	69.6	56.7
TB	32.6	39.1	23.6	42.1	34.3
FL	58.0	69.6	42.1	74.9	61.0
WC	47.2	56.7	34.3	61.0	49.7
Panel C: $B\mathbf{eR}^T$ matrix					
NS	31.8	38.1	23.0	41.0	33.4
OC	31.8	38.1	23.0	41.0	33.4
TB	31.8	38.1	23.0	41.0	33.4
FL	31.8	38.1	23.0	41.0	33.4
WC	31.8	38.1	23.0	41.0	33.4
Panel D: $B\mathbf{Re}^T$ matrix					
NS	31.8	31.8	31.8	31.8	31.8
OC	38.1	38.1	38.1	38.1	38.1
TB	23.0	23.0	23.0	23.0	23.0
FL	41.0	41.0	41.0	41.0	41.0
WC	33.4	33.4	33.4	33.4	33.4
Panel E: $\boldsymbol{\Omega}$ matrix*					
NS	.0452	−.0403	−.0309	−.0073	.0331
OC	−.0403	.0377	.0285	.0066	−.0325
TB	−.0309	.0285	.0239	.0082	.0296
FL	−.0073	.0066	.0082	.0054	−.0130
WC	.0331	−.0325	−.0296	−.0130	.0420

* $\boldsymbol{\Omega}=-[\mathbf{V}^{-1}(C\mathbf{ee}^T+A\mathbf{RR}^T-B\mathbf{eR}^T-B\mathbf{Re}^T)/\boldsymbol{\Delta}-\mathbf{I}]\mathbf{V}^{-1}$.

Exhibit 4
Solving for Optimal Portfolio Weights Using Linear Programming

Panel A: Objective function and constraints

Max* $\mu = 11.22x_1 + 13.47x_2 + 8.14x_3 + 14.49x_4 + 11.8x_5$

subject to**

$$
\begin{aligned}
x_1 \quad & + .0073\Gamma_4 \quad - .0331\Gamma_5 = -.081690 \\
x_2 \quad & - .0066\Gamma_4 \quad + .0325\Gamma_5 = .123679 \\
x_3 \quad & - .0082\Gamma_4 \quad + .0296\Gamma_5 = .800007 \\
x_4 \quad & - .0054\Gamma_4 \quad + .0130\Gamma_5 = .231699 \\
x_5 \quad & + .0130\Gamma_4 \quad - .0420\Gamma_5 = -.073690
\end{aligned}
$$

x_4 and $\Gamma_4 \geq 0$***
x_5 and $\Gamma_5 \geq 0$***

Panel B: Solution using linear programming

$x_1 = -.02361526$
$x_2 = .06665697$
$x_3 = .74807304$
$x_4 = .20889022$
$x_5 = 0.0$
$\Gamma_4 = 0.0$
$\Gamma_5 = 1.75452390$
$\mu = 9.7491$

*The x_1, x_2, x_3, x_4 and x_5 in the objective function are the optimal weights for the five asset types. It should be noted that x_4 and x_5 are the weights for the two real estate assets. The coefficients in the objective function are the mean returns of the asset types (see that last column vector in Panel A of Exhibit 1).

**Short sale restrictions are applied only to real estate assets (x_4 and x_5 must be non-negative). Since there is no restriction on the short sale of financial assets (x_1, x_2 and x_3 can be negative), Γ_1, Γ_2 and Γ_3 take the zero values. The coefficients of Γ_4 and Γ_5 take on values from the last two column vectors of the Ω matrix (see panel E of Exhibit 3). The vector on the RHS of the constraint equations is the $\mathbf{y}(\mu)$ vector (see Panel B of Exhibit 2).

***These two constraints ensure that the weights of real estate assets are non-negative.

expected portfolio return ($\mathbf{xR}^T$) subject to equation 7 ($\mathbf{x}_r \geq 0$) and equation 14 ($\mathbf{x} = \mathbf{y}(\mu) + \mathbf{\Omega\Gamma}$). In our example, there is no restriction on the short sale of financial assets (x_1, x_2 and x_3 can be negative), and therefore, Γ_1, Γ_2 and Γ_3 take zero values.

Panel A of Exhibit 4 reports the objective function and the constraints using parameters derived from the previous exhibits. Panel B of Exhibit 4 reports the optimal weight of each asset type derived from the linear programming technique. The portfolio weights for the two real estate assets are non-negative ($x_4 = .21$ and $x_5 = 0$).[7] The optimal weights for the financial assets, however, are either negative ($x_1 = -.02$) or positive ($x_2 = .07$ and $x_3 = .75$). This optimal solution satisfies the stated constraint that short sales are allowed for financial assets, but are prohibited for real estate assets. Note that the expected portfolio return ($\mu = 9.75$) derived from the linear programming solution is identical to the mean return used as the input to derive the $\mathbf{y}(\mu)$ vector (see Exhibit 2). This equality must hold in every application.

Conclusions

This paper adds a new dimension to the real estate diversification literature. Past studies examine real estate diversification benefits without explicitly considering the constraint on short sales of real estate assets. When this constraint is neglected, the diversification benefits of real estate assets will be overestimated. Using the mean-variance efficient paradigm and including a constraint on short sales of real estate assets, this study explicitly develops a simple but powerful technique for mixed-asset portfolio selection. With this technique, the diversification benefits of real estate assets can be correctly estimated and portfolio managers can easily and correctly construct efficient portfolios that include real estate assets.

This paper contributes to the literature by investigating the impact that short sale constraints have on the formation of mean-variance efficient portfolios. Since the Capital Asset Pricing Model (CAPM) has its root in portfolio theory, further investigation of the relationship between the short sale constraint and real estate asset pricing is strongly encouraged. An empirical study that uses historical data to quantify the impact of short sale constraints on the diversification benefits of real estate is also warranted.

Notes

[1]Using the mean-variance efficient paradigm, Grissom et al. [3] also examine the diversification benefits of including real estate properties in portfolios.

[2]Dybvig [1] indicates that the mean-variance efficient frontier could be kinked when short sales are constrained.

[3]There might be some nonlinear programming packages available to solve constrained nonlinear problems. To use this type of package, the user must be quite sophisticated in the area.

[4]A notation with a bold face refers to either a vector or a matrix. A notation with a " ~ " represents a random variable. A notation with a superscript "T" denotes the transpose vector or matrix.

[5]As shown by Pratt [7], the mean-variance approach is consistent with the constant risk-aversion utility. Either the assumption of a mean-variance utility function or a normal distribution of asset returns will result in the same portfolio formulation as in equations (5) to (7).

[6]For the time series properties of the data, see Tables A1 to A7 in Webb et al.'s study.

[7]The intuitive explanation as to why the weight of the WAC Index must be zero can be found in footnote 8 of Vandell [8]. However, Vandell's proposed method is difficult to implement if there are more than two assets in the portfolio that require short sale adjustments.

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